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A NOTE ON BARGAINING SETS IN ATOMLESS ECONOMIES

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Dedicated to Professor Carlos Hervés-Beloso on the occasion of his 70th birthday

Abstract. We consider an atomless exchange economy with a finite number of commodities. We introduce a new notion of bargaining set and show that it characterizes competitive allocations and, at the same time, it is robust to restrictions on the sizes of objecting and counterobjecting coalitions. Both results do not hold simultaneously for the bargaining sets defined by [18] and [23] for which we provide some new remarks.

1. **Introduction.** The core of an economy is defined as the set of feasible allocations that no coalition of agents can object to by proposing an alternative redistribution of their initial resources that makes each of its members better off. This notion does not take into account the reaction of other agents that might form a new coalition and counter the objection. [3] introduces the concept of bargaining set in a game theory framework with the idea that an objection is credible or justified only if no other coalition in the economy reacts to it and proposes an alternative action. Objections that are counterobjected are not justified and then they must be disregarded. Since fewer objections are allowed and blocking is harder, the bargaining set contains the core. [18] adapts the notion of bargaining set to atomless exchange economies as the set of all feasible allocations with no justified objections and proves that it coincides with the set of competitive allocations and a fortiori with the core.[1] An alternative notion of bargaining set has been introduced by [23]. The main difference between the definition of [18] and that of [23] is related to the notion of counterobjection and more precisely to what agents outside an objection look at. Indeed, [23] requires the allocations involved in objections and counterobjections to be feasible for the set of all consumers, whereas in [18] such allocations only need to be attainable for the corresponding coalitions. Furthermore, agents in counterobjections improve their welfare with respect to the objection to which they counter and not with respect to the original allocation as instead required by [18].[2] Because of this, [23]'s bargaining set is strictly larger than the core and does not characterize

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[1]Refinements of [18]'s bargaining set are provided by [4, 25] and [10] among others.

[2]See [24] and Remark 3.8 in this paper for a formal comparison between the two bargaining set notions.

competitive allocations. Following the terminology used in [19] and [17], we refer to [18] and [23] bargaining sets respectively as the *local* and *global* bargaining set.

We focus on three fundamental questions for cooperative solutions in exchange economies: the existence, the equivalence with the set of competitive allocations, and the robustness of the coalition formation mechanism to restrictions on their size. The core allows to answer positively to each of them. Indeed, it is well known since [2] and [1] that in perfectly competitive markets the core is non-empty and it coincides with the set of competitive allocations. With the idea that the measure of a coalition can be interpreted as the cost of information and communication needed for its formation, [9, 20] and [22] provide further characterizations of the core imposing restrictions to coalition formation.[3] Precisely, [20] shows that any allocation that is not blocked by coalitions with "small" size is competitive. [9] further restricts the objecting coalitions to those which are formed by a finite collection of subcoalitions whose agents are "similar" to each other. Finally, [22] completes the analysis and characterizes competitive allocations, and a fortiori core allocations, as those that are not objected by coalitions with an arbitrarily "large" measure.

[19] and [17] apply these ideas to the case of bargaining sets. Precisely, [19] extends [20]'s theorem and shows that if one restricts the coalitions that can enter into the objection and counterobjection mechanism to those whose measure is arbitrarily small, then the local bargaining set becomes strictly larger than the original one, whereas the corresponding global bargaining set remains unaltered.[4] [17] imposes on the objection and counterobjection mechanism restrictions similar to those in [9] and [22]. It is shown that, under both types of restrictions, the global bargaining set remains the same while the local bargaining set changes. Therefore, in atomless economies, the local bargaining set coincides with the set of competitive allocations but it becomes strictly larger if we impose restrictions on the coalition formation. At the same time, in atomless economies the global bargaining set is unaffected by limitations on the size of the coalitions allowed to form, but it does not coincide with the set of competitive allocations. The existence question is not an issue for both bargaining sets. However, contrary to the core, neither the local nor the global bargaining set positively answers all three questions above.

In this paper, we propose an alternative notion of bargaining set characterized by two main variations with respect to that of [18]. First, we weaken the blocking mechanism assuming that an agent accepts to join an objection or a counterobjection only if she gets a bundle as good as those consumed by agents with her same characteristics (same initial endowment and preferences). Roughly speaking, we consider only objections and counterobjections that "treat equals equally" because they are based on equitable agreements between agents. Second, we require that counterobjections must engage a non-negligible group of agents from the objection they counter. According to [18], in fact, a counterobjecting coalition may include no agent of the objection. In this case, the counterobjection is not really against

[3]Several extensions of these results have been provided in the literature. [14] studies continuum economy with infinitely many commodities; [16] and [15] consider pure exchange economy with asymmetric information and with, respectively, a finite number of commodities and infinitely many commodities; [12] analyzes asymmetric information economies with information sharing rules for which the information of each trader depends on the coalition she belongs to.

