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A NOTE ON THE RANGE OF VECTOR MEASURES

Niccolò Urbinati — Hans Weber

Dedicated to Professor Paolo de Lucia on the occasion of his 80th birthday

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ABSTRACT. We give another proof for Kluvanek and Knowles' characterization of Liapounoff measures [KLUVANEK, I.—KNOWLES, G.: *Vector Measures and Control Systems.* North-Holland Mathematics Studies 20, Amsterdam, 1976] and of the fact that the range of an exhaustive measure with values in a complete locally convex space is relatively weakly compact.

1. Introduction

Liapounoff's famous theorem about the range of measures says that the range of any non-atomic σ-additive $\mathbb{R}^n$-valued measure defined on a σ-algebra is convex and compact. The validity of such a theorem also characterizes finitely dimensional Banach spaces, see [1: Corollary 6, p. 265]. More generally, Wnuk [10] proved: If X is an F-normed linear space such that any non-atomic σ-additive measure $\mu\colon \mathcal{A} \to X$ defined on a σ-algebra is convex or compact, then X is finitely dimensional. On the other hand, for a σ-additive measure $\mu\colon \mathcal{A} \to E$ defined on a σ-algebra with values in a Hausdorff complete locally convex linear space, Kluvanek and Knowles give in [4: Theorem V.1.1] a necessary and sufficient condition for the range of μ to be convex and weakly compact. The proof of this theorem is rather involved. Much simpler is the proof in the Banach-space-valued case given in [1: IX.1.4]. The reason why the Banach-space-valued case is easier to handle is that in this situation a vector measure has a real control measure by a theorem of Bartle, Dunford and Schwartz.

In this note we give another proof of Kluvanek and Knowles' version of Liapounoff's convexity theorem. Using the Frechét-Nikodým-approach we decompose an E-valued measure μ as $\mu = \sum_{\gamma\in\Gamma} \mu_\gamma$ where each μ_γ has a real control measure. Based on such a decomposition the proof can be reduced to the case of measures admitting a control measure and in this case the proof can be done as in the Banach space-valued case. Moreover, we give in Theorem 5.3 a *finitely additive* version of Kluvanek and Knowles' theorem.

Another theorem in which we are interested in says that any exhaustive E-valued measure has relatively weakly compact range. As Tweddle [7] showed, this is an easy consequence of James' characterization of weakly compact sets. But James' theorem is rather deep and therefore it is of interest to give a proof of this theorem about the range of measures based on less deep tools. A proof not based on James' theorem can be found in [4]. Much easier is Bartle-Dunford-Schwartz' proof in the Banach space-valued case, see [1: Corollary I.1.6]. In this note we prove the theorem

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about the relative weak compactness of the range of an exhaustive measure μ again using the decomposition μ as $\mu = \sum_{\gamma\in\Gamma} \mu_\gamma$ where the μ_γ's are measures admitting a control measure, and thus we can reduce the proof to the case of measures admitting a control measure which can be done as in the Banach space-valued case.

This note is organized as follows. In Section 2 we present the mentioned theorems of Tweddle and of Kluvanek and Knowles in the special case of measures admitting a control measure. For completeness and convenience of the reader we include their short proofs which are almost identical with the proofs of Diestel and Uhl [1] in the Banach space-valued case. In Section 3 we study uniform summability. In Section 4 we prove a decomposition theorem for a measure μ of the type $\mu = \sum_{\gamma\in\Gamma} \mu_\gamma$ where μ_γ are measures admitting a control measure and the system of all $(y_\gamma)_{\gamma\in\Gamma}$ with y_γ belonging to the range of μ_γ is uniformly summable. In Section 5 we use the results of Sections 3 and 4 to reduce the proof of the main Theorems 5.1 and 5.3 to the case of measures admitting a control measure, the case presented in Section 2.

Notation. Throughout, let (E, ρ) be a Hausdorff complete locally convex linear space with continuous dual E^*.

$\mathcal{A}$ stands for an algebra of subsets of a nonempty Ω. If $\mathcal{R}$ is a subring of $\mathcal{A}$, then $S(\mathcal{R}) := \operatorname{span}\{\chi_{\mathrm{A}} : \mathrm{A} \in \mathcal{R}\}$ denotes the linear subspace generated by the characteristic functions χ_A, $A \in \mathcal{R}$, and $B(\mathcal{R})$ denotes the closure of $S(\mathcal{R})$ in the Banach space of all bounded functions $f\colon \Omega \to \mathbb{R}$ endowed with the sup-norm $\|f\|_s := \sup_{x\in\Omega} |f(x)|$.

Let R be a Boolean ring (not necessarily with unit). As usual we denote the symmetric difference (addition), infimum (multiplication), supremum, difference, natural order by $\triangle, \wedge, \vee, \smallsetminus, \leq$, respectively. If R is a Boolean algebra, i.e., a Boolean ring with unit, we denote the unit by e and the complement of any $x \in R$ by $x' := e \smallsetminus x$.

We say that a finitely additive function $\mu\colon R \to E$ is a *measure*. μ is called *exhaustive* if $\lim_{n\to\infty} \mu(a_n) = 0$ for any disjoint sequence $(a_n)_{n\in\mathbb{N}}$ in R. We set $N(\mu) := \{a \in R : \forall x \in [0, a]\ \mu(x) = 0\}$.

For any homomorphism ϕ we denote its kernel by $N(\phi)$.

