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Generalized coalitions and bargaining sets☆

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ABSTRACT

We introduce new notions of bargaining set for mixed economies which rest on the idea of generalized coalitions (Aubin, 1979) to define objections and counter-objections. We show that the bargaining set defined through generalized coalitions coincides with competitive allocations under assumptions which are weak and natural in the mixed market literature. As a further result, we identify some additional properties that a generalized coalition must satisfy to object an allocation.

1. Introduction

The core of an economy is defined as the set of feasible allocations that are not blocked or objected by any coalition. The possibility for other agents to react to this objection and propose a new counter-objection is not taken into account. Aumann and Maschler (1964) propose a new solution concept according to which objections that are counter-objected are not credible and therefore they should be neglected. Mas-Colell (1989) adapts this notion to atomless economies and defines the bargaining set as an enlargement of the core containing all the feasible allocations against which it is impossible to raise an objection with no counter-objections. Mas-Colell (1989) proves the equivalence between the set of competitive equilibria and the bargaining set under assumptions that are close to those used to prove the existence of competitive equilibria and the Core–Walras equivalence theorem respectively in Aumann (1966, 1964). The key idea of Mas Colell's proof consists in characterizing credible objections as those that are price supported. This allows him to conclude that the set of competitive allocations and the bargaining set coincide and are equivalent to the core in atomless economies.

It is clear that when we move on to the case of finite economies the previous equivalences are no longer true. In this case, in fact, the core and, a fortiori, the bargaining set, strictly contains the set of competitive allocations. Furthermore, Anderson et al. (1997) show that, whereas the core shrinks to the set of competitive allocations after a sufficiently large number of replicas, the bargaining set does not. A similar investigation is conducted by Shitovitz (1989) in mixed markets, i.e. economies in which the measure space of agents have both atoms and an atomless sector. An atom of a measure space (T, Σ, m) is a set $A \in \Sigma$ with positive measure such that $m(A \setminus B) = 0$ or $m(B) = 0$ for every other $B \subseteq A$ and it represents a non-negligible agent in the market. For example, an atom can be representative of a trader who concentrates in his hands an initial ownership of commodities that is sufficiently large with respect to the total market endowment, as in the case of monopolistic or, more generally, oligopolistic markets. Or else, even though the initial endowment is spread over a continuum of negligible traders, an atom can be representative of a group of traders that decide to act as a single player, as in the case of cartels, syndicates, or similar institutions. Notice that in a mixed market the set of agent T is the disjoint union of an atomless section T_0 and the atomic part T_1. This allows to view as special case of mixed markets both atomless economies (once T_1 is empty) and finite economies (when T_0 is null and T_1 finite). Shitovitz (1989) proves that, if in addition to certain assumptions there exists a commodity owned by only one of the atoms (*veto player*), then the core coincides with the bargaining set and it strictly contains the set of competitive allocations. He also illustrates an example of mixed market outside the class mentioned above and satisfying the sufficient hypotheses for the Core–Walras equivalence theorem (Shitovitz, 1973) in which the bargaining set is strictly larger than the core.

The previous conclusions seem to suggest that it is the core, rather than the set of competitive equilibria, to be compared with

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the bargaining set in models comprising atoms. In this work, instead, we go back to the original idea in Mas-Colell (1989), with the aim of characterizing the bargaining set also in mixed markets by means of competitive equilibria. Our approach consists in relaxing the class of coalitions that can form an objection and/or a counter-objection according to the veto mechanism of Aubin (1979). We allow agents to join a coalition with a partial participation rate, rather than to decide only whether to join or not. Basically, we enlarge the class of potential blocking coalitions to the so-called *generalized coalitions*. A generalized coalition is a measurable function γ from the space of agents T to the unit interval [0, 1] with non-null support. Intuitively, the value $\gamma(t)$ represents the share of resources employed by agent t in the formation of the coalition γ. We define four variants of the bargaining set depending on which class of coalitions is involved in objections and/or counter-objections and we study the relations among them (Proposition 2.13). In particular we show that all four bargaining set variants coincide when the economy is atomless, the familiar framework of Mas-Colell (1989) (Proposition 3.7). Our main result states the equivalence between the set of competitive allocations and a bargaining set in mixed economies in which atoms have convex preferences (Theorem 1), an assumption quite common in the literature of mixed markets (see for example Hildenbrand, 1974; Shitovitz, 1973; Greenberg and Shitovitz, 1986; Pesce, 2014; Bhowmik and Graziano, 2015; Avishay, 2019, among the others). Our theorem extends to mixed economies the Mas-Colell's equivalence theorem since, as already noticed, once the set of atoms is null, a mixed market reduces to be an atomless economy. From a technical point of view, we closely follow the approach of Mas-Colell (1989), since we identify the notion of competitive objection as the one on which to focus attention. Indeed, even if competitive objections are defined as particular objections with a specific property and hence constitute only a part of the set of all possible objections, they are the only ones to consider when dealing with the bargaining set. Precisely, we prove that in order to show that an allocation belongs to the bargaining set it is enough to verify that there are no competitive objections against it. From this result we derive the existence of a competitive equilibrium in a mixed market under quite mild conditions (Corollary 3.6) as done by Mas-Colell (1989) for atomless economies. On the other hand, contrary to Mas-Colell (1989), since the measure space of agents we consider is not necessarily non-atomic, we cannot conclude that the correspondence defined as the integral of the demand net trade set has convex values. For this reason we work with its convex hull, a needless step in Mas-Colell's setting thanks to Lyapunov–Richter's Theorem. A further contribution of this paper is the identification of certain properties that a generalized coalition γ has to satisfy to object. Indeed, we show that for an allocation f outside the bargaining set there exists a competitive objection characterized by full participation of negligible traders as well as of traders which are strictly better off (Proposition 4.3).

Summing up, our analysis contributes to two literatures: the one that studies bargaining sets in exchange economies and the literature on mixed markets. Recently Hervés-Beloso et al. (2018), Hervés-Estévez and Moreno-García (2018a) and Hervés-Estévez and Moreno-García (2018b) study the notion of bargaining set in finite economies. They allow generalized coalitions to form objections and counter-objections and obtain the Mas-Colell's equivalence theorem for finite economies. Their result follows from ours since, as earlier observed, even a finite economy can be viewed as a special case of mixed market. At the same time, our work differs from the previous contributions in many respects. We consider four variants of the bargaining set among which only one is an extension to mixed markets of the definition they adopt. We obtain the equivalence theorem directly via the notion of competitive objections and we use existence and welfare theorem arguments, whereas in the above papers the equivalence is obtained by associating to the finite economy a continuum economy with a finite number of types of agents.[1] On the other hand, they also allow for production, they also investigate on the bargaining set of replica economies and analyze how the restriction on the formation of coalitions may impact on the bargaining set. We defer the analysis of our model to address these research questions to future works.

The paper is organized as follows: in Section 2 we introduce the economic model and the main definitions. Our main theorem is presented in Section 3 whereas further results and concluding remarks are stated respectively in Section 4 and Section 5. All the proofs are collected in the Appendix.

2. The model and main definitions

2.1. The economic model

We consider an exchange economy $\mathcal{E}$ with a finite number N of different commodities. The commodity space is therefore the positive orthant $\mathbb{R}^N_+$ of the N-dimensional Euclidean space while $\Delta := \{p \gg 0 : \sum_{i=1}^N p_i = 1\}$ is the set of all *price systems*. We use the symbol $\mathcal{P}$ to denote the set of all total pre-orders on $\mathbb{R}^N_+$ that are continuous and strictly monotone[2] and consider it endowed with the product topology. As usual, for $\succcurlyeq \in \mathcal{P}$, the relations $\succ$ and $\sim$ denote the irreflexive and symmetric components of $\succcurlyeq$ respectively.

The agents in the economy are represented as the points of a σ-additive, complete probability space (T, Σ, m). Each agent $t \in T$ is characterized by an initial bundle of resources $e(t) \in \mathbb{R}^N_+$ and a preference relation $\succcurlyeq_t \in \mathcal{P}$. An economy is thus represented as the measurable map:

$$\mathcal{E}: T \to \mathcal{P} \times \mathbb{R}^N_+$$

defined by the relation $\mathcal{E}(t) := (\succcurlyeq_t, e(t))$, where $e: T \to \mathbb{R}^N_+$ is an integrable function and $\succcurlyeq_t \in \mathcal{P}$ for all $t \in T$. We assume that $\int e(t)dt \gg 0$ meaning that each good is present in the market.