[4]This is saying that allowing only small coalitions to raise objections and counterobjections does not change the set of feasible allocations without justified global objections, but it does change the set of those without justified local objections.

the objection but rather against the original allocation.[5] We show that our new notion of bargaining set positively answers all three questions above. Indeed, it is non-empty, it coincides with the set of competitive allocations and it remains unaltered if restrictions on the size of blocking coalitions are imposed.

The paper is organized as follows. In Section 2 we present the model and the assumptions needed throughout the paper. In Section 3 we analyze the notions of bargaining sets defined by [18] and [23] and characterize justified objections as core allocations of some suitable associated economies. In Section 4 we introduce a new notion of bargaining set and illustrate its properties. In Section 5 we collect some final remarks.

2. **The model.** The economy $\mathcal{E}$ consists of a finite-dimensional commodity space $\mathbb{R}^m_+$ and an atomless finite measure space of consumers (T, Σ, λ). The set T represents all individual traders and Σ is the collection of all groups that are able or allowed to trade. For $S \in \Sigma$, $\lambda(S)$ is the size (or weight) of the group S. A *coalition* is an economically relevant group of agents, i.e., a set in Σ with positive measure. Every agent $t \in T$ is characterized by a preference relation $\succeq_t$ on $\mathbb{R}^m_+$ and an endowment bundle $\omega(t) \in \mathbb{R}^m_{++}$.[6] The irreflexive and symmetric components of $\succeq_t$ are $\succ_t$ and $\sim_t$ respectively. The economy is then defined as the collection

$$\mathcal{E} = \left\{ (T, \Sigma, \lambda), \mathbb{R}^m_+, (\succeq_t, \omega(t))_{t \in T} \right\}.$$

Throughout the paper, we make the following assumptions:

(*i*) $\omega \colon T \to \mathbb{R}^m_{++}$ is an integrable function;

(*ii*) agents' preferences are strictly monotone, continuous, total preorders on $\mathbb{R}^m_+$;[7]

(*iii*) agents' preferences are measurable in the sense that $\{t : v \succeq_t w\} \in \Sigma$ for every $v, w \in \mathbb{R}^m_+$.

Assumption (*i*) is about the boundedness of endowments and ensures that each commodity is present in the market, whereas (*ii*) requires the usual continuity assumption on preferences and that agents prefer consuming more of every commodity. Finally, (*iii*) is a technical hypothesis which is standard in the literature (see for example [1] and [13] for a comparison with other measurability conditions).

We say that two agents s, t are of the *same type*, and write $s \sim t$, if they have identical preferences and endowments. Under our assumptions, the equivalence relation $\sim$ is measurable in the sense that $\{s : t \sim s\} \in \Sigma$ for every $t \in T$. We call *type of agents* an equivalence class in the quotient $T/_\sim$, i.e., a set formed by all the agents that share a given preference relation and endowment bundle. We assume that for a.e. $t \in T$ the set of agents of the same type of t has positive measure, i.e., $\lambda(\{s : s \sim t\}) > 0$. This implies that there are countably many types of agents, that we denote by $(K_n)_n$, such that $\lambda(K_n) > 0$ and $T = \bigcup_n K_n$ up to null sets.

An *allocation* is an integrable function $x \colon T \to \mathbb{R}^m_+$ assigning to each consumer $t \in T$ a consumption bundle $x(t) \in \mathbb{R}^m_+$. A coalition S *attains* an allocation x if $\int_S x \, d\lambda \leq \int_S \omega \, d\lambda$, i.e., if the amount of resources that x assigns to the agents in S does not exceed their initial endowments. If x is attained by the grand coalition

[5] [25] first requires the non-empty intersection between objecting and counterobjecting coalitions (see also [24] and [21] among others).

[6] Following standard notations, for x and y in $\mathbb{R}^m_+$, we write $x \geq y$ to mean $x_i \geq y_i$ for all $i = 1, \ldots, m$; $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ to mean $x_i > y_i$ for all $i = 1, \ldots, m$. We set $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m_+ : x \gg 0\}$, where 0 is the null vector (the context will make clear the use of 0).

[7] A binary relation $\succeq$ on $\mathbb{R}^m_+$ is continuous if the sets $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open. It is strictly monotone if $x > y$ implies $x \succ y$.

T we say that x is *feasible*. We denote the set of feasible allocations by $\mathcal{F}$. An allocation x has the *equal treatment property* (ETP for short) on a coalition S if $x(t) \succeq_t x(s)$ for every $t, s \in S$ of the same type. If x has the ETP on the whole T we simply say that it has the ETP. Given a coalition S, we write $\mathcal{M}_S$ for the set of allocations with the ETP on S and $\mathcal{M}$ for $\mathcal{M}_T$.

Given a price vector $p \in \mathbb{R}^m_+ \setminus \{0\}$, the *budget set* of consumer t at p is $\beta(t,p) = \left\{x \in \mathbb{R}^m_+ : p \cdot x \leq p \cdot \omega(t)\right\}$. A feasible allocation x is *competitive* at the price p if $x(t)$ maximizes $\succeq_t$ on the set $\beta(t,p)$ for a.e. $t \in T$, i.e., if $x(t) \in \beta(t,p)$ and $x(t) \succeq_t v$ for every $v \in \beta(t,p)$. We denote the set of competitive allocations by $\mathcal{W}$. Since agents of the same type maximize their preferences on the same budget sets, a competitive allocation always satisfies the ETP and so $\mathcal{W} \subseteq \mathcal{M}$. Under the conditions above, the existence of competitive allocations follows from [2], and so $\mathcal{W}$ is always non-empty.