2. The range of vector-valued measures admitting a control measure

Let $\mu\colon \mathcal{A} \to E$ be a bounded measure. Then there is a unique linear map $T_\mu\colon S(\mathcal{A}) \to E$ with $T_\mu(\chi_A) = \mu(A)$, $A \in \mathcal{A}$. Since $\{f \in S(\mathcal{A}) : 0 \leq f \leq 1\} = \operatorname{co}\{\chi_A : A \in \mathcal{A}\}$, we have:

$$\{T_\mu(f) : f \in S(\mathcal{A}),\ 0 \leq f \leq 1\} = \operatorname{co}\mu(\mathcal{A}).$$

Hence $\{T_\mu(f) : f \in S(\mathcal{A}),\ \|f\|_s \leq 1\}$ is contained in $\operatorname{co}\mu(\mathcal{A}) - \operatorname{co}\mu(\mathcal{A})$ and is therefore bounded. It follows that $T_\mu\colon (S(\mathcal{A}), \|\cdot\|_s) \to E$ is a continuous linear map and has therefore a continuous linear extension on $(B(\mathcal{A}), \|\cdot\|_s)$, which we denote again by T_μ. We also write $\int f \mathrm{d}\mu := T_\mu(f)$ for $f \in B(\mathcal{A})$. By a continuity argument one sees that $x^*(\int f \mathrm{d}\mu) = \int f \mathrm{d}(x^* \circ \mu)$ for any $x^* \in E^*$ and $f \in B(\mathcal{A})$.

Remark 2.1. If $\mathcal{N}$ is an ideal in $\mathcal{A}$ contained in $N(\mu)$ and $\phi\colon B(\mathcal{A}) \to B(\mathcal{A})/B(\mathcal{N})$ denotes the quotient map, then $B(\mathcal{N}) \subseteq N(T_\mu)$ and the linear map $\hat{T}_\mu\colon (B(\mathcal{A}), \|\cdot\|_s)/B(\mathcal{N}) \to E$ uniquely determined by $\hat{T}_\mu \circ \phi = T_\mu$ is continuous, too.

Let $\lambda\colon \mathcal{A} \to [0,+\infty[$ be a positive measure and $\mu\colon \mathcal{A} \to E$ a vector measure. We say that λ is a *control measure* for μ (in symbols $\mu \ll \lambda$) if $\lim\limits_{n\to\infty} \mu(A_n) = 0$ whenever $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ with $\lim\limits_{n\to\infty} \lambda(A_n) = 0$. In this case, μ is exhaustive and therefore bounded. A slight generalization of a theorem of Bartle, Dunford and Schwartz says that $\mu\colon \mathcal{A} \to E$ has a control measure if μ is exhaustive and E is metrizable; hereby the control measure can be chosen σ-additive if μ is σ-additive. If $\mathcal{A}$ is a σ-algebra and λ and μ are σ-additive, then $\mu \ll \lambda$ if and only if $N(\lambda) \subseteq N(\mu)$.

If $\mathcal{A}$ is a σ-algebra and $\lambda\colon \mathcal{A} \to [0,+\infty[$ a σ-additive measure then $B(\mathcal{A})$ is the space of all bounded measurable functions and $B(N(\lambda))$ is the space of all functions of $B(\mathcal{A})$ which are λ-almost everywhere (λ-a.e.) equal to 0. This means that $(B(\mathcal{A}), \|\cdot\|_s)/B(N(\lambda))$ can be identified with the Banach space $L_\infty(\lambda)$, which is the continuous dual of $L_1(\lambda)$.

The proof of the following two propositions is taken from the book of Diestel and Uhl [1: Corollary I.2.7 and Theorem IX.1.4] (there formulated for Banach space-valued measures).

Proposition 2.2. *Let $\mu\colon \mathcal{A} \to E$ be a measure on a σ-algebra and $\lambda\colon \mathcal{A} \to [0,+\infty[$ be a σ-additive control measure for μ. Then:*

(a) $\hat{T}_\mu\colon (L_\infty(\lambda), \sigma(L_\infty(\lambda), L_1(\lambda))) \to (E, \sigma(E, E^*))$ *is continuous (with respect to (w.r.t.) the weak* topology on $L_\infty(\lambda)$ and the weak topology on E).*

(b) $\overline{\text{co}}\mu(\mathcal{A}) = \{\int f \mathrm{d}\mu : f \in B(\mathcal{A}),\ 0 \le f \le 1\}$ *is weakly compact, hence $\mu(\mathcal{A})$ is relatively weakly compact.*

P r o o f. We identify functions which are equal λ-a.e..

(a) Let $(f_\gamma)_{\gamma\in\Gamma}$ be a net in $L_\infty(\lambda)$ weak* converging to 0 and $x^* \in E^*$. We show that $\lim\limits_{\gamma\in\Gamma} x^*(\int f_\gamma \mathrm{d}\mu) = 0$. Let g be the Radon-Nikodým derivative of $x^* \circ \mu$ w.r.t. λ. Then:

$$x^*\left(\int f_\gamma \mathrm{d}\mu\right) = \int f_\gamma \mathrm{d}(x^* \circ \mu) = \int f_\gamma g \mathrm{d}\lambda \to 0.$$

(b) (i) We first observe that $\{\int f \mathrm{d}\mu : g \in B(\mathcal{A}),\ 0 \le f \le 1\}$ is contained in the closure of $\{\int f \mathrm{d}\mu : f \in S(\mathcal{A}),\ 0 \le f \le 1\}$ which coincides with $\overline{co}\mu(\mathcal{A})$.

(ii) Identifying functions which are equal λ-a.e., the unit ball B of $B(\mathcal{A})$ becomes by the Alaoglou theorem a weak* compact subset of $L_\infty(\lambda)$, hence $T_\mu(B)$ is weakly compact by (a). Consequently $\{\int f \mathrm{d}\mu : f \in B(\mathcal{A}),\ 0 \le f \le 1\}$ is a weakly compact and convex set containing $\mu(\mathcal{A})$, therefore it contains $\overline{\text{co}}\mu(\mathcal{A})$ which proves the remaining inclusion. □

The following proposition is a special case of Kluvanek-Knowles' version of Liapounoff's convexity theorem [4: Theorem V.1.1].