Since we do not require m to be non-atomic, it is allowed the presence in Σ of m-atoms, i.e. sets $A \in \Sigma$ with non-zero measure which are such that $m(A \setminus B) = 0$ or $m(B) = 0$ for every other $B \subseteq A$. According to the atomless-atomic decomposition of measures, T can be partitioned into an *atomless component* representative of an ocean of negligible traders that we denote by T_0, and the *atomic component* $T_1 := T \setminus T_0$, which is the union of an at most countable family $\{A_1, A_2, \dots, A_k, \dots\}$ of disjoint atoms. With an abuse of notation we still denote by T_1 the collection $\{A_1, A_2, \dots, A_k, \dots\}$ and we write $A \in T_1$ instead of $A \subseteq T_1$. Being $\mathcal{E}: T \to \mathcal{P} \times \mathbb{R}^N_+$ a measurable map, for any $A \in T_1$ and $t, s \in A$ we necessarily have $\mathcal{E}(s) = \mathcal{E}(t)$. Therefore, every agent in A is endowed with the same preference relation $\succcurlyeq_A$ and the same initial bundle of resources e_A. This allows the usual interpretation that each atom can be considered as a single individual concentrating in his hands a large amount of the total initial endowment (oligopolistic agent) or as a group of individuals deciding to act only together (cartels, syndicates).

An *allocation* is an integrable function $f: T \to \mathbb{R}^N_+$ and it is said to be *feasible* if $\int f(t)\, dt \le \int e(t)\, dt$. The set of all allocations is denoted by $\mathcal{M}(\mathcal{E})$.

[1] For a similar construction see Husseinov (1994).

[2] A binary relation $\succcurlyeq$ on $\mathbb{R}^N_+$ is *continuous* if the sets $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open. It is *strictly monotone* if $y \succ x$ whenever $y \ge x$ and $x \neq y$.

Definition 2.1. A feasible allocation $f \in \mathcal{M}(\mathcal{E})$ is **competitive** or **Walrasian** if there is a price system $p \in \Delta$ such that, for almost all $t \in T$, $p \cdot f(t) \leq p \cdot e(t)$ and $p \cdot x > p \cdot e(t)$ whenever $x \succ_t f(t)$.

We use the symbol $\mathcal{W}(\mathcal{E})$ to indicate the set of competitive allocations in $\mathcal{E}$.

2.2. The objection mechanism

Following Aubin (1979), we allow agents to participate in coalitions using only a part of their resources. This way of considering participation in a coalition actually leads to an enlargement of the class of ordinary coalitions. Formally, a *coalition* is any element of Σ with positive measure. Whereas, a *generalized coalition* is any couple (S, γ) where $\gamma : T \to [0, 1]$ is a non-null integrable function and S is its support, i.e. the set $\{t \in T : \gamma(t) > 0\}$. We denote by $\mathcal{F}$ the collection of all generalized coalitions and we observe that, by pairing each coalition $S \in \Sigma$ with its correspondent characteristic function[3] χ_S, the σ-algebra Σ can be viewed as a subset of $\mathcal{F}$. In what follows, for the sake of the exposition, we call *standard* or *crisp coalitions* the elements of Σ with positive measure.

Definition 2.2. Given an allocation $f \in \mathcal{M}(\mathcal{E})$, a generalized coalition (S, γ) **objects** or **improves upon** f if there is an allocation $g \in \mathcal{M}(\mathcal{E})$ such that:

(i) $\int \gamma(t)g(t)\, dt \leq \int \gamma(t)e(t)\, dt$,
(ii) $g(t) \succcurlyeq_t f(t)$ for almost every $t \in S$,
(iii) $m(\{t \in S : g(t) \succ_t f(t)\}) > 0$.

In this case, the triple (S, γ, g) is said to be an **Aubin-objection** to f and we denote by $\mathcal{O}_A(f)$ the set of all the Aubin-objections to f.

The weighted veto mechanism based on generalized coalitions extends the ordinary one by assigning more power to coalitions. In fact, as it has been mentioned before, any standard coalition can be paired to its characteristic function and identified with a generalized coalition. This allows us to adapt Definition 2.2 and say that (S, g) is a *standard* (or *crisp*) *objection* against f if (S, χ_S, g) is an Aubin-objection against f. In this case we write $(S, \chi_S, g) \in \mathcal{O}(f)$, where $\mathcal{O}(f)$ denotes the set of all the standard objections against f. Observe that the inclusion $\mathcal{O}(f) \subseteq \mathcal{O}_A(f)$ always holds.

Definition 2.3. A feasible allocation f is in the **Aubin core** if $\mathcal{O}_A(f) = \emptyset$. We denote by $C_A(\mathcal{E})$ the Aubin core of the economy $\mathcal{E}$. Whereas f belongs to the **core** of $\mathcal{E}$, denoted by $C(\mathcal{E})$, if $\mathcal{O}(f) = \emptyset$.

Remark 2.4. In Definition 2.2(*ii*) a weak improvement formulation is considered, while in the definition of the Aubin-core and of the core, usually, coalitional improvement is formulated as a strong notion. However, under continuity and monotonicity, weak and strong improvement are equivalent. In particular, from the inclusion $\mathcal{O}(f) \subseteq \mathcal{O}_A(f)$ we deduce the well known inclusion $C_A(\mathcal{E}) \subseteq C(\mathcal{E})$.

Definition 2.5. Let f be an allocation and (S, γ, g) be an Aubin objection to f, i.e. $(S, \gamma, g) \in \mathcal{O}_A(f)$. A generalized coalition (Q, δ) **counter-objects** (S, γ, g) if there is an allocation h such that:

(i) $\int \delta(t)h(t)\, dt \leq \int \delta(t)e(t)\, dt$,
(ii) $h(t) \succ_t g(t)$ for almost every $t \in Q \cap S$,
(iii) $h(t) \succ_t f(t)$ for almost every $t \in Q \setminus S$.

In this case, the triple (Q, δ, h) is said to be an **Aubin-counter-objection** to (S, γ, g) and we write $(Q, \delta, h) \in \mathcal{CO}_A(S, \gamma, g)$, denoting by $\mathcal{CO}_A(S, \gamma, g)$ the set of all Aubin-counter-objections to (S, γ, g).

As done for objections, among all the counter-objections to (S, γ, g) we call (Q, h) a *standard* (or *crisp*) counter-objection to (S, γ, g) if (Q, χ_Q, h) is an Aubin-counter-objection to (S, γ, g). We denote by $\mathcal{CO}(S, \gamma, g)$ the set of all standard counter-objections to (S, γ, g) and we identify it with a subset of $\mathcal{CO}_A(S, \gamma, g)$, i.e. $\mathcal{CO}(S, \gamma, g) \subseteq \mathcal{CO}_A(S, \gamma, g)$.

We stress that, being $T = T_0 \cup T_1$, any atom $A \in T_1$ is treated as a single individual, thus it can belong to S and prefer h to g or it can be outside S and prefer h to f. The same is not guaranteed in the corresponding atomless economy obtained by splitting each atom into a continuum of negligible individual (as in Greenberg and Shitovitz, 1986), because in that case there might be two non-null groups of identical agents t of A so that one objects being in S and the other does not being outside S.

Remark 2.6 (*The Allocation Induced by an Objection*)**.** Whenever (S, γ, g) is an Aubin-objection against $f \in \mathcal{M}(\mathcal{E})$, our interest in the allocation g is limited to its restriction to S, that is the support of γ. Indeed, outside S, g can take any value (it can even be unbounded) and still it does not affect (S, γ)'s capacity of objecting the allocation f. To improve this idea, we introduce the notion of *allocation generated by* the Aubin-objection (S, γ, g) as the function $\tilde{g} \in \mathcal{M}(\mathcal{E})$ defined by

$$\tilde{g}(t) := \begin{cases} g(t) & \text{if } t \in S, \\ f(t) & \text{otherwise.} \end{cases}$$

Notice that in defining a counter-objection to the Aubin-objection (S, γ, g) it is the support S of γ that is taken into account rather than the function γ itself. Hence, the only thing that determines whether or not (S, γ, g) is counter-objected is the pair (S, g) and, consequently, the allocation $\tilde{g}$ generated by the objection.

The following proposition clarifies the role of the allocation $\tilde{g}$ induced by an objection.

Proposition 2.7. *Let (S, γ, g) be an Aubin-objection against f. Then, (S, γ, g) is Aubin-counter-objected by a generalized coalition (Q, δ) if and only if there is an allocation h such that (Q, δ, h) is an Aubin-objection to the allocation $\tilde{g}$ induced by (S, γ, g).*

Proof. See Appendix A.1.