3. **The *Local* and *Global* bargaining set notions.** In this section we recall the notions of objection and counterobjection as defined by [18] and [23], and study the consequent definitions of bargaining set.

We consider the same terminology used in [19] and [17] and refer to the bargaining set of [18] and [23] respectively as *local* and *global* bargaining set.

Definition 3.1. Given an allocation x, a **local objection** to x is a pair (B, y) consisting of a coalition B and an integrable function $y : B \to \mathbb{R}^m_+$ such that

$(O1)$ B attains y, i.e., $\int_B (y - \omega) d\lambda \leq 0$;
$(O2)$ $y(t) \succeq_t x(t)$ for a.e. $t \in B$;
$(O3)$ $\lambda\left(\{t \in B : y(t) \succ_t x(t)\}\right) > 0$.

We denote by $Ob_\ell(x)$ the set of local objections to x. The **core** of the economy $\mathcal{E}$, denoted by $\mathcal{C}$, is defined as the set of feasible allocations against which there is no local objection, that is

$$\mathcal{C} = \{x \in \mathcal{F} : Ob_\ell(x) = \emptyset\}.$$

Definition 3.2. Given an allocation x, a **global objection** to x is a pair (B, y) consisting of a coalition B and an allocation $y : T \to \mathbb{R}^m_+$ that satisfies $(O1) - (O3)$ above and, in addition, y is feasible (i.e., $y \in \mathcal{F}$). We denote by $Ob_g(x)$ the set of global objections to x.

Remark 3.3. It is clear that, given an allocation x,

(i) if $(B, y) \in Ob_\ell(x)$, then $(B, y\chi_B + \omega\chi_{T\setminus B}) \in Ob_g(x)$;
(ii) if $(B, z) \in Ob_g(x)$, then $(B, z|_B) \in Ob_\ell(x)$,

where $z|_B$ denotes the restriction of the feasible allocation z to B and $y\chi_B + \omega\chi_{T\setminus B}$ is the allocation $\widetilde{y}$ assigning y to members of B and ω to agents outside B, i.e.,

$$\widetilde{y} = \begin{cases} y(t), & \text{if } t \in B \\ \omega(t), & \text{otherwise.} \end{cases}$$

By Remark 3.3 it follows that, given an allocation x,

$$Ob_\ell(x) = \emptyset \quad \Leftrightarrow \quad Ob_g(x) = \emptyset. \tag{1}$$

Thus by (1), even though the sets of local and global objections are different, they define the same core as the class of feasible allocations against which there is no local nor global objection.

The bargaining set grounds on a two-step veto mechanism, in the sense that, given an objection (B, y) against an allocation x, it allows some other agents to form a new coalition C and counter to the objection (B, y) by proposing an alternative redistribution of their initial resources z. The pair (C, z) is a counterobjection to (B, y) and the bargaining set is defined as the class of feasible allocations whose only objections are in turn counterobjected.

We now recall the notion of *counterobjection* as defined in [18].

Definition 3.4. Let x be an allocation and $(B, y) \in Ob_\ell(x)$. A **local counterobjection** to (B, y) consists of a pair (C, z) where C is a coalition and z is an integrable function $z : C \to \mathbb{R}^m_+$ such that:

$(C1)$ C attains z, i.e., $\int_C (z - \omega)\, d\lambda \leq 0$;
$(C2_\ell)$ $z(t) \succ_t y(t)$ for all $t \in C \cap B$;
$(C3_\ell)$ $z(t) \succ_t x(t)$ for all $t \in C \setminus B$.

The set of local counterobjections to (B, y) is denoted by $Cob^x_\ell(B, y)$. A local objection is **justified** if it has no local counterobjection. The **local bargaining set**, defined by [18] and denoted by BS_ℓ, is the class of all feasible allocations that have no justified local objection, i.e.,

$$BS_\ell = \{x \in \mathcal{F} : \ (B, y) \in Ob_\ell(x) \Rightarrow Cob^x_\ell(B, y) \neq \emptyset\}.$$

Remark 3.5. According to the above definition, the counterobjecting coalition C may include no agent of B, that is $C \cap B$ may be empty. In this case, $C \setminus B = C$ and the only relevant conditions are $(C1)$ and $(C3_\ell)$, so that (C, z) is actually just an objection to x rather than a counterobjection to (B, y). To avoid this situation, [25] requires that there is non-empty intersection between C and B, that is $C \cap B \neq \emptyset$. In Section 4 we propose a new notion of bargaining set that keeps this condition and we show that in atomless economies the non-empty intersection condition does not affect the bargaining set (see (5)).