Proposition 2.3. *Besides the assumption of* Proposition 2.2 *suppose that for every $A \in \mathcal{A} \setminus N(\lambda)$ there exists a function $g \in B(\mathcal{A}) \setminus B(N(\lambda))$ such that $\int g \mathrm{d}\mu = 0$ and the support* $\mathrm{supp}(g)$ *is contained in A. Then $\mu(\mathcal{A})$ is convex and weakly compact.*

P r o o f. Let $B_0 := \{f \in B(\mathcal{A}) : 0 \le f \le 1\}$. In view of Proposition 2.2(b) it is enough to prove that $T_\mu(B_0) \subseteq \mu(\mathcal{A})$. Let $y_0 \in T_\mu(B_0)$. Again, as in the Proposition 2.2, we identify functions which are equal λ-a.e.. Thus, by Proposition 2.2(a), $K := T_\mu^{-1}(y_0) \cap B_0$ becomes a weak* compact subset of $L_\infty(\lambda)$ and has therefore an extreme point f_0 by the Krein-Milman theorem. We show that $f_0(x) = \chi_A(x)$ λ-a.e. for some $A \in \mathcal{A}$. Suppose that this is not true. Then there is an $\varepsilon > 0$ such that $\lambda(A_\varepsilon) > 0$ where $A_\varepsilon := \{x \in \Omega : \varepsilon \le f_0(x) \le 1 - \varepsilon\}$. By assumption there is a $g \in B(\mathcal{A}) \setminus B(N(\lambda))$ such that $\mathrm{supp}(g) \subseteq A_\varepsilon$ and $\int g \mathrm{d}\mu = 0$. We may assume that $\|g\|_s \le \varepsilon$. Then $f_0 \pm g \in K$, a contradiction to the fact that f_0 is an extreme point of K. This shows that $f_0(x) = \chi_A(x)$ λ-a.e. for some $A \in \mathcal{A}$. Hence $y_0 = \int f_0 \mathrm{d}\mu = \mu(A) \in \mu(\mathcal{A})$. □

In Proposition 2.3 the idea to use Krein-Milman (and the so called *extreme point* technique) to prove $T_\mu(B_0) = \mu(\mathcal{A})$ is present in Lindenstrauss' [5] approach to the classical Liapounoff's theorem and was already employed by Uhl [8] in order to generalize the theorem to measures of finite variation whose values lie in a Banach space which is reflexive or is a separable dual space.

3. Uniformly summable families

Let $G := (G, +, \tau)$ be a Hausdorff topological commutative group, Γ an index set and $\mathcal{F}(\Gamma)$ the system of all finite subsets of Γ.

A family $(x_\gamma)_{\gamma\in\Gamma} \in G^\Gamma$ is called *summable* if the net of finite partial sums $(\sum_{\gamma\in F} x_\gamma)_{F\in\mathcal{F}(\Gamma)}$ converges to some $x \in G$. We write then $s((x_\gamma)_{\gamma\in\Gamma}) := \sum_{\gamma\in\Gamma} x_\gamma := x$. The set $\ell_1(\Gamma, G)$ of all summable families of G^Γ is a subgroup of G^Γ and $s\colon \ell_1(\Gamma, G) \to G$ is a group homomorphism.

Definition 3.1. We call a subset $A \subseteq G^\Gamma$ *uniformly summable* if $A \subseteq \ell_1(\Gamma, G)$ and for any 0-neighborhood U in G there exists $F_0 \in \mathcal{F}(\Gamma)$ such that for any $F \in \mathcal{F}(\Gamma)$ with $F_0 \subseteq F$ and any $a = (a_\gamma)_{\gamma\in\Gamma} \in A$ one has $s(a) - \sum_{\gamma\in F} a_\gamma \in U$ (i.e., $\sum_{\gamma\in\Gamma\setminus F} a_\gamma \in U$).

Using that G has a 0-neighborhood base consisting of closed sets one immediately obtains:

Proposition 3.2. *A subset A of G^Γ is uniformly summable if and only if:*

(1) $A \subseteq \ell_1(\Gamma, G)$ *and*

(2) *for any 0-neighborhood U in G there exists $F_0 \in \mathcal{F}(\Gamma)$ such that $\sum_{\gamma\in F} a_\gamma \in U$ whenever $(a_\gamma)_{\gamma\in\Gamma} \in A$ and $F \in \mathcal{F}(\Gamma)$ with $F \cap F_0 = \emptyset$.*

Corollary 3.3 (Cauchy's criterion). *Let G be complete and $A \subseteq G^\Gamma$. Then A is uniformly summable if and only if condition* (2) *of* Proposition 3.2 *is satisfied.*

Proposition 3.4. *Let G be complete and $A \subseteq G^\Gamma$ uniformly summable. Then the closure $\overline{A}$ of A in $(G, \tau)^\Gamma$ is uniformly summable, too.*

Proof. We use Cauchy criterion 3.3. Let U be a closed 0-neighborhood in G and F_0 be chosen according to Proposition 3.2. Let $F \in \mathcal{F}(\Gamma)$ with $F \cap F_0 = \emptyset$. Then $\{\sum_{\gamma\in F} a_\gamma : (a_\gamma)_{\gamma\in\Gamma} \in \overline{A}\}$ is contained in the closure of $\{\sum_{\gamma\in F} a_\gamma : (a_\gamma)_{\gamma\in\Gamma} \in A\}$ and therefore in $\overline{U} = U$ by the choice of F_0. Thus $\overline{A}$ is uniformly summable by Corollary 3.3. □

The next example shows that in Proposition 3.4 the completeness assumption cannot be cancelled. For this reason we have to be careful when further on we consider uniform summability w.r.t. the weak topology of a locally convex linear space.

Remark 3.5. Suppose that $a = (a_n)_{n\in\mathbb{N}}$ is a sequence in G which is not summable in G but summable in the completion of G. Let $a_{n,m} = a_m$ if $m \le n$ and $a_{n,m} = 0$ if $m > n$. Then $A := \{(a_{n,m})_{m\in\mathbb{N}} : n \in \mathbb{N}\}$ is uniformly summable in G, but the closure $\overline{A}$ of A in G is not so (since $a \in \overline{A}$).