Remark 2.8. From Proposition 2.7 we deduce that an Aubin-objection (S, γ, g) against $f \in \mathcal{M}(\mathcal{E})$ has no Aubin-counter-objection if the induced allocation $\tilde{g}$ belongs to the Aubin core. The converse might not be true because, although $\tilde{g}$ cannot be objected by any generalized coalition, it might be not feasible. However, if a standard objection (S, χ_S, g) against $f \in \mathcal{M}(\mathcal{E})$ has no Aubin counter-objection then the induced allocation $\tilde{g}$ restricted to S, that coincides with g, belongs to the Aubin core of $\mathcal{E}$ restricted to S. A similar result has been obtained by Hervés-Beloso et al. (2018)[Proposition 3.1] for production economies with a finite number of agents and in terms of standard counter-objections and core.

Remark 2.9. In a similar fashion we can observe that an Aubin-counter-objection (Q, δ, h) induces a new allocation $\tilde{h}$. Precisely, when $(S, \gamma, g) \in \mathcal{O}_A(f)$, $\tilde{g}$ is the induced allocation and $(Q, \delta, h) \in$

[3] For any $S \in \Sigma$, the characteristic function of S is the function $\chi_S : T \to [0, 1]$ that assigns 1 to each $t \in S$ and 0 to every other t outside S.

$\mathcal{CO}_A(S,\gamma,g)$, that is $(Q,\delta,h)\in\mathcal{O}_A(\tilde g)$ (by Proposition 2.7), we can define a new allocation $\tilde h$ by

$$\tilde h(t) := \begin{cases} h(t) & \text{if } t\in Q,\\ \tilde g(t) & \text{otherwise.}\end{cases}$$

Similarly to Dutta et al. (1989), we can iterate one more step the counter-objection process and define an Aubin-counter-objection to (Q,δ,h) as an Aubin-objection to $\tilde h$.

2.3. The bargaining sets

A *bargaining set* is the collection of all feasible allocations against which it is impossible to raise an objection that is not counter-objected itself. Different notions of bargaining sets can therefore be obtained by specifying which classes of objections and counter-objections are allowed at each time.

In particular, with the definitions given above, we can introduce four different versions of the bargaining set depending on whether or not Aubin or standard objections and counter-objections are considered.

Definition 2.10. Given a feasible allocation $f\in\mathcal{M}(\mathcal{E})$, we say that:

- $f\in BS_{ss}$ if all the standard objections to f have a standard counter-objection.
- $f\in BS_{as}$ if all the Aubin-objections to f have a standard counter-objection.
- $f\in BS_{sa}$ if all the standard objections to f have an Aubin-counter-objection.
- $f\in BS_{aa}$ if all the Aubin-objections to f have an Aubin-counter-objection.

Remark 2.11. The notions of BS_{ss} and BS_{aa} are an extension to mixed markets respectively of the definitions of bargaining set given by Mas-Colell (1989) and by Hervés-Estévez and Moreno-García (2018a) (see also Hervés-Estévez and Moreno-García, 2018b; Hervés-Beloso et al., 2018). In general, the relationship between the two sets BS_{aa} and BS_{ss} is unclear. Indeed, on the one hand reducing the set of potential objecting coalitions enlarges the bargaining set, whereas, on the other hand, reducing the set of potential coalitions that can counter-object restricts it. Nevertheless, the trivial inclusions $\mathcal{CO}(S,\gamma,g)\subseteq\mathcal{CO}_A(S,\gamma,g)$ and $\mathcal{O}(f)\subseteq\mathcal{O}_A(f)$, that hold whenever $f\in\mathcal{M}(\mathcal{E})$ and $(S,\gamma,g)\in\mathcal{O}_A(f)$, can be used to prove the following inclusions:

$$BS_{as}\subseteq BS_{aa}\subseteq BS_{sa} \quad\text{and}\quad BS_{as}\subseteq BS_{ss}\subseteq BS_{sa}.$$

Furthermore by Definition 2.10:

$$C_A(\mathcal{E})\subseteq BS_{as}\subseteq BS_{aa} \quad\text{and}\quad C(\mathcal{E})\subseteq BS_{ss}\subseteq BS_{sa}.$$

Our goal is to determine the relations between these four notions of bargaining sets and the set of competitive allocations $\mathcal{W}(\mathcal{E})$. In this perspective, we first focus on those allocations against which it is not possible to raise any objection at all.

Proposition 2.12. *For any f feasible allocation let us consider the following statements:*

(1) *There is a $p\gg 0$ such that $p\cdot x\geq p\cdot e(t)$ for almost every $t\in T$ and every $x\in\mathbb{R}^N_+$ for which $x\succ_t f(t)$.*
(2) *There is no Aubin-objection against f.*

Then condition (1) implies condition (2). If, in addition, $\succ_t$ is convex for every $t\in T_1$, then the conditions (1) and (2) are equivalent.

Proof. See Appendix A.2.

The above proposition establishes the well known relation between the set of competitive allocations and the Aubin core. A Walrasian allocation cannot be objected, hence $\mathcal{W}(\mathcal{E})\subseteq C_A(\mathcal{E})\subseteq C(\mathcal{E})$. Conversely, if atoms have convex preferences, any allocation not objected by a generalized coalition is competitive, i.e., $\mathcal{W}(\mathcal{E})=C_A(\mathcal{E})\subseteq C(\mathcal{E})$.[4] Consequently, a Walrasian allocation must belong to each of the bargaining sets we have defined. Furthermore, a bargaining set shrinks whenever we allow a larger set of objections or a smaller set of counter-objections. Summing up, we can state the following general result.

Proposition 2.13. *The following inclusions always hold.*

- $\mathcal{W}(\mathcal{E})\subseteq BS_{as}\subseteq BS_{ss}\subseteq BS_{sa}$.
- $\mathcal{W}(\mathcal{E})\subseteq BS_{as}\subseteq BS_{aa}\subseteq BS_{sa}$.

The above inclusions may be strict. This can be shown by means of the bargaining set B_{as}. In fact, as the next example shows, it may happen that there is a feasible non-Walrasian allocation that belongs to the Aubin core of a finite economy and hence to the set BS_{as}, making the inclusion $\mathcal{W}(\mathcal{E})\subseteq B_{as}$ strict.

Example 2.14. Consider an exchange economy with two goods ($N=2$) and three agents ($T=\{1,2,3\}$) whose characteristics are given as follows:

$$\begin{array}{lll} e(t)=(0,1) & u_t(x(t),y(t))=x^2(t)+y^2(t) & \text{for } t=1,2 \text{ and}\\ e(3)=(1,1) & u_3(x(3),y(3))=x^2(3)+y(3). & \end{array}$$

We now show that the initial endowment e is an Aubin core allocation and a fortiori it belongs to BS_{as}, whereas it is not a Walrasian allocation.

Assume to the contrary the existence of an alternative allocation (x,y) and a generalized coalition (S,γ) such that

$$\begin{cases} x^2(1)+y^2(1)>1 & \text{if } \gamma(1)>0 \ (\text{or } 1\in S),\\ x^2(2)+y^2(2)>1 & \text{if } \gamma(2)>0 \ (\text{or } 2\in S),\\ x^2(3)+y(3)>2 & \text{if } \gamma(3)>0 \ (\text{or } 3\in S),\\ \gamma(1)x(1)+\gamma(2)x(2)+\gamma(3)x(3)\leq\gamma(3) & \\ \gamma(1)y(1)+\gamma(2)y(2)+\gamma(3)y(3)\leq & \\ \quad\gamma(1)+\gamma(2)+\gamma(3). & \end{cases}$$

First notice that for $t=1,2$

$$(x(t)+y(t))^2\geq x^2(t)+y^2(t)>1 \quad\Rightarrow\quad x(t)+y(t)>1.$$

Hence, $\gamma(3)>0$, which implies that $x(3)\leq 1$, and $\gamma(1),\gamma(2)$ can not be both null. Furthermore, from $x(3)\leq 1$, it follows that $x(3)+y(3)\geq x^2(3)+y(3)>2$ and hence $x(3)+y(3)>2$. By summing the last two inequalities in the system above, we get the following contradiction.

$$\gamma(1)+\gamma(2)+2\gamma(3)<\gamma(1)[x(1)+y(1)]+\gamma(2)[x(2)+y(2)] + \gamma(3)[x(3)+y(3)]\leq\gamma(1)+\gamma(2)+2\gamma(3).$$

Hence, $e\in C_A(\mathcal{E})\subseteq BS_{as}$. We now show that e is not a Walrasian allocation. To this end, let (p,q) be any price system of Δ and consider agent 3.