Remark 3.6. As discussed in [18, Remark 1], the definition of counterobjection can be weakened by requiring strict preference only for a positive measure subset of the counterobjecting coalition. With this change, even if the set of counterobjections is formally larger, the set of justified objections (and hence the bargaining set) remains unaltered. A similar argument does not apply to objections: if one considers only objections in which all the deviating agents have strict preferences then the core does not change, but the bargaining set may become significantly larger. See [24] for a formal comparison of the bargaining sets generated by these different classes of objections.

[23] proposes an alternative definition of bargaining set with the idea that

(i) an objection is a *"global"* redistribution of commodities among all the agents in the economy,
(ii) agents in counterobjections improve their welfare with respect to the objection and not with respect to the original allocation.

We recall the formal definition of [23] (see also [19] and [17]).

Definition 3.7. Let x be an allocation and $(B, y) \in Ob_g(x)$. A **global counterobjection** to (B, y) consists of a pair (C, z) where C is a coalition and z is a feasible allocation (i.e., $z \in \mathcal{F}$) such that:

$(C1)$ C attains z, i.e., $\int_C (z - \omega)\, d\lambda \leq 0$;
$(C2_g)$ $z(t) \succ_t y(t)$ for all $t \in C$.

We denote by $Cob_g^x(B,y)$ the set of global counterobjections to (B,y). A global objection is **justified** if it has no global counterobjection. The **global bargaining set**, defined by [23] and denoted by BS_g, is the class of all feasible allocations that have no justified global objection, i.e.,

$$BS_g = \{x \in \mathcal{F} : \ (B,y) \in Ob_g(x) \Rightarrow Cob_g^x(B,y) \neq \emptyset\}.$$

Remark 3.8. The main difference between the two concepts is that [23] imposes that the allocations involved in objections and counterobjections are feasible for the set of all consumers, whereas in [18] such allocations are attainable only for the corresponding coalitions. Clearly, if there is no objection against an allocation, a fortiori, there is no justified (local nor global) objection against it. Thus, both BS_ℓ and BS_g contain the core, and hence the set of competitive allocations, so that no existence issue arises. Furthermore, [18] shows that the BS_ℓ coincides with the set of competitive allocation $\mathcal{W}$ and a fortiori with the core $\mathcal{C}$. Whereas, [23] shows that this is not the case for the global bargaining set, which is in general larger than the core. Therefore,

$$\emptyset \neq \mathcal{W} = \mathcal{C} = BS_\ell \subseteq BS_g \tag{2}$$

where the last inclusion can be strict.

3.1. **Justified objections and the core.** This section is devoted to illustrate the relationships between justified objections and the core. We characterize allocations that can be used to raise justified *global* objections as those in the core of the economy $\mathcal{E}$, whereas allocations used to raise justified *local* objections as those in the core of a suitably defined economy with production. These results do not need any assumption on the agents' measure space, so they hold in any economy regardless of the number of consumers.

Proposition 3.9. *Let x be an allocation and (B,y) be a global objection to x. Then, (B,y) is justified if and only if y belongs to the core of the economy $\mathcal{E}$.*

Proof. Let (B,y) be a justified global objection to x and assume to the contrary that y is not in the core of $\mathcal{E}$. Being y feasible, there exists a pair (C,z) such that C is a coalition that attains the allocation z and $z(t) \succ_t y(t)$ for a.e. $t \in C$ (see Remark 3.6). Define $\widetilde{z} = z\chi_C + \omega\chi_{T\setminus C}$ which is feasible and such that $\widetilde{z}|_C = z$. Thus, $(C,\widetilde{z})$ is a global counterobjection to (B,y), which is a contradiction. Viceversa, let y be a core allocation of the economy $\mathcal{E}$ and suppose, by a way of contradiction, that (B,y) is not justified. Then there exists a global counterobjection (C,z) to (B,y), that is such that C attains z and $z(t) \succ_t y(t)$ for all $t \in C$. This means that C improves upon y via z contradicting the fact that y is in the core of the economy $\mathcal{E}$. □

For local objections, instead, we only know that if (B,y) is justified then y is in the core of the economy $\mathcal{E}|_B$ defined as the restriction of $\mathcal{E}$ to the coalition B (see Proposition 3.1 in [11] for production economies with a finite number of agents). The converse may not be true (see also [6, Remark 2.8] for a similar observation). However, we show below that justified local objections can be characterized as core allocations of a suitably defined economy with production. To this end, given an allocation x and a local objection (B,y) to x, we can define a production economy

$$\mathcal{E}_x^B = \left\{(B,\Sigma_B,\lambda_B), \mathbb{R}_+^m, (\succeq_t, \omega(t))_{t\in B}, Y_x^B\right\},$$

where (B, Σ_B, λ_B) is the measure space restricted to B and its subcoalitions; for every $t \in B$, $\succeq_t$ and $\omega(t)$ are the same as in the original economy $\mathcal{E}$, and Y_x^B is the common production set defined as follows:

$$Y_x^B = \left[\bigcup_{S \subseteq T \setminus B} \left\{ \int_S (\widetilde{x} - \omega) d\lambda \ : \quad \widetilde{x}(t) \succeq_t x(t) \text{ for all } t \in S \right\} \cup \{0\} \right] + \mathbb{R}_+^m.$$