Theorem 3.6. *Denote by τ_p the topology on $\ell_1(\Gamma, G)$ induced by the product topology of $(G, \tau)^\Gamma$. Let $A \subseteq G^\Gamma$ be uniformly summable. Then $s|_A$ is uniformly continuous w.r.t. τ_p.*

Proof. Let U be a symmetric 0-neighborhood in G and $F \in \mathcal{F}(\Gamma)$ such that $\sum_{\gamma \in \Gamma \setminus F} a_\gamma \in U$ for all $a = (a_\gamma)_{\gamma \in \Gamma} \in A$. Let $n := |F|$ and V a 0-neighborhood in G with

$$V^{(n)} := V + \cdots + V(n \text{ } times) \subseteq U.$$

Then $W := \{(x_\gamma)_{\gamma \in \Gamma} \in \ell_1(\Gamma, G) : \forall \gamma \in F \quad x_\gamma \in V\}$ is a 0-neighborhood in $(\ell_1(\Gamma, G), \tau_p)$. Let $x = (x_\gamma)_{\gamma \in \Gamma}$ and $y = (y_\gamma)_{\gamma \in \Gamma}$ be families in A such that $x - y \in W$. Then:

$$s(x) - s(y) = \sum_{\gamma \in F} (x_\gamma - y_\gamma) + \sum_{\gamma \in \Gamma \setminus F} x_\gamma - \sum_{\gamma \in \Gamma \setminus F} y_\gamma \in V^{(n)} + U - U \subseteq U^{(3)}.$$

□

Corollary 3.7. *Let $A_\gamma \subseteq G$ for $\gamma \in \Gamma$ such that $A := \prod_{\gamma \in \Gamma} A_\gamma$ is uniformly summable.*

(a) *If A_γ is compact for every $\gamma \in \Gamma$, then $s(A)$ is compact.*

(b) *If G is complete and A_γ is relatively compact for every $\gamma \in \Gamma$, then $s(A)$ is relatively compact.*

Proof. (a) If the A_γ's are compact, then A is compact by Tychonoff's theorem. Hence the continuous image $s(A)$ is compact.

(b) By Proposition 3.4 the closure $\overline{A} = \prod_{\gamma \in \Gamma} \overline{A}_\gamma$ is uniformly summable. Therefore $s(\overline{A})$ is compact. Thus $s(A)$ is relatively compact. □

We now consider uniformly summable families in Hausdorff topological linear spaces.

Proposition 3.8. *Let F be a Hausdorff topological linear space. Then $\ell_1(\Gamma, F)$ is a linear space and $s\colon \ell_1(\Gamma, F) \to F$ is a linear map. Therefore, if A_γ are convex subsets of F and if $A := \prod_{\gamma \in \Gamma} A_\gamma \subseteq \ell_1(\Gamma, F)$, then $s(A)$ is convex.*

Theorem 3.9. *Let (E, ρ) be a complete Hausdorff topological locally convex linear space, A_γ relatively weakly compact subsets of E for $\gamma \in \Gamma$ and $A := \prod_{\gamma \in \Gamma} A_\gamma$ uniformly summable. Then $s(A)$ is relatively weakly compact.*

Proof. (i) We first prove that $\prod_{\gamma \in \Gamma} \overline{\text{co}}(A_\gamma)$ is uniformly summable. Let U be a closed convex 0-neighborhood in E and $F_0 \in \mathcal{F}(\Gamma)$ chosen according to condition (2) of Proposition 3.2, i.e., $\sum_{\gamma \in F} A_\gamma \subseteq U$ for any $F \in \mathcal{F}(\Gamma)$ with $F \cap F_0 = \emptyset$. For such an F we then have:

$$\sum_{\gamma \in F} \overline{\text{co}}(A_\gamma) \subseteq \overline{\sum_{\gamma \in F} \text{co}\, A_\gamma} = \overline{\text{co}} \sum_{\gamma \in F} A_\gamma \subseteq \overline{\text{co}} U = U.$$

Now apply Corollary 3.3.

(ii) Let K_γ be the closure of A_γ w.r.t. $\sigma := \sigma(E, E^*)$. Then K_γ is weakly compact. Since $K := \prod_{\gamma \in \Gamma} K_\gamma \subseteq \prod_{\gamma \in \Gamma} \overline{\text{co}}(A_\gamma)$, by (i) K is uniformly summable w.r.t. ρ and therefore w.r.t. σ. We now apply Corollary 3.7(a) with $(G, \tau) := (E, \sigma)$. Thus $s(K)$ is a compact subset of (E, σ), i.e., $s(A)$ is relatively weakly compact. □

4. Measures and FN-topologies

Let u be an *FN-topology* on R, i.e., a group topology on $(R, \triangle)$ which admits a 0-neighborhood base consisting of solid [1] sets. u is called *exhaustive* if $x_n \to 0$ (u) for any disjoint sequence $(x_n)_{n\in\mathbb{N}}$ in R, or equivalently, if every monotone net is Cauchy in (R, u), see [9: Proposition 2.4]. u is called *order continuous* if every decreasing net order converging to 0 converges topologically to 0 in (R, u), or equivalently, if every monotone net order converging to some $x \in R$ converges topologically to x in (R, u).

Proposition 4.1 ([9: Proposition 4.12]). *Let u be an exhaustive Hausdorff FN-topology on R such that (R, u) is complete (as a uniform space). Then $(R, \leq)$ is a complete Boolean algebra and u is order continuous.*

We recall the proof: If $(x_\gamma)_{\gamma\in\Gamma}$ is an increasing net in R, then $(x_\gamma)_{\gamma\in\Gamma}$ is a Cauchy net, hence converges by assumption to some x in (R, u). It follows that $x = \sup x_\gamma$ (see [9: Corollary 1.8 and Proposition 1.9]). □

If u is a Hausdorff FN-topology on R, then (R, u) is a dense subring of a Boolean ring $(\tilde{R}, \tilde{u})$ endowed with a Hausdorff complete FN-topology. $(\tilde{R}, \tilde{u})$ is then the *completion* of (R, u).