If $p>q$, the bundle $(x(3),y(3))=\left(0,\frac{p+q}{q}\right)$ is such that

$$\begin{cases} u_3(x(3),y(3))=\frac{p}{q}+1>2=u_3(1,1) \quad\text{and,}\\ px(3)+qy(3)=p+q=(p,q)\cdot(1,1).\end{cases}$$

[4] For the equivalence with the standard core stronger conditions on T_1 are needed as proved by Shitovitz (1973) and Greenberg and Shitovitz (1986) among others (see also Pesce, 2010 for asymmetric information economies and Basile et al., 2016 for economies with public goods).

If $p \leq q$, the bundle $(x(3), y(3)) = (2, 0)$ is such that

$$\begin{cases} u_3(x(3), y(3)) = 4 > 2 = u_3(1, 1) & \text{and,} \\ px(3) + qy(3) = 2p \leq p + q = (p, q) \cdot (1, 1). \end{cases}$$

The example above proves that with no further assumption $\mathcal{W}(\mathcal{E}) \subsetneq BS_{aa}$ and $\mathcal{W}(\mathcal{E}) \subsetneq BS_{ss}$. In the next section we look for sufficient conditions to the equivalence between the set of competitive allocations and the bargaining set in mixed economies.

3. Equivalence results

Throughout this section we consider the following additional assumption on preferences which is standard in the literature on mixed markets (see for instance Hildenbrand, 1974).

Assumption 3.1. For all $A \in T_1$, $\succ_A$ is convex.

Note that, when Assumption 3.1 is met, Proposition 2.12 ensures that Walrasian allocations are all and only those that cannot be objected by any generalized coalition. Our main goal is now to characterize the Walrasian allocations as the only allocations for which all the Aubin objections are counter-objected by a generalized coalition, i.e. $\mathcal{W}(\mathcal{E}) = BS_{aa}$. To this end, following Mas-Colell (1989), we consider a specific class of objections obtained with the imposition of a price system p.

Definition 3.2 (*Competitive Objections*)**.** Let $f \in \mathcal{M}(\mathcal{E})$. An Aubin objection to $f(S, \gamma, g) \in \mathcal{O}_A(f)$ is **competitive** if there exists a price system $p \gg 0$ such that for every $x \in \mathbb{R}^N_+$ and almost every $t \in T$ we have:

- $p \cdot x \geq p \cdot e(t)$ whenever $t \in S$ and $x \succ_t g(t)$.
- $p \cdot x \geq p \cdot e(t)$ whenever $t \notin S$ and $x \succ_t f(t)$.

Suppose that (S, γ, g) is an objection against f and that $\tilde{g}$ is the allocation it induces. The next lemma shows that (S, γ, g) is competitive if and only if $\tilde{g}$ satisfies the condition (2) in Proposition 2.12.

Lemma 3.3. *Let (S, γ, g) be an Aubin objection against a feasible allocation $f \in \mathcal{M}(\mathcal{E})$ and let $\tilde{g}$ be the allocation it induces. Then, (S, γ, g) is competitive if and only if there is no Aubin-objection against $\tilde{g}$.*

Proof. See Appendix A.3.

By combining Proposition 2.7 and Lemma 3.3 we derive the following key result, which generalizes Propositions 1 and 3 of Mas-Colell (1989).

Proposition 3.4. *Let f be a feasible allocation and $(S, \gamma, g) \in \mathcal{O}_A(f)$. Then (S, γ, g) is competitive if and only if there is no Aubin-counter-objection against it.*

Proposition 3.4 implies that even though competitive objections represent only a small portion of all possible way that a generalized coalition can object an allocation f, they are the only one about which we should be concerned in the study of the bargaining sets. Therefore, in order to prove that an allocation f belongs to BS_{aa} (or BS_{sa}) it is sufficient to show that no competitive Aubin (or standard) objection can be raised against f. The next proposition is a generalization of Proposition 2 in Mas-Colell (1989) to mixed markets with Aubin objections.

Proposition 3.5. *Assume that $e(t) \gg 0$ for almost all $t \in T$, and let f be a feasible non-Walrasian allocation. Then there is an Aubin-objection against f that is competitive.*

Proof. See Appendix A.3.

As consequences of Proposition 3.5 we get the existence of a Walrasian allocation for a mixed economy. The same has been proved in Mas-Colell (1989) for atomless economies. Furthermore, Proposition 3.5 allows us to derive the desired equivalence theorem.

Corollary 3.6. *Assume that $e(t) \gg 0$ for almost all $t \in T$, then there exists a Walrasian allocation in the mixed economy $\mathcal{E}$.*

Proof. See Appendix A.3.

It is worthwhile to note that the existence of Walrasian allocation in a mixed market can be obtained under weaker assumptions. In particular, the requirement $e(t) \gg 0$ for almost all $t \in T$, which is used for Proposition 3.5, can be weakened with the condition $\int e(t)dt \gg 0$ (see Remark A.3 and Theorem 2 of Hildenbrand, 1974, page 151). An alternative existence proof is recently provided by D'Agata (2005), who dispenses with Assumption 3.1 and imposes a condition on the measure of the atoms (see Theorem 2 of D'Agata, 2005).

Theorem 1. *Assume that $e(t) \gg 0$ for almost all $t \in T$, then $\mathcal{W}(\mathcal{E}) = BS_{aa}$.*

Proof. See Appendix A.3.

Mas-Colell (1989) proves the equivalence $\mathcal{W}(\mathcal{E}) = BS_{ss}$ when (T, Σ, m) is an atomless measure space. More precisely, he shows that whenever $T = T_0$ and $f \in \mathcal{M}(\mathcal{E})$ is a feasible non-Walrasian allocation, it is always possible to find a standard objection against f that is competitive (Mas-Colell, 1989, Proposition 2). In our framework, we can use this property together with Proposition 3.4 to extend Mas-Colell's main result and prove that in atomless economies all the notions of bargaining set we gave are actually equivalent.

Proposition 3.7. *Suppose that $T = T_0$. Then $\mathcal{W}(\mathcal{E}) = BS_{as} = BS_{aa} = BS_{ss} = BS_{sa}$.*

Remark 3.8. Going back to the series of inclusions proved in Proposition 2.13, Theorem 1 ensures that under Assumption 3.1 we always have the following:[5]

$$\mathcal{W}(\mathcal{E}) = C_A(\mathcal{E}) = BS_{as} = BS_{aa} \subseteq BS_{ss} \subseteq BS_{sa}.$$

Hence, in particular, we derive the equilibria existence and the Aubin-Core–Walras equivalence theorem in economies with countably many agents.[6] However, the inclusion $BS_{aa} \subseteq BS_{ss}$ might be strict. This can be proved moving from the examples provided by Shitovitz (1989) who considers the notion of the *veto player* as an atom who is the unique owner of a certain good. In one example, Shitovitz (1989) describes an economy with no veto player and two atoms with the same initial endowment and same utility function in which the bargaining set is strictly larger than the core which, on the other hand, coincides with the set of Walrasian allocations by Shitovitz (1973). Hence,

$$W(\mathcal{E}) = C_A(\mathcal{E}) = B_{as} = BS_{aa} = C(\mathcal{E}) \subsetneq BS_{ss}.$$

A second example illustrates an economy with a veto player in which the core coincides with the bargaining set and strictly contains the set of Walrasian allocations. Hence,

$$W(\mathcal{E}) = C_A(\mathcal{E}) = B_{as} = BS_{aa} \subsetneq C(\mathcal{E}) = BS_{ss}.$$

5 Notice that Example 2.14 does not fulfill the assumptions of Theorem 1.

6 The Aubin-core equivalence is proved by Noguchi (2000) when the commodity space is infinite dimensional and assuming convexity of preferences.

From Theorem 1 it also follows that our bargaining set is consistent according to the notion of Dutta et al. (1989), for which each objection in a "chain" of objections is tested in the same way as its predecessor (see also Remark 2.9).[7]

4. Further characterizations

Competitive objections have been shown to play a key role in the study of bargaining sets. In this section we look for some further characterizations of competitive objections under Assumption 3.1. To this end, the following new notations are needed.

Let f be a feasible allocation. For every price vector $p \in \Delta$, we denote by $\eta(t, p)$ the demand set for the agent $t \in T$ and define $C(p)$ and $D(p)$ as follows:

$$C(p) := \{t : \eta(t, p) \succ_t f(t)\}, \quad D(p) := \{t : \eta(t, p) \succcurlyeq_t f(t)\}.$$

Intuitively, an agent t belongs to $C(p)$ (respectively $D(p)$) if she strictly (respectively weakly) prefers what she can obtain by trading $e(t)$ at price p over the bundle $f(t)$.