An allocation in the economy $\mathcal{E}_x^B$ is an integrable function $z : B \to \mathbb{R}_+^m$ which is called feasible with respect to a subcoalition C of B, if $\int_C (\omega - z) d\lambda \in Y_x^B$. This means that C, seen as a coalition in the original economy $\mathcal{E}$, attains z or that there exist a $S \subseteq T \setminus B$ and an integrable function $\widetilde{x} : S \to \mathbb{R}_+^m$ such that

$$\int_C (\omega - z) d\lambda \geq \int_S (\widetilde{x} - \omega) d\lambda \text{ and } \widetilde{x}(t) \succeq_t x(t) \text{ for all } t \in S.$$

The definitions of objections and core adapt naturally to this framework.

Proposition 3.10. *Let x be an allocation of $\mathcal{E}$ and (B, y) be a local objection to x. Then, (B, y) is justified in the economy $\mathcal{E}$ if and only if y is in the core of the economy $\mathcal{E}_x^B$.*

Proof. Let (B, y) be a justified local objection against x and assume by contradiction that y does not belong to the core of the economy $\mathcal{E}_x^B$. First note that y is feasible in $\mathcal{E}_x^B$ since B attains y.

Then, there exists (D, h) such that $D \subseteq B$, $h(t) \succ_t y(t)$ for a.e. $t \in D$, and $\int_D (\omega - h) d\lambda \in Y_x^B$. If D attains h in the economy $\mathcal{E}$, then (D, h) is a local counterobjection to (B, y) in $\mathcal{E}$, and this is a contradiction. Then, there exist a $S \subseteq T \setminus B$ and an integrable function $\widetilde{x} : S \to \mathbb{R}_+^m$ such that

$$\int_D (\omega - h) d\lambda \geq \int_S (\widetilde{x} - \omega) d\lambda \tag{3}$$

and

$$\widetilde{x}(t) \succeq_t x(t) \text{ for all } t \in S. \tag{4}$$

Consider the coalition $C = D \cup S$ and the allocation $z = h\chi_D + \widetilde{x}\chi_S$ and note that:

(a) $\int_C (z - \omega) d\lambda = \int_D (h - \omega) d\lambda + \int_S (\widetilde{x} - \omega) d\lambda \leq 0$ because of (3);
(b) $z(t) \succ_t y(t)$ for a.e. $t \in C \cap B$ because $C \cap B = D$ and $z|_D = h$;
(c) $z(t) = \widetilde{x}(t) \succeq_t x(t)$ for a.e. $t \in C \setminus B = S$ because of (4).

By Remark 3.6, this means that from (C, z) one can construct a local counterobjection to (B, y), which is a contradiction.

For the converse, assume that y is a core allocation for the economy $\mathcal{E}_x^B$ and, to the contrary, that there exists a local counterobjection (C, z) to (B, y) in $\mathcal{E}$. This means that

(a) $\int_C (z - \omega) d\lambda \leq 0$,
(b) $z(t) \succ_t y(t)$ for a.e. $t \in C \cap B$,
(c) $z(t) \succ_t x(t)$ for a.e. $t \in C \setminus B$.

Let us distinguish two cases: $\lambda(C \cap B) > 0$ and $\lambda(C \cap B) = 0$.

If $\lambda(C \cap B) > 0$, let $D = C \cap B$ and note that the pair $(D, z|_D)$ objects y in the economy $\mathcal{E}_x^B$. Indeed, consider $S = C \setminus B \subseteq T \setminus B$ and $\widetilde{x} = z|_{C \setminus B}$, and note that $\int_D (\omega - z) d\lambda = \int_{C \cap B} (\omega - z) d\lambda = \int_C (\omega - z) d\lambda - \int_{C \setminus B} (\omega - z) d\lambda \geq \int_{C \setminus B} (z - \omega) d\lambda = \int_S (\widetilde{x} - \omega) d\lambda$ because of (a) and $\widetilde{x}(t) = z(t) \succ_t x(t)$ for a.e. $t \in S = C \setminus B$ because of (c). Then, $\int_D(\omega - z)d\lambda \in Y_x^B$. Furthermore, $z(t) \succ_t y(t)$ for a.e. $t \in D$ because of (b). Therefore, y is not in the core of the economy $\mathcal{E}_x^B$ which is a contradiction.

If $\lambda(C \cap B) = 0$, then $C \setminus B = C$. By $(a),(c)$ and the monotonicity of preferences, the allocation z' defined as $z'(t) = z(t) + \frac{1}{\lambda(C)}\int_C(\omega - z)d\lambda$ for all $t \in C$ is such that

(a') $\int_C(z' - \omega)d\lambda = 0$,
(c') $z'(t) \succ_t x(t)$ for a.e. $t \in C$.