Proposition 4.2 ([9: Theorem 6.1]). *Let u be an exhaustive Hausdorff FN-topology on R and $(\tilde{R}, \tilde{u})$ the completion of (R, u). Then $\tilde{R}$ is a complete Boolean algebra and $\tilde{u}$ is order continuous.*

This immediately follows from Proposition 4.1 and the fact that with u also $\tilde{u}$ is exhaustive, see [9: Proposition 2.5].

Let $\mu\colon R \to E$ be a measure. Then the sets $\{a \in R : \mu([0, a]) \subseteq U\}$, where U is a 0-neighborhood in E, form a 0-neighborhood base for the weakest FN-topology on R making μ (uniformly) continuous. This topology is called the *μ-topology*. Obviously μ is exhaustive if and only if the μ-topology is exhaustive. Adapting the terminology of [4: p. 71] we call μ *closed* if (R, μ-topology) is complete. To compare the assumptions of Theorem 5.3 and of Proposition 2.3 we mention the well-known fact that $\mu\colon \mathcal{A} \to E$ is closed if $\mathcal{A}$ is a σ-algebra and μ has a σ-additive control measure.

A measure $\mu\colon R \to E$ induces a measure $\hat{\mu}\colon \hat{R} \to E$ on the quotient $\hat{R} := R/N(\mu)$ where $\hat{\mu}(\xi) = \mu(x)$ if $x \in \xi \in \hat{R}$. Obviously $N(\hat{\mu}) = \{0\}$; moreover $\hat{\mu}$ is exhaustive or closed if and only if μ is exhaustive or closed, respectively.

Proposition 4.3. *Let $\mu\colon R \to E$ be an exhaustive measure, u the μ-topology, $(\tilde{R}, \tilde{u})$ the completion of $\hat{R} := (R, u)/N(\mu)$ and $\tilde{\mu}\colon (\tilde{R}, \tilde{u}) \to E$ the continuous extension of $\hat{\mu}$ where $\hat{\mu}(\xi) = \mu(x)$ for $x \in \xi \in \hat{R}$. Then $\tilde{R}$ is a complete Boolean algebra and $\tilde{u}$ is order continuous. Moreover, $\mu(R)$ is a dense subset of $\tilde{\mu}(\tilde{R})$ and $\tilde{\mu}$ is a completely additive measure, i.e., $(\tilde{\mu}(\xi_\gamma))_{\gamma\in\Gamma}$ is summable for any disjoint family $(\xi_\gamma)_{\gamma\in\Gamma}$ in $\tilde{R}$ and $\tilde{\mu}(\sup\limits_{\gamma\in\Gamma} \xi_\gamma) = \sum\limits_{\gamma\in\Gamma} \tilde{\mu}(\xi_\gamma)$.*

Proof. The first assertion follows from Proposition 4.2. Obviously, $\mu(R) = \hat{\mu}(\hat{R})$ and $\hat{\mu}(\hat{R})$ is dense in $\tilde{\mu}(\tilde{R})$. The complete additivity of $\tilde{\mu}$ follows from the fact that $\tilde{\mu}$ is continuous w.r.t. an order continuous FN-topology. □

Remark 4.4. One easily verifies that under the assumptions and notations of Proposition 4.3 the $\tilde{\mu}$-topology agrees with $\tilde{u}$ (see [9: Remark 5.1.9]).

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[1] A subset U of R is *solid* if $x \leq y \in U$ implies $x \in U$.

Theorem 4.5. *Let $\mu\colon R\to E$ be a closed exhaustive measure. Then there is a family $d_\gamma\in R$ of almost disjoint*[2] *elements and $x^*_\gamma\in E^*$, $\gamma\in\Gamma$, such that the measures $\mu_\gamma\colon R\to E$ defined by $\mu_\gamma(x)=\mu(x\wedge d_\gamma)$ have the following properties:*

(1) *$\mu_\gamma\ll|x^*_\gamma\circ\mu|$ for $\gamma\in\Gamma$;* [3]

(2) *$\mu=\sum\limits_{\gamma\in\Gamma}\mu_\gamma$ (i.e., for any $x\in R$ the family $(\mu_\gamma(x))_{\gamma\in\Gamma}$ is summable and $\mu(x)=\sum\limits_{\gamma\in\Gamma}\mu_\gamma(x)$);*

(3) *$\prod\limits_{\gamma\in\Gamma}\mu_\gamma(R)$ is uniformly summable;*

(4) *$\mu(R\wedge a)=s(\prod\limits_{\gamma\in\Gamma}\mu_\gamma(R\wedge a))$ for every $a\in R$ where s has the same meaning as in* Section 3.

Proof. Passing to the quotient $\hat R=R/N(\mu)$ we may assume that $N(\mu)=\{0\}$. Indeed, if $\pi\colon R\to\hat R$ is the quotient map, $\hat\mu\colon\hat R\to E$ the measure determined by $\hat\mu\circ\pi=\mu$ and if $\hat\mu=\sum\limits_{\gamma\in\Gamma}\hat\mu_\gamma$ is a desired decomposition of $\hat\mu$, then with $\mu_\gamma=\hat\mu_\gamma\circ\pi$ we get the desired decomposition $\mu=\sum\limits_{\gamma\in\Gamma}\mu_\gamma$ of μ.