Remark 4.1. Being agents' preferences continuous and monotone, the set $C(p)$ defined above coincides with the set $\{t \in T : \exists\, v \text{ for which } v \succcurlyeq_t f(t) \text{ and } p \cdot v < p \cdot e(t)\}$ defined in Mas-Colell (1989). Furthermore, by Proposition 2.12, $\mathcal{O}_A(f) \neq \emptyset$ if and only if $m(C(p)) > 0$ for every $p \in \Delta$.

We can now prove the following result.

Proposition 4.2. *Let f be a feasible allocation such that the set $\mathcal{O}_A(f)$ is not empty. Then (S, γ, g) is a competitive Aubin-objection against f if and only if there is a price $p \in \Delta$ such that:*

(1) $g(t) \in \eta(t, p)$ *for almost every* $t \in S$,
(2) $C(p) \subseteq S \subseteq D(p)$,
(3) $\int_S \gamma(t)(g(t) - e(t))\, dt = 0$.

Proof. See Appendix A.4.

Pursuing the interest in the bargaining set BS_{aa} we now present a second result that allows us to focus on a smaller class of Aubin-objections.

Proposition 4.3. *Let $f \in \mathcal{M}(\mathcal{E})$ be such that $f \notin BS_{aa}$. Then there is a competitive Aubin-objection (S, γ, g) against f such that:*

1. γ *is a simple function,*
2. $\gamma(t) = 1$ *for every* $t \in S \cap T_0$,
3. $\gamma(t) = 1$ *for every* $t \in S$ *such that* $g(t) \succ_t f(t)$.

The Proposition above follows directly from the Proof of Proposition 3.5 where we consider a feasible $f \notin BS_{aa}$ and find a competitive Aubin-objection (S, γ, g) to it. In Step 3 of the Proof, in fact, we show how the function γ meets all the conditions of Proposition 4.3.

Remark 4.4. Proposition 4.3 says that, as in the case of atomless economies, small agents fully participate in a competitive objection. The same is true for traders which are strictly better off whereas agents which are indifferent can object with any participation rate. Notice that, even though Proposition 4.2 does not give a full description of all competitive Aubin-objections, Proposition 4.3 can be considered as a characterization of BS_{aa}. Indeed, define a new class of Aubin-objections $\mathcal{O}_A^*(f)$ formed by all the $(S, \gamma, g) \in \mathcal{O}_A(f)$ such that γ satisfies all the three conditions in Proposition 4.3. Similarly to what we did in Sub Section 2.3, we can define a new bargaining set BS^*_{aa} containing all attainable $f \in \mathcal{M}(\mathcal{E})$ such that every $(S, \gamma, g) \in \mathcal{O}_A^*(f)$ has an Aubin-counter-objection. Being $\mathcal{O}_A^*(f)$ strictly smaller than $\mathcal{O}_A(f)$, we would expect that BS^*_{aa} contains BS_{aa}. However, the result in Proposition 4.3 guarantees that whenever $f \notin BS_{aa}$ we can always find a $(S, \gamma, g) \in \mathcal{O}_A^*(f)$ that cannot be counter-objected. Hence, $W(\mathcal{E}) = BS_{aa} = BS^*_{aa}$. A fortiori this equivalence still holds if we impose similar restrictions even on the counter-objections. Precisely, given $f \in \mathcal{M}(\mathcal{E})$ and $(S, \gamma, g) \in \mathcal{O}_A^*(f)$, we can define a new class of Aubin-counter-objections to (S, γ, g), denoted by $\mathcal{CO}_A^*(S, \gamma)$, imposing full participation to the non-atomic part and allowing partial participation only to the atoms of (S, γ), i.e. $\mathcal{CO}_A^*(S, \gamma) = \{(Q, \delta, h) \in \mathcal{CO}_A(S, \gamma) : \delta(t) = 1 \text{ for every } t \in Q \cap T_0\}$. Let BS^{**}_{aa} be the corresponding bargaining set of all the feasible $f \in \mathcal{M}(\mathcal{E})$ such that every $(S, \gamma, g) \in \mathcal{O}_A^*(f)$ has an Aubin-counter-objection in $\mathcal{CO}_A^*(S, \gamma)$. Note that, being $\mathcal{O}(f) \subseteq \mathcal{O}_A^*(f) \subseteq \mathcal{O}_A(f)$ and $\mathcal{CO}(S, \gamma) \subseteq \mathcal{CO}_A^*(S, \gamma) \subseteq \mathcal{CO}_A(S, \gamma)$, we have that $W(\mathcal{E}) \subseteq BS^{**}_{aa} \subseteq BS^*_{aa}$ and hence $W(\mathcal{E}) = C_A(\mathcal{E}) = BS_{as} = BS_{aa} = BS^*_{aa} = BS^{**}_{aa} \subseteq BS_{ss} \subseteq BS_{sa}$.

5. Concluding remarks

We have based our definitions of bargaining sets on a two-step veto mechanism that allows agents to participate to the formation of coalitions with any portion of their endowments. It is possible, however, to modify this process by imposing some restrictions on the class of generalized coalitions that can raise objections and counter-objections and to analyze the conditions under which the corresponding veto-mechanisms can be used to characterize the set of Walrasian allocations. As we have observed in Remark 4.4, Proposition 4.3 can be considered as a contribution in this direction, when we limit the participation rates of individual agents to the objections and counter-objections. We sketch here below in three final remarks further possible restrictions that can be pursued in our specific framework.

Remark 5.1. The veto mechanism defining our bargaining set can be modified considering only coalitions in which agents' participation rates are smaller than a given threshold. For every $\varepsilon \in (0, 1)$ let $\mathcal{F}_\varepsilon$ be the set formed by those $(S, \gamma) \in \mathcal{F}$ such that $\gamma(t) \leq \varepsilon$ for almost every $t \in S$. If (S, γ, g) is an Aubin-objection to an allocation f, $(S, \varepsilon\gamma, g)$ is a new Aubin-objection to f raised by a coalition in $\mathcal{F}_\varepsilon$ and every Aubin-counter-objection to (S, γ, g) is an Aubin-counter-objection to $(S, \varepsilon\gamma, g)$. Hence, from Theorem 1 we can say that for every feasible and non-Walrasian allocation there exists an Aubin-objection that is raised by a coalition in $\mathcal{F}_\varepsilon$ and that cannot be counter-objected. Similar ideas can be found in Hervés-Beloso et al. (2018) and Hervés-Estévez and Moreno-García (2018b).

Different scenarios follow if we impose that a generalized coalition is allowed to raise objections and counter-objections if and only if belongs to $\mathcal{F}_{\mathbb{Q}} := \{(S, \gamma) \in \mathcal{F} : \gamma(t) \in \mathbb{Q} \text{ for all } t \in S\}$. This kind of restriction, that has been explored to study replica economies via generalized coalitions, leaves the Aubin-core unaltered but describes a bargaining set that is strictly larger than $BS_{aa}(\mathcal{E})$. It is in fact possible to prove with standard arguments that a feasible allocation is Walrasian if and only if there are no coalitions in $\mathcal{F}_{\mathbb{Q}}$ that can object it. At the same time, the examples presented in Hervés-Estévez and Moreno-García (2018a), page 334, show that there can be a non-Walrasian, feasible allocation f such that every Aubin-objection to f raised by a coalition in $\mathcal{F}_{\mathbb{Q}}$ can be counter-objected.

[7] We thank an anonymous referee for pointing this out.

Remark 5.2. Other developments can be obtained by limiting the size of the generalized coalitions entitled to raise objections and counter-objections. To this end, we stress that a substantial difference between the core and Mas-Colell's bargaining set prevails even in the case of atomless economies. The examples in Schjødt and Sloth (1994) prove that it is not possible to put a bound on the size of competitive, and hence justified, standard objections. In other words, when we limit the size of the standard coalitions that are allowed to raise objections and counter-objections in an atomless economy, we leave the core unaltered and we define a bargaining set that is strictly larger than $BS_{ss}(\mathcal{E})$.

In our framework we can work on the definition of "size" of an objection and obtain new characterizations of Walrasian allocations, in contrast with the results proved in Schjødt and Sloth (1994). Formally, we can refer to the size of a generalized coalition (S,γ) as the integral $\int_S \gamma(t)\,dt$ and observe that for every $\varepsilon \in (0,1)$, all the coalitions in the set $\mathcal{F}_\varepsilon$ defined above have size smaller than or equal to ε (see also Bhowmik and Graziano, 2015). Then, using the same arguments discussed before, one can prove that for any choice of $\varepsilon \in (0,1)$ it is always possible to block a feasible and non-Walrasian allocation with a generalized coalition that has size smaller than ε and that is not counter-objected. Similar arguments do not hold if we allow exclusively objections raised by large coalitions (compare Examples 1 and 2 in Hervés-Estévez and Moreno-García, 2015).