Monotonicity and continuity imply the existence of a $\varepsilon \in (0,1)$ and $C' \subseteq C$ such that $\varepsilon z'(t) \succ_t x(t)$ for a.e. $t \in C'$. Consider the allocations $z'' = \varepsilon z'\chi_{C'} + z'\chi_{C \setminus C'}$ and $y'(t) = y(t) + \frac{(1-\varepsilon)}{\lambda(B)}\int_{C'} z'd\lambda$ for all $t \in B$. Note that $z''(t) \succ_t x(t)$ for a.e. $t \in C \subseteq T \setminus B$. Furthermore,

(i) $y'(t) \succ_t y(t)$ for a.e. $t \in B$,
(ii) $\int_B(\omega - y')d\lambda \in Y_x^B$. Indeed,

$$\begin{aligned}
\int_B(\omega - y')d\lambda &= \int_B(\omega - y)d\lambda - (1-\varepsilon)\int_{C'} z'd\lambda \\
&\geq 0 - (1-\varepsilon)\int_{C'} z'd\lambda \\
&= \int_C(z' - \omega)d\lambda - (1-\varepsilon)\int_{C'} z'd\lambda \\
&= \int_{C\setminus C'} z'd\lambda + \int_{C'} \varepsilon z'd\lambda - \int_C \omega d\lambda \\
&= \int_C(z'' - \omega)d\lambda,
\end{aligned}$$

with $z''(t) \succ_t x(t)$ for a.e. $t \in C \subseteq T \setminus B$. In other words, $\int_B(\omega - y')\,d\lambda \in Y_x^B$ and so B, seen as a coalition in $\mathcal{E}_x^B$, attains y'. This means that (B, y') is an objection to y in the economy $\mathcal{E}_x^B$, which contradicts the fact that y is a core allocation in $\mathcal{E}_x^B$. □

4. **A new notion of Bargaining set.** We now introduce a variation of Mas-Colell's bargaining set which is based on a weaker mechanism of objections and counterobjections. Intuitively, we assume that an agent accepts to join an objection (or a counterobjection) only if she is promised a bundle at least as good as those consumed by her peers, i.e., by the agents of her same type.

Definition 4.1. Given an allocation x, an **objection*** to x is a pair (B, y) such that $(B, y) \in Ob_\ell(x)$ and $y \in \mathcal{M}_B$.

Denote by $Ob^*(x)$ the set of objections* to x, it is clear that $Ob^*(x) \subseteq Ob_\ell(x)$. On the other hand, it can be proved that, if x has the ETP, from a local objection $(B, y) \in Ob_\ell(x)$ we can construct an objections* (B, y') to x. This follows from the Lemma in [5] (see also Theorem 3.8 in [7]). We define the set of feasible allocations with the ETP against which there is no objection* as the *core** and we denote it by $\mathcal{C}^*$, i.e.,

$$\mathcal{C}^* = \{x \in \mathcal{F} \cap \mathcal{M} : Ob^*(x) = \emptyset\}.$$

Clearly, $\mathcal{W} \subseteq \mathcal{C}^*$, and so it is non-empty under our assumptions.

Definition 4.2. Let x be an allocation and (B, y) be an objection* to x. A **counterobjection*** to (B, y) consists of a pair (C, z) where C is a coalition and z is an integrable function $z : C \to \mathbb{R}_+^m$ such that:

$(C1)$ C attains z, i.e., $\int_C (z-\omega)\, d\lambda \leq 0$;
$(C2^*)$ $z(t) \succeq_t y(t)$ for all $t \in C \cap B$ and $\lambda(\{t \in C \cap B : z(t) \succ_t y(t)\}) > 0$;
$(C3^*)$ $z(t) \succ_t x(t)$ for all $t \in C \setminus B$;
$(C4^*)$ $z \in \mathcal{M}_C$;
$(C5^*)$ for all $t \in C$, $z(t) \succeq_t y(s)$ if $s \in B \setminus C$ and $s \sim t$.

Note that $(C2^*)$ implies that
$(C6^*)$ $\lambda(C \cap B) > 0$.

Thus, in addition to the notion of local counterobjection (see Definition 3.4) we ask that no agent in the counterobjection* is envious of what her peers in C receive from z, what her peers in B receive from y and what the others receive from x (see $(C4^*)$ and $(C5^*)$). Furthermore, as in [25], we impose the existence of a non-null set of objecting agents that join also the counterobjection (see $(C6^*)$). It is not immediate, however, whether considerations similar to those of Remark 3.6 hold for counterobjections*: given a counterobjection* to an objection*, it is unclear whether this can be modified into a new counterobjection* whose members are all strictly better off and keep the ETP.

We denote the set of counterobjections* to (B, y) by $Cob^*_x(B, y)$. An objection* is **justified** if it has no counterobjection*. The **bargaining set***, denoted by BS^*, is the class of all feasible allocations satisfying the ETP that have no justified objection*, i.e.,

$$BS^* = \{x \in \mathcal{F} \cap \mathcal{M} : \ (B, y) \in Ob^*(x) \Rightarrow Cob^*_x(B, y) \neq \emptyset\}.$$

It is clear that the bargaining set* contains the core*, i.e., $\mathcal{C}^* \subseteq BS^*$. Furthermore, we now show the relation between justified objections* and allocations in the core* of a restricted economy. As for results in Section 3.1, the following proposition does not need any assumption on the agents' measure space so it holds in any economy.