We now assume that $N(\mu)=\{0\}$. Therefore R is a complete Boolean algebra and the μ-topology u is order continuous (see Proposition 4.2). Let $x^*\in E^*$. Since $x^*\circ\mu$ is continuous w.r.t. u, $N(x^*\circ\mu)$ is a closed ideal, hence $N(x^*\circ\mu)=[0,a(x^*)]$ for some $a(x^*)\in R$. Let $b(x^*)=a(x^*)'$ be the complement of $a(x^*)$. Since

$$\bigcap_{x^*\in E^*}[0,a(x^*)]=\bigcap_{x^*\in E}N(x^*\circ\mu)=N(\mu)=\{0\},$$

we have $\inf\limits_{x^*\in E^*}a(x^*)=0$, thus $\sup\limits_{x^*\in E^*}b(x^*)=e$. Therefore there exists a disjoint family $(d_\gamma)_{\gamma\in\Gamma}$ in R and a family $(x^*_\gamma)_{\gamma\in\Gamma}$ in E^* such that $d_\gamma\le b(x^*_\gamma)$ for $\gamma\in\Gamma$ and $\sup\limits_{\gamma\in\Gamma}d_\gamma=e$. Let $\mu_\gamma(x):=\mu(x\wedge d_\gamma)$ for $x\in R$ and $\gamma\in\Gamma$. Then

$$N(|x^*_\gamma\circ\mu|)=N(x^*_\gamma\circ\mu)=[0,b(x^*)']\subseteq[0,d'_\gamma]=N(\mu_\gamma).$$

Therefore $\mu_\gamma\ll|x^*_\gamma\circ\mu|$ since with μ also $x^*_\gamma\circ\mu$ and $|x^*_\gamma\circ\mu|$ are completely additive.

(2) immediately follows from the complete additivity of μ: If $x\in R$, then

$$\mu(x)=\mu(\sup_{\gamma\in\Gamma}x\wedge d_\gamma)=\sum_{\gamma\in\Gamma}\mu(x\wedge d_\gamma)=\sum_{\gamma\in\Gamma}\mu_\gamma(x).$$

To prove (3) let U be a 0-neighborhood in E. Then $U^*:=\{a\in R:\mu([0,a])\subseteq U\}$ is a 0-neighborhood in (R,u). Since u is order continuous, the net $\sup\limits_{\gamma\in F}d_\gamma$, where F is a finite subset of Γ, converges to e. Therefore there exists a finite $F_0\subseteq\Gamma$ such that $s'\in U^*$ where $s:=\sup\limits_{\gamma\in F_0}d_\gamma$. Let F be a finite subset of $\Gamma\setminus F_0$ and $a_\gamma\in R$, $\gamma\in\Gamma$. Then $\sup\limits_{\gamma\in F}a_\gamma\wedge d_\gamma\le s'$, hence

$$\sum_{\gamma\in F}\mu_\gamma(a_\gamma)=\sum_{\gamma\in F}\mu(a_\gamma\wedge d_\gamma)=\mu(\sup_{\gamma\in F}a_\gamma\wedge d_\gamma)\in U.$$

This proves $\sum\limits_{\gamma\in F}\mu_\gamma(R)\subseteq U$.

[2] i.e., $d_\alpha\wedge d_\beta\in N(\mu)$ for different indexes $\alpha,\beta\in\Gamma$.

[3] For a real-valued measure ν, the total variation of ν is denoted by $|\nu|$.

(4) By (3)we have $\prod_{\gamma\in\Gamma} \mu_\gamma(R\wedge a) \subseteq \ell_1(\Gamma, E)$ and (2) then implies the inclusion $\subseteq$ in (4). For the inclusion $\supseteq$ let $a \in R$ and $y_\gamma \in \mu_\gamma(R\wedge a)$ for $\gamma\in\Gamma$. If $a_\gamma \in R$ with $y_\gamma = \mu_\gamma(a_\gamma \wedge a)$, then

$$s((y_\gamma)_{\gamma\in\Gamma}) = \sum_{\gamma\in\Gamma} \mu(a\wedge a_\gamma \wedge d_\gamma) = \mu(a \wedge \sup_{\gamma\in\Gamma} a_\gamma \wedge d_\gamma) \in \mu(R\wedge a).$$

□

From Theorem 4.5 one can also deduce a similar decomposition theorem for an exhaustive measure $\mu\colon R \to E$ which is not necessarily closed: Let $\tilde\mu$ be chosen as in Proposition 4.3 and $\tilde\mu = \sum_{\gamma\in\Gamma} \tilde\mu_\gamma$ a decomposition according to Theorem 4.5. If $\pi\colon R \to R/N(\mu)$ the quotient map and $\mu_\gamma := \tilde\mu_\gamma \circ \pi$, then $\mu = \sum_{\gamma\in\Gamma} \mu_\gamma$ is a decomposition satisfying (1), (2), (3) of 4.5 and instead of (4) the weaker condition that $\mu(R\wedge a)$ is a dense subset of $s(\prod_{\gamma\in\Gamma} \mu_\gamma(R\wedge a))$ for every $a \in R$.

5. The range of vector-valued measures

We first give a new proof of the following well-known result; more precisely, new is the reduction to the special case presented in Proposition 2.2(b).

Theorem 5.1. *Let $\mu\colon R\to E$ be an exhaustive measure. Then the range $\mu(R)$ is relatively weakly compact.*