Remark 5.3. A further possible restriction deals with the preferences of the objecting agents. Indeed, a fundamental point in the proof of Proposition 4.3 is that for an objection (S,γ,g) to an allocation f there may be a group of agents in S that are indifferent between g and f. The equivalence $W(\mathcal{E}) = BS_{aa}(\mathcal{E})$, in fact, is no longer true if we strengthen the definition of objections and require that almost every $t \in S$ is such that $g(t) \succ_t f(t)$.

Following the idea presented in Mas-Colell (1989)[Remark 2], one can overcome this obstacle with the introduction of a new notion of bargaining set in which the preferences in the definition of objections are strict and the idea of a "leader" a' la Aumann–Maschler is adapted from Geanakoplos (1978).

Formally, given any $\varepsilon > 0$ we say that (S,γ,g) is an ε-Aubin objection to f with leader K if $g(t) \succ_t f(t)$ for every $t \in S$ and K is a non-null subset of S such that $\int_K \gamma(t)\,dt < \varepsilon$. A feasible $f \in \mathcal{M}(\mathcal{E})$ belongs to the ε-Aubin bargaining set $\varepsilon - BS_{aa}(\mathcal{E})$ if for every ε-Aubin-objection to f with leader K there exists a counter-objection (Q,δ,h) such that $m(Q \cap K) > 0$.

Under the Assumptions of Theorem 1, following the same procedure as in Mas-Colell (1989)[Remark 2], it can be shown that if there exists a competitive Aubin-objection to a feasible allocation $f \in \mathcal{M}(\mathcal{E})$ then f does not belong to the ε-Aubin bargaining set. This is enough to conclude that $W(\mathcal{E}) = \varepsilon - BS_{aa}(\mathcal{E})$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

A.1. Proof of Proposition 2.7

Proof. The first implication directly follows from the definition of the allocation $\tilde{g}$ induced by (S,γ,g). For the converse we use continuity and strict monotonicity of agents' preferences. Precisely, let $h \in \mathcal{M}(\mathcal{E})$ be such that $(Q,\delta,h) \in \mathcal{O}_A(\tilde{g})$, that is

(i) $\int \delta(t)h(t)dt \le \int \delta(t)e(t)dt$
(ii) $h(t) \succsim_t \tilde{g}(t)$ for almost all $t \in Q$
(iii) $m(Q') > 0$, where $Q' = \{t \in Q \,:\, h(t) \succ_t \tilde{g}(t)\}$.

If $m(Q \setminus Q') = 0$, then (Q,δ,h) is an Aubin-counter-objection to (S,γ,g). Otherwise, by (iii) and continuity, there exists $\varepsilon \in (0,1)$ and $\tilde{Q} \subseteq Q'$ such that $m(\tilde{Q}) > 0$ and $\varepsilon h(t) \succ_t \tilde{g}(t)$ for almost all $t \in \tilde{Q}$. Define

$$\tilde{h}(t) := \begin{cases} \varepsilon h(t) & \text{if } t \in \tilde{Q}, \\ h(t) + \frac{1-\varepsilon}{m(Q\setminus\tilde{Q})}\int_{\tilde{Q}} \delta(t)h(t)dt & \text{if } t \in Q \setminus \tilde{Q}. \end{cases}$$

Notice that by (i), we have that

$$\begin{aligned}\int \delta(t)\tilde{h}(t)dt &= \int_{\tilde{Q}} \delta(t)\varepsilon h(t)dt + \int_{Q\setminus\tilde{Q}} \delta(t)h(t)dt \\ &\quad + \int_{Q\setminus\tilde{Q}} \delta(t)\left[\frac{(1-\varepsilon)}{m(Q\setminus\tilde{Q})}\int_{\tilde{Q}} \delta(t)h(t)dt\right]dt \le \\ &\le \varepsilon \int_{\tilde{Q}} \delta(t)h(t)dt + \int_{Q\setminus\tilde{Q}} \delta(t)h(t)dt \\ &\quad + (1-\varepsilon)\int_{\tilde{Q}} \delta(t)h(t)dt = \int \delta(t)h(t)dt \le \\ &\le \int \delta(t)e(t)dt, \text{ that is}\end{aligned}$$

(1) $\int \delta(t)\tilde{h}(t)dt \le \int \delta(t)e(t)dt$.

Furthermore, by strict monotonicity and (ii), $\tilde{h}(t) \succ_t \tilde{g}(t)$ for almost all $t \in Q$, which means, by the definition of $\tilde{g}$, that

(2) $\tilde{h}(t) \succ_t g(t)$ for almost all $t \in Q \cap S$
(3) $\tilde{h}(t) \succ_t f(t)$ for almost all $t \in Q \setminus S$.

Therefore, the Aubin-objection (S,γ,g) against f is Aubin-counter-objected by $(Q,\delta,\tilde{h})$. This concludes the proof. ■

A.2. Proof of Proposition 2.12

We divide the proof of Proposition 2.12 in two separated statements, one for each implication.

Lemma A.1 (*Proposition 2.12, First Implication*)**.** *Suppose that f is a feasible allocation and $p \gg 0$ such that $p \cdot x \ge p \cdot e(t)$ for almost every $t \in T$ and every $x \in \mathbb{R}^N_+$ for which $x \succsim_t f(t)$. Then there is no Aubin-objection against f.*

Proof. First of all let us observe that since $p \gg 0$, by the continuity and strict monotonicity of preferences, the inequality $p \cdot e(t) < p \cdot x$ holds for almost every $t \in T$ and $x \in \mathbb{R}^N_+$ for which $x \succ_t f(t)$.

Suppose, by the way of contradiction, that there exist $g \in \mathcal{M}(\mathcal{E})$ and (S,γ) are such that $(S,\gamma,g) \in \mathcal{O}_A(f)$. Hence, $\int \gamma(t) g(t)dt \le \int \gamma(t)e(t)dt$; $g(t) \succsim_t f(t)$ for almost every $t \in S$ and $S' := \{t \in S \,:\, g(t) \succ_t f(t)\}$ has positive measure. Using the assumption on p we obtain the following inequalities:

$$p \cdot \int_{S\setminus S'} \gamma(t)(g(t)-e(t))\,dt = \int_{S\setminus S'} \gamma(t)\,p\cdot(g(t)-e(t))\,dt \ge 0,$$

$$p \cdot \int_{S'} \gamma(t)(g(t)-e(t))\,dt = \int_{S'} \gamma(t)\,p\cdot(g(t)-e(t))\,dt > 0$$

which together imply

$$p \cdot \int_S \gamma(t)g(t)\,dt > p \cdot \int_S \gamma(t)e(t)\,dt,$$

that contradicts the inequality $\int \gamma(t)g(t)\,dt \le \int \gamma(t)e(t)\,dt$. ■

Lemma A.2 (*Proposition 2.12, Second Implication*)**.** *Suppose that Assumption 3.1 is satisfied and let f be a feasible allocation such that there are no Aubin-objections against it. Then there is $p \gg 0$ such that $p \cdot e(t) \le p \cdot x$ for almost every $t \in T$ and $x \in \mathbb{R}^N_+$ for which $x \succcurlyeq f(t)$.*

Proof. By the continuity of preferences it is enough to find a $p \gg 0$ such that $p \cdot x \ge p \cdot e(t)$ for almost every $t \in T$ and every $x \in \mathbb{R}^N_+$ for which $x \succ_t f(t)$.