Proposition 4.3. *Let x be an allocation and (B, y) be an objection* to x. If (B, y) is justified then y belongs to the core* of the restricted economy $\mathcal{E}|_B$.*

Proof. Let (B, y) be a justified objection* to x. In particular, B attains y meaning that y is feasible in $\mathcal{E}|_B$. Assume to the contrary that y in not in the core* of $\mathcal{E}|_B$. Since $y \in \mathcal{M}_B$, this means that there exists (C, z) such that $C \subseteq B$, $z \in \mathcal{M}_C$, C attains z, $z(t) \succeq_t y(t)$ for a.e. $t \in C$ and $\lambda(\{t \in C : z(t) \succ_t y(t)\}) > 0$. We show that (C, z) is a counterobjection* to (B, y) and we reach a contradiction. Note that, since $C \subseteq B$, $\lambda(C \cap B) = \lambda(C) > 0$ and $C \setminus B = \emptyset$. Thus, we only need to show that (C, z) also satisfies condition $(C5^*)$ of Definition 4.2. To this end, take $t \in C$ and $s \in B \setminus C$ where s and t are of the same type ($s \sim t$). Since $y \in \mathcal{M}_B$ and $z(t) \succeq_t y(t)$ for a.e. $t \in C$, we have that $z(t) \succeq_t y(t) \succeq_t y(s)$. This completes the proof. □

The converse implication of Proposition 4.3 does not hold in general. Moreover, the arguments in Proposition 3.10 do not apply to this new class of objections, and so they cannot be used to study justified objections* within special production economies.

Remark 4.4. The notion of bargaining set* can be formalized for economies with more general measure spaces (not necessarily atomless) and compared to the core therein. Once every type of agents consists of a single individual, the equal treatment property (ETP) becomes irrelevant and the core is a subset of BS^*. But, in general, since it is well known that there are core allocations without the ETP (see [8]), comparisons between the BS^* and the core are not straightforward (see [7] for further observations). Nevertheless, we show below that BS^* always contains the set of competitive allocations $\mathcal{W}$ so that, under the assumptions of [1], it contains the core $\mathcal{C}$ too.

Proposition 4.5. *The bargaining set* contains the set of competitive allocations, i.e., $\mathcal{W} \subseteq BS^*$, and so it is non-empty.*

Proof. Let x be a competitive allocation. We have already observed that $\mathcal{W} \subseteq \mathcal{F} \cap \mathcal{M}$. Furthermore, being $\mathcal{W} \subseteq \mathcal{C}$, $Ob_\ell(x) = \emptyset$ and a fortiori $Ob^*(x) = \emptyset$ since $Ob^*(x) \subseteq Ob_\ell(x)$. This means that $x \in BS^*$. $\square$

Proposition 4.5 also follows from the inclusions $\mathcal{W} \subseteq \mathcal{C}^*$ and $\mathcal{C}^* \subseteq BS^*$.

Remark 4.6. Since, $Ob^*(x) \subseteq Ob_\ell(x)$ for any allocation x and $Cob^*_x(B,y) \subseteq Cob^x_\ell(B,y)$ and any objection* $(B,y) \in Ob^*(x)$, the relationship between BS^* and BS_ℓ is not immediate. In fact, a simultaneous reduction on the set of objections and counterobjections causes, respectively, an enlargement and a reduction of the bargaining set. They both contain the set of competitive allocations and thus, under standard assumptions, they are both non-empty.

Remark 4.7. Our notion of bargaining set has some similarities also with the notion of global bargaining set due to [23]. Given an allocation x, an objection (B,y) to x and a counterobjection (C,z) to (B,y), recall that the main difference between the notion of counterobjection due to [18] and [23] concerns the feasiblity of z and merely agents in $C \setminus B$, because for [18] these agents $t \in C \setminus B$ are such that $z(t) \succ_t x(t)$, whereas [23] requires z feasible and such that $z(t) \succ_t y(t)$. Our notion keeps some features of both, because we require that agents in $C \setminus B$ do not envy only their counterparts in the objection to which they counter. In other words, we impose a comparison between z and y, as done in [23], only among agents of the same type and for the rest z is compared with x, as in [18]. We show that this is enough to extend Schmeidler's theorem as done in [19] for the global bargaining set (Theorem 4.12), as well as to characterize competitive allocations (Theorem 4.9), as done for local bargaining set by [18].

4.1. **Equivalence with competitive equilibria.**

In this section we show that our notion of bargaining set coincides with the set of competitive allocations. To this end, the following notion due to [18] is needed.

Definition 4.8. An objection (B,y) to x is **Walrasian** if there exists a price system $p \neq 0$ such that, for a.e. $t \in T$:

(i) $p \cdot v \geq p \cdot \omega(t)$ for $v \succeq_t y(t)$, $t \in B$;
(ii) $p \cdot v \geq p \cdot \omega(t)$ for $v \succeq_t x(t)$, $t \notin B$.