Proof. Choose $\tilde R$, $\tilde u$, $\tilde\mu$ as in Proposition 4.3. Since $\mu(R) \subseteq \tilde\mu(\tilde R)$, we may assume that $(R, u, \mu) = (\tilde R, \tilde u, \tilde \mu)$, i.e., we may assume that the μ-topology u is Hausdorff and order continuous (hence exhaustive), moreover that (R, u) is complete (as uniform space), $(R, \le)$ is a complete Boolean algebra and μ is completely additive. In particular, μ is a closed exhaustive measure (see also Remark 4.4). Let $\mu = \sum_{\gamma\in\Gamma} \mu_\gamma$ be a decomposition of μ according to Theorem 4.5. We first show that $\mu_\gamma(R)$ is relatively weakly compact for any $\gamma \in \Gamma$. By the Loomis-Sikorski representation theorem [6: 29.1] there is a σ-algebra $\mathcal{A}$ and a Boolean epimorphism $\pi\colon \mathcal{A} \to R$ such that the kernel $N(\pi)$ is a σ-ideal in $\mathcal{A}$, i.e., R is isomorphic to the quotient $\mathcal{A}/N(\pi)$. Let $x^*_\gamma \in E^*$ as in Theorem 4.5. Then $\lambda_\gamma := |x^*_\gamma \circ \mu| \circ \pi\colon \mathcal{A} \to [0, +\infty[$ is a σ-additive control measure for the measure $\nu_\gamma := \mu_\gamma \circ \pi\colon \mathcal{A} \to E$. Therefore the range of ν_γ is relatively weakly compact by Proposition 2.2, i.e., $\mu_\gamma(R) = \nu_\gamma(\mathcal{A})$ is relatively weakly compact.

We now can apply Theorem 3.9. The relative weak compactness of $\mu_\gamma(R)$, $\gamma\in\Gamma$, implies in view of (3) and (4) of Theorem 4.5 that $\mu(R)$ is relatively weakly compact. □

In the proof of 5.3 we are interested in the relationship of the integral w.r.t. μ and the integral w.r.t. $\hat\mu$ where $\pi\colon \mathcal{A} \to \hat{\mathcal{A}}$ is an epimorphism onto an algebra of sets and $\mu = \hat\mu \circ \pi$. This relationship is an immediate consequence of a result contained in [2: 45.D], summarized in (1) of the following lemma.

Lemma 5.2. *Let $\pi\colon \mathcal{A} \to \hat{\mathcal{A}}$ be a Boolean epimorphism onto an algebra $\hat{\mathcal{A}}$ of sets, $\hat\mu\colon \hat{\mathcal{A}} \to E$ a bounded measure and $\mu = \hat\mu\circ\pi$.*

(1) *Then there exists a unique continuous Riesz epimorphism*[4] $\overline{\pi}\colon B(\mathcal{A}) \to B(\hat{\mathcal{A}})$ *such that $\overline{\pi}(\chi_A) = \chi_{\pi(A)}$ for any $A \in \mathcal{A}$.*

[4] A Riesz epimorphism is a surjective linear map which is also a lattice homomorphism.

(2) $\int g \mathrm{d}\mu = \int \overline{\pi}(g) \mathrm{d}\hat{\mu}$ *for any* $g \in B(\mathcal{A})$.

Proof. (1) is contained in [2: 45.D].

(2) Since $\mu = \hat{\mu} \circ \pi$, we have $T_\mu(\chi_A) = (T_{\hat{\mu}} \circ \overline{\pi})(\chi_A)$ for any $A \in \mathcal{A}$. By linearity and continuity of T_μ, $T_{\hat{\mu}}$, $\overline{\pi}$ we obtain $T_\mu(g) = (T_{\hat{\mu}} \circ \overline{\pi})(g)$ for any $g \in B(\mathcal{A})$. □

The following theorem is essentially a reformulation of [4: Theorem V.1.1] on page 82. In Theorem 5.3 we have replaced the assumption of [4: V.1.1] that μ is a σ-additive measure on a σ-algebra by the weaker condition that μ is an exhaustive finitely additive measure on an algebra. Furthermore, new is here the proof to reduce (1) ⇒ (5) of Theorem 5.3 to the special case presented in Proposition 2.3.

Theorem 5.3. *Let* $\mu\colon \mathcal{A} \to E$ *be a closed exhaustive measure and* $\mathcal{N} := N(\mu)$. *Then the following conditions are equivalent:*

(1) *For every* $A \in \mathcal{A} \setminus \mathcal{N}$ *there exists a function* $g \in B(\mathcal{A}) \setminus B(\mathcal{N})$ *such that* $\int g \mathrm{d}\mu = 0$ *and* $\operatorname{supp}(g) \subseteq A$.

(2) *For every* $A \in \mathcal{A} \setminus \mathcal{N}$ *there exists a function* $g \in S(\mathcal{A}) \setminus S(\mathcal{N})$ *such that* $\int g \mathrm{d}\mu = 0$ *and* $\operatorname{supp}(g) \subseteq A$.

(3) *For every* $A \in \mathcal{A} \setminus \mathcal{N}$ *there are sets* $B, C \subseteq \mathcal{A}$ *contained in* A *such that* $\mu(B) = \mu(C)$ *and* $B \triangle C \notin \mathcal{N}$.

(4) $\mu(\mathcal{A} \cap A)$ *is convex for every* $A \in \mathcal{A}$.

(5) $\mu(\mathcal{A} \cap A)$ *is convex and weakly compact for every* $A \in \mathcal{A}$.

Proof. (5) ⇒ (4) and (2) ⇒ (1) are obvious.

(4) ⇒ (3): Let $A \in \mathcal{A} \setminus \mathcal{N}$ and $B \in \mathcal{A} \cap A$ with $\mu(B) = \frac{1}{2}\mu(A)$. Then by choosing $C := A \setminus B$ the condition (3) is satisfied.

(3) ⇒ (2): If A, B, C are taken as in (3), then $g := \chi_B - \chi_C \in S(\mathcal{A}) \setminus S(\mathcal{N})$, $\int g \mathrm{d}\mu = 0$ and $\operatorname{supp}(g) \subseteq A$.

We now prove the main implication (1) ⇒ (5): Let $\mu = \sum\limits_{\gamma \in \Gamma} \mu_\gamma$ be a decomposition of μ according to Theorem 4.5. If we can prove that $\mu_\gamma(\mathcal{A} \cap A)$ is convex and weakly compact for every $\gamma \in \Gamma$ and $A \in \mathcal{A}$ then Corollary 3.7(a) and Proposition 3.8 together with Theorem 4.5 yield that $\mu(\mathcal{A} \cap A)$ is convex and weakly compact. Since the μ_γ's have a control measure as stated in (1) of Theorem 4.5, it is enough to prove (1) ⇒ (5) under the additional assumption that $\lambda := |x^* \circ \mu|$ is a control measure of μ for some $x^* \in E^*$ which we assume in the following. If $\mathcal{A}$ was a σ-algebra and μ σ-additive one could immediately apply Proposition 2.3 to finish the proof. In the finitely additive case we have still to employ Lemma 5.2.