Let us define a correspondence $\varphi: T \to 2^{\mathbb{R}^N}$ by setting $\varphi(t) := \{x \in \mathbb{R}^N_+ : x \succ_t f(t)\} - \{e(t)\}$ for every $t \in T$. By the monotonicity assumption φ is a non-empty valued correspondence which admits integrable selections. This implies that the set

$$K := \left\{ \int_S \gamma(t)\varphi(t)\, dt : (S, \gamma) \in \mathcal{F} \right\}$$

is non-empty. We now prove that K is convex. To this end, define the sets

$$K_0 := \left\{ \int_{S\cap T_0} \gamma(t)\varphi(t)\, dt : (S, \gamma) \in \mathcal{F} \right\} \quad \text{and}$$

$$K_1 := \left\{ \int_{S\cap T_1} \gamma(t)\varphi(t)\, dt : (S, \gamma) \in \mathcal{F} \right\},$$

and notice that $K = K_0 + K_1$, where K_0 is convex thanks to Lyapunov–Richter's Theorem. To conclude our claim is then enough to show that K_1 is convex as well. This follows from Assumption 3.1. Indeed, let $\alpha y_1 + (1-\alpha)y_2$ be a convex combination of two elements y_1 and y_2 of K_1. Then, there exist $(S_1, \gamma_1), (S_2, \gamma_2) \in \mathcal{F}$ and $z_1, z_2 \in \mathcal{M}(\mathcal{E})$ such that

$$y_1 = \int_{S_1\cap T_1} \gamma_1(t)z_1(t)dt \quad \text{and} \quad y_2 = \int_{S_2\cap T_1} \gamma_2(t)z_2(t)dt,$$

where $z_i(t) \in \varphi(t)$ for $i = 1, 2$ and for almost all $t \in S_i$. Define now the generalize coalition γ, with support $S_1 \cup S_2$, and the function $y : (S_1 \cup S_2) \cap T_1 \to \mathbb{R}^N_+$ as follows:

$$\gamma(t) = \alpha\gamma_1(t) + (1-\alpha)\gamma_2(t)$$
$$y(t) = \frac{\alpha\gamma_1(t)}{\gamma(t)}z_1 + \frac{(1-\alpha)\gamma_2(t)}{\gamma(t)}z_2(t).$$

From Assumption 3.1, $y(t) \in \varphi(t)$ for almost all $t \in (S_1 \cup S_2) \cap T_1$, and hence $\int_{(S_1\cup S_2)\cap T_1} \gamma(t)y(t)dt \in K_1$. Since

$$\alpha y_1 + (1-\alpha)y_2 = \int_{(S_1\cup S_2)\cap T_1} \gamma(t)y(t)dt \in K_1,$$

the set K_1 is convex and so is K. Now, being $\mathcal{O}_A(f)$ empty, the sets K and $-\mathbb{R}^N_+$ are disjoint and can therefore be separated by a hyperplane. That is, there exists a $p \ge 0$, $p \ne 0$ such that $p \cdot \int_S \gamma(t)(g(t) - e(t))\, dt \ge 0$ whenever $(S, \gamma) \in \mathcal{F}$, $g \in \mathcal{M}(\mathcal{E})$ and $g(t) \succ_t f(t)$ for almost every $t \in S$. We conclude that

$$p \cdot e(t) \le p \cdot x \text{ for almost every } t \in T \text{ and } x \in \mathbb{R}^N_+ \text{ for which } x \succ_t f(t). \tag{1}$$

We only need to show that $p \gg 0$. To this end, first observe that $\int f(t)dt = \int e(t)dt$, otherwise from monotonicity $(T, 1, f + \int[e(t) - f(t)]dt) \in \mathcal{O}_A(f)$ for any $y \in \mathbb{R}^\ell_+ \setminus \{0\}$. This together with (1) implies that $p \cdot f(t) = p \cdot e(t)$ for almost all $t \in T$. Since by assumption $\int e(t)dt \gg 0$ and $p \ge 0$, $p \ne 0$, it follows that $m(\{t \in T : p \cdot e(t) > 0\}) > 0$. Take t such that $p \cdot f(t) = p \cdot e(t) > 0$ and notice that continuity implies that $p \cdot x > p \cdot e(t)$ whenever $x \succ_t f(t)$. Now, assume to the contrary that $p^h = 0$ for some $h \in \{1, \dots, N\}$ and consider the bundle g defined as $g^h = f^h(t)$ if $p^h > 0$ and $g^h = f^h(t) + y$, with $y > 0$, if $p^h = 0$. Then, $p \cdot g = p \cdot f(t)$, but by monotonicity $g \succ_t f(t)$ and hence $p \cdot g > p \cdot e(t) = p \cdot f(t)$, which is a contradiction. This proves that $p \gg 0$. ∎

A.3. Proofs of Section 3

Proof of Lemma 3.3. One implication directly follows from Proposition 2.12. Precisely, if there is no Aubin-objection against $\tilde{g}$, then there exists a price $p \gg 0$ such that $p \cdot x(t) \ge p \cdot e(t)$ for almost every $t \in T$ and every $x \in \mathbb{R}^N_+$ for which $x \succcurlyeq_t \tilde{g}$. By definition of the allocation $\tilde{g}$, it follows that (S, γ, g) is competitive. Conversely, let (S, γ, g) be competitive and assume to the contrary that (Q, δ, h) is an Aubin-objection against $\tilde{g}$. Thus,

(i) $\int \delta(t)h(t)dt \le \int \delta(t)e(t)dt$,
(ii) $h(t) \succcurlyeq_t \tilde{g}(t)$ for almost every $t \in Q$,
(iii) $h(t) \succ_t \tilde{g}(t)$ for almost all $t \in Q' \subseteq Q$ with $m(Q') > 0$.

By definition of $\tilde{g}$, since (S, γ, g) is competitive, from (ii) it follows that $p \cdot h(t) \ge p \cdot e(t)$ for almost all $t \in Q$, with a strict inequality for almost all $t \in Q'$ because agents' preferences are continuous and strictly monotone and $p \gg 0$. Therefore, $\int \delta(t)p \cdot h(t)dt > \int \delta(t)p \cdot e(t)dt$ which contradicts (i) above. ∎

We now show Proposition 3.5. To this end, we divide the proof in several steps, in order to simplify it and to make further comparisons with Mas-Colell's original work.

Proof of Proposition 3.5. Let us fix a feasible allocation $f \in \mathcal{M}(\mathcal{E})$ that is not Walrasian. As in Paragraph 4, for every $p \in \Delta$ we denote by $\eta(t, p)$ the demand set for the agent $t \in T$ and use $C(p)$ and $D(p)$ for the sets:

$$C(p) := \{t : \eta(t, p) \succ_t f(t)\}, \quad D(p) := \{t : \eta(t, p) \succcurlyeq_t f(t)\}.$$

Being f non-Walrasian it must be that $m(C(p)) > 0$.

Step 1: Let us call $\varphi: \Delta \times T \to 2^{\mathbb{R}^N}$ the correspondence that assigns to each $p \in \Delta$ and $t \in T$ the set:

$$\varphi(p, t) := \begin{cases} \eta(p, t) - e(t), & \text{if } t \in C(p), \\ \eta(p, t) - e(t) \cup \{0\}, & \text{if } t \in D(p) \setminus C(p), \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then define $\Phi: \Delta \to 2^{\mathbb{R}^N}$ as the integral of φ, i.e. the map $\Phi(p) := \int_T \varphi(p, t)\, dt$, and observe that Φ satisfies all of the following properties: (i) Φ is upper hemicontinuous and bounded from below, (ii) Φ is non-empty for every $p \in \Delta$, (iii) The Walras' law prevails, which is saying that $p \cdot v = 0$ for every $p \in \Delta$ and $v \in \Phi(p)$, (iv) for every sequence $(p_n)_n \subset \Delta$ converging to some $p \notin \Delta$ we have $\|x_n\| \to \infty$ whenever $(x_n)_n$ is such that $x_n \in \Phi(p_n)$ for every $n \in \mathbb{N}$.[8] We can therefore apply the weak form of Gale-Debreu-Nikaido Lemma to Φ and obtain the existence of a $\overline{p} \in \Delta$ such that $0 \in co\Phi(\overline{p})$ (a proof can be found, for example in Hildenbrand 1974, Lemma 1 page 150).

Step 2: Since $0 \in co\Phi(\overline{p})$ and the latter was defined as $\int \varphi(\overline{p}, t)\, dt$, we can take g_1, g_2 be integrable selections of $\varphi(\overline{p}, \cdot)$ and $\theta \in [0, 1]$ such that

$$\theta \int g_1(t)\, dt + (1-\theta) \int g_2(t)\, dt = 0.$$

By the Lyapunov–Richter's Theorem $\int_{T_0} \theta g_1(t) + (1-\theta)g_2(t)\, dt = \int_{T_0} g_0(t)$ for some integrable selection g_0 of $\varphi(\overline{p}, \cdot)$. Call $S_0 := \{t \in T_0 : g_0(t) + e(t) \in \eta(\overline{p}, t)\}$ and, for $i = 1, 2$, put $S_i := \{t \in T_1 :$

8 The proof of (i)–(iv) is almost identical to what is done for the proof of the existence of a competitive equilibrium. See for example Mas-Colell (1985), page 270.