Let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{N}$ be the quotient map. By Stone's representation theorem $\mathcal{A}/\mathcal{N}$ is isomorphic to an algebra $\hat{\mathcal{A}}$ of sets. To simplify the notation we identify $\mathcal{A}/\mathcal{N}$ with $\hat{\mathcal{A}}$, thus $\pi\colon \mathcal{A} \to \hat{\mathcal{A}}$ becomes an epimorphism onto $\hat{\mathcal{A}}$. Since by Proposition 4.1 $\mathcal{A}/\mathcal{N}$ ($= \hat{\mathcal{A}}$) is a complete Boolean algebra, by the representation theorem of Loomis-Sikorski there is a σ-algebra $\mathcal{A}_0$ and an epimorphism $\pi_0\colon \mathcal{A}_0 \to \hat{\mathcal{A}}$ such that $\mathcal{N}_0 := N(\pi_0)$ is a σ-ideal. Define measures $\hat{\mu}$, $\hat{\lambda}$, μ_0, λ_0 on $\hat{\mathcal{A}}$ and $\mathcal{A}$, respectively, by $\hat{\mu} \circ \pi = \mu = \hat{\mu}_0 \circ \pi_0$ and $\hat{\lambda} \circ \pi = \lambda = \hat{\lambda}_0 \circ \pi_0$. Then $\hat{\lambda}$ and λ_0 are control measures, respectively, for $\hat{\mu}$ and μ_0. By Proposition 4.3 $\hat{\mu}$ and $\hat{\lambda}$ are completely additive, therefore μ_0 and λ_0 are σ-additive. Let $A_0 \in \mathcal{A}_0 \setminus N(\mu_0)$, $\hat{A} := \pi_0(A_0)$ and $A \in \mathcal{A}$ with $\pi(A) = \hat{A}$. Chose $\overline{\pi}$ as in Lemma 5.2, and analogously let $\overline{\pi_0}\colon B(\mathcal{A}_0) \to B(\hat{\mathcal{A}})$ be the continuous Riesz epimorphism with $\overline{\pi_0}(\chi_A) = \chi_{\pi_0(A)}$ for $A \in \mathcal{A}_0$. By assumption there is a function $g \in B(\mathcal{A}) \setminus B(\mathcal{N})$ such that $\int g \mathrm{d}\mu = 0$ and $\operatorname{supp}(g) \subseteq A$. Therefore $|g| \leq k\chi_A$ for some $k \in \mathbb{N}$. Let $g_0 \in B(\mathcal{A}_0)$

with $\overline{\pi_0}(g_0) = \overline{\pi}(g)$. Replacing g_0 by $(-k\chi_{A_0} \vee g_0) \wedge k\chi_{A_0}$ we may assume that $|g_0| \leq k\chi_{A_0}$, i.e., $\mathrm{supp}(g_0) \subseteq A_0$. Moreover, $g \in B(\mathcal{A}) \setminus B(\mathcal{N})$ implies $\overline{\pi_0}(g_0) = \overline{\pi}(g) \neq 0$, hence $g_0 \in B(\mathcal{A}_0) \setminus B(\mathcal{N}_0)$. Finally $\int g_0 \mathrm{d}\mu_0 = \int \overline{\pi_0}(g_0) \mathrm{d}\hat{\mu} = \int \overline{\pi}(g) \mathrm{d}\hat{\mu} = \int g \mathrm{d}\mu = 0$. Now Proposition 2.3 yields that $\mu_0(\mathcal{A}_0)$ is convex and weakly compact. Hence $\mu(\mathcal{A}) = \mu_0(\mathcal{A}_0)$ is convex and weakly compact. For any $A \in \mathcal{A}$ the same argument can be applied to $\mu|_{\mathcal{A} \cap A}$. Thus $\mu(\mathcal{A} \cap A)$ is convex and weakly compact. □

With the notation of Remark 2.1, condition (1) of Theorem 5.3 means exactly that $\hat{T}_\mu \colon B(\mathcal{A})/B(\mathcal{N}) \to E$ is not injective. If under the assumption of Theorem 5.3 μ is non-atomic, then $B(\mathcal{A})/B(\mathcal{N})$ infinite dimensional. Therefore condition (1) of Theorem 5.3 is obviously satisfied if μ is non-atomic and $\dim E < +\infty$; in this way Kluvanek and Knowles [4] deduce Liapounoff's classical convexity theorem from their Theorem V.1.1.

We would like to give a comment to the theorem of Wnuk mentioned in the introduction which says that an F-space X is finite dimensional if any σ-additive non-atomic measure $\mu \colon \mathcal{A} \to X$ defined on a σ-algebra has compact and convex range. In this theorem the metrizability assumption cannot be cancelled. Indeed if $X = c_{00}$ is the space of all real sequences which are eventually 0 endowed with the box topology τ_b, then (c_{00}, τ_b) is a complete locally convex linear space (see [3: section 6.6]). If $\mu \colon \mathcal{A} \to (c_{00}, \tau_b)$ is a non-atomic σ-additive measure on a σ-algebra, then μ is bounded; therefore $\mu(\mathcal{A})$ is contained in a finite dimensional subspace of c_{00}. By the classical version of Liapounoff's theorem, $\mu(\mathcal{A})$ is both convex and compact.

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Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Università di Udine
Via delle scienze 206
33100 Udine
ITALY

E-mail: urbinati.niccolo@spes.uniud.it
hans.weber@uniud.it