$g_i(t)+e(t) \in \eta(\overline{p},t)\}$. Now define the allocation $g\colon T \to \mathbb{R}^N_+$ by:

$$g(t) := \begin{cases} g_0(t)+e(t), & \text{if } t \in S_0, \\ g_1(t)+e(t), & \text{if } t \in S_1 \setminus S_2, \\ \theta g_1(t)+(1-\theta)g_2(t)+e(t), & \text{if } t \in S_1 \cap S_2, \\ g_2(t)+e(t), & \text{if } t \in S_2 \setminus S_1, \\ 0 & \text{otherwise.} \end{cases}$$

For all t, $g(t) \in \eta(\overline{p},t)$ if $t \in S := S_0 \cup S_1 \cup S_2$ and $g(t) = 0$ if $t \notin S$. In particular, this is true because when $t \in S_1 \cap S_2 \subset T_1$, $g_1(t)+e(t)$ and $g_2(t)+e(t)$ are both in $\eta(\overline{p},t)$ and the latter is convex since $\succcurlyeq_t$ is convex by assumption.

Step 3: Define γ as the function $\chi_{S_0} + \theta\chi_{S_1} + (1-\theta)\chi_{S_2}$. We claim that (S,γ,g) is a competitive Aubin-objection to f. First we observe that (S,γ,g) is an Aubin-objection to x: in fact, by construction, S is the non-null support of γ and:

$$\int \gamma(t)(g(t)-e(t))\,dt = \int_{T_0} g_0(t)\,dt + \int_{T_1} \theta g_1(t)+(1-\theta)g_2(t)\,dt = \\ = \theta \int g_1(t)\,dt + (1-\theta)\int g_2(t)\,dt = 0.$$

Furthermore, $S \subseteq D(\overline{p})$ and hence $g(t) \succcurlyeq_t f(t)$, for all $t \in S$. Finally, since $C(\overline{p}) \subset S$ and $m(C(\overline{p})) > 0$ (because f is non-competitive) $m(\{t \in S : g(t) \succ_t f(t)\}) > 0$. We are only left to prove that (S,γ,g) is competitive: for any $x \in \mathbb{R}^N_+$, if $x \succcurlyeq_t g(t)$ for some $t \in S$, then $x \succcurlyeq_t \eta(\overline{p},t)$ and so $\overline{p}\cdot x \geq \overline{p}\cdot e(t)$. Similarly, if $x \succcurlyeq_t f(t)$ for some $t \notin S$ then $x \succcurlyeq_t \eta(\overline{p},t)$ and so $\overline{p}\cdot x \geq \overline{p}\cdot e(t)$. We conclude that (S,γ,g) is a competitive Aubin-objection to f. ■

Remark A.3. The proof of Proposition 3.5 follows from a close range the proof of Proposition 2 in Mas-Colell 1989 for the case of a non-atomic economy. In particular, Step 1 is identical to what is done by Mas-Colell with the only exception that, since our measure space of agents is not necessarily non-atomic, we could not conclude that the correspondence Φ has convex values. This is the reason why, in Step 2, we had to move from Φ to its convex hull, an expedient that was unnecessary in Mas-Colell's settings thanks to Lyapunov–Richter's Theorem. Once the triple (S,γ,g) is defined, in Step 3 the proof that (S,γ,g) is a competitive Aubin-objection follows, with the necessary changes in register, the last part of Mas-Colell's proof. The assumption $e(t) \gg 0$ for almost all $t \in T$ ensures that given any $p \in \Delta$ and any coalition S, with $C(p) \subseteq S \subseteq D(p)$, the aggregate initial endowment over S is strictly positive, i.e. $\int_S e(t)dt \gg 0$, and consequently that the correspondence Φ is upper hemicontinuous.

Proof of Corollary 3.6. If the initial endowment e is a Walrasian allocation, then the Corollary is trivially proved. If e is not Walrasian, then by Proposition 3.5 there exists an Aubin-objection (S,γ,x) against e which is competitive. In particular:

(i) $\int \gamma(t)x(t)dt \leq \int \gamma(t)e(t)dt$,
(ii) $x(t) \succcurlyeq_t e(t)$ for almost all $t \in S$,
(iii) $x(t) \succ_t e(t)$ for almost all $t \in S' \subseteq S$, with $m(S') > 0$.

By the proof of Proposition 3.5, we can assume that $\gamma(t) = 1$ for almost all $t \in (T_0 \cap S) \cup S'$.

Consider now the allocation $y(t) = \gamma(t)x(t) + (1-\gamma(t))e(t)$ for all $t \in T$ and notice that by (i) it is feasible. Since (S,γ,x) is a competitive Aubin-objection against e, there exists $p \in \Delta$ such that

(1) $p\cdot z \geq p\cdot e(t)$ whenever $t \in S$ and $z \succcurlyeq_t x(t)$
(2) $p\cdot z \geq p\cdot e(t)$ whenever $t \notin S$ and $z \succcurlyeq_t e(t)$.

We show that y is a Walrasian allocation supported by p. First note that from (1) and (i) it follows that $p\cdot x(t) = p\cdot e(t)$ for almost all $t \in S$. Hence, by definition of y we have that $p\cdot y = p\cdot e(t)$ for almost all $t \in T$. Suppose now the existence of a bundle z preferred to $y(t)$ by some agent t, i.e. $z \succ_t y(t)$. If $t \notin S$, then $\gamma(t) = 0$ and $z \succ_t y(t) = e(t)$. Thus, from (2), $p\cdot z \geq p\cdot e(t)$. Actually, $p\cdot z > p\cdot e(t)$ because $\succcurlyeq_t$ is continuous and $e(t) \gg 0$. If $t \in (T_0 \cap S) \cup S'$, then $\gamma(t) = 1$ and $z \succ_t y(t) = x(t)$, whereas for $t \in (S \setminus S') \cap T_1$, $z \succ_t y(t) \sim_t x(t)$. Thus, from (1), $p\cdot z \geq p\cdot e(t)$. Again continuity of $\succcurlyeq_t$ and $e(t) \gg 0$ imply that $p\cdot z > p\cdot e(t)$. This completes the proof. ■

Proof of Theorem 1. Let f be a feasible non-Walrasian allocation. Proposition 3.5 implies the existence of an Aubin-objection to f which is Walrasian and which, in addition, has no Aubin-counter-objection because of Proposition 3.4. Then $f \notin BS_{aa}$, that is $BS_{aa} \subseteq W(\mathcal{E})$. The other inclusion is given by Proposition 2.13. ■

A.4. Proof of Proposition 4.2

Proof of Proposition 4.2. First notice that, since $\mathcal{O}_A(f)$ is non-empty, it is $m(C(p)) > 0$ for every $p \in \Delta$.

Let us assume that (S,γ,g) is competitive and call p the relative price system. Point (1) follows directly from the definition of competitive allocation. If point (2) is violated then either $m(C(p)\setminus S) > 0$ or $m(S \setminus D(p)) > 0$. In the first case let $t \in C(p)\setminus S$ and take $x \in \eta(t,p)$; by definition we have $x \succ_t f(t)$ and $p\cdot x \leq p\cdot e(t)$ and so (S,γ,g) is not competitive. On the other hand, for $t \in S \setminus D(p)$ then $f(t) \succ_t g(t)$ contradicts the fact that (S,γ,g) is an Aubin-objection against f. To prove point (3) suppose that $x := \int_S \gamma(t)(g(t)-e(t))\,dt < 0$ and define $h \in \mathcal{M}(\mathcal{E})$ by:

$$h(t) = g(t) - \frac{x}{\int_S \gamma(t)\,dt}.$$

But then (S,γ,h) constitutes an Aubin-counter-objection to (S,γ,g) in contradiction to Proposition 3.4.

Suppose now that (S,γ,g) satisfies conditions (1), (2) and (3). We first need to prove that $(S,\gamma,g) \in \mathcal{O}_A(f)$. The requirement for which $\int_S \gamma(t)g(t)\,dt \leq \int_S \gamma(t)e(t)\,dt$, is guaranteed by point (3). For every $t \in S$ we have that $t \in D(p)$ (point (2)) and $g(t) \in \eta(t,p)$ (point (1)), meaning that $g(t) \succcurlyeq_t f(t)$. Furthermore, from (1) and (2) we also derive that $\{t \in S : g(t) \succ_t f(t)\} = C(p)$ and has non-zero measure. To prove that (S,γ,g) is competitive we pick $x \in \mathbb{R}^N_+$ and observe that, being $g(t) \in \eta(t,p)$, if $x \succcurlyeq_t g(t)$ for some $t \in S$ we must have $p\cdot x \geq p\cdot e(t)$. On the other hand, if $x \succcurlyeq_t f(t)$ for some $t \notin S$, the inclusion $t \notin C(p)$ allows us to write $x \succcurlyeq_t \eta(t,p)$ so that $p\cdot x \geq p\cdot e(t)$. ■

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