Theorem 4.9. *An allocation belongs to the bargaining set* if and only if it is a competitive allocation, i.e., $\mathcal{W} = BS^*$.*

Proof. The inclusion $\mathcal{W} \subseteq BS^*$ follows from Proposition 4.5. For the converse, let x be a feasible allocation with the ETP that is not competitive. By Proposition 2 in [18], there is a Walrasian objection (B,y) against x. Remark 5 of [18] ensures that, if x has the ETP, every allocation used to produce a Walrasian objection to x has the ETP too. But this implies that $(B,y) \in Ob^*(x)$. From Proposition 1 in [18] any Walrasian objection is justified, then $Cob^x_\ell(B,y) = \emptyset$. Being

$Cob_x^*(B,y) \subseteq Cob_\ell^x(B,y)$, we have that $Cob_x^*(B,y)$ is empty too. Therefore, x is a feasible allocation with the ETP (i.e., $x \in \mathcal{M}$) against which there is an objection* (B,y) with no counterobjection*. We conclude that x is not in BS^*. □

From Theorem 4.9 above and Theorem 1 in [18] we get that

$$\mathcal{W} = \mathcal{C}^* = \mathcal{C} = BS^* = BS_\ell \subseteq BS_g, \tag{5}$$

where the last inclusion can be strict.

The notion of objection* is introduced in the companion paper [7], where we define the Equitable Bargaining set, denoted by BS_e, as a solution concept that combines the concern for equitability with the need to prevent possible deviations from agents seeking better opportunities. We show that the Equitable Bargaining set coincides with that of [18] when the economy is atomless. Moreover, we provide two sets of conditions for economies with market imperfections that apply to finite economies and to mixed market economies. In the first case, our conditions imply that the Equitable Bargaining set is a subset of the core, and so it converges to the set of competitive allocations if the economy is replicated. In the second case, we show the equivalence with the set of competitive allocations in mixed markets. There are two differences between BS^* and BS_e that regard the notion of counterobjections. In [7] we do not impose that counterobjections include some members of the objections they counter; moreover counterobjecting agents are all strictly better off. On the one hand, condition $(C6^*)$ makes counterobjecting more difficult and consequently the bargaining set becomes smaller. On the other hand, requiring that only a non-negligible group of counterobjecting agents are strictly better off makes the set of justified objections smaller with a consequent enlargement of the bargaining set. Therefore BS^* and BS_e are, in general, not comparable. However, thanks to Proposition 4.1 in [7] and Theorem 4.9, they coincide in atomless economies. In order to test whether an allocation is not in the Equitable Bargaining set we can use a notion of objection that is more general than the Walrasian objection due to [18] (see Definition 4.8). We call *weakly Walrasian* this class of objections. Proposition 1 in [18] ensures that any Walrasian objection is justified. The same does not hold for a weakly Walrasian objection that, by Lemma 3.12 in [7], always has an empty intersection with any of its counterobjections.[8] Consequently, weakly Walrasian objections are justified if one considers in the counterobjecting mechanism condition $(C6^*)$.

4.2. **Bargaining set with small coalitions.** [9,20] and [22] show that, in atomless economies, restrictions on the size of blocking coalitions do not change the core. Then, not all coalitions must be considered to block a non-competitive allocation. In particular, [20] proves that if an allocation is blocked by a certain coalition, then it is also blocked by a subcoalition with an arbitrarily small measure. [19] attempts to extend this result to the bargaining set. It is demonstrated that if only coalitions of arbitrarily small size are allowed to object and counterobject then the global bargaining set remains the same whereas the local bargaining set gets larger. In what follows, we show that this kind of restriction on coalition formation has no impact on the bargaining set* too. To this end, we adopt the same terminology and notation used in [19]: given an allocation x and a positive number $\delta > 0$, we say that an objection* (B,y) to x is a δ-objection* to x, in symbol $(B,y) \in Ob_\delta^*(x)$,

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[8] Althought, Proposition 3.13 in [7] shows that if a group of agents accepts to raise a weakly Walrasian objection then this can be extended to some larger justified objection.

if $\lambda(B) \leq \delta$. Similarly, given $\varepsilon > 0$, a counterobjection* (C, z) to (B, y) is said to be a ε-counterobjection* to (B, y), in symbol $(C, z) \in Cob^*_{\varepsilon,x}(B, y)$, if $\lambda(C) \leq \varepsilon$. An objection* (B, y) to x is ε-justified if $Cob^*_{\varepsilon,x}(B, y) = \emptyset$. A feasible allocation x belongs to the $\varepsilon\delta$-bargaining set, in symbol $x \in BS^*_{\varepsilon\delta}$, if it satisfies the ETP and there is no ε-justified δ-objection* to x, i.e.,

$$BS^*_{\varepsilon\delta} = \{x \in \mathcal{F} \cap \mathcal{M} : \ (B, y) \in Ob^*_\delta(x) \Rightarrow Cob^*_{\varepsilon,x}(B, y) \neq \emptyset\}.$$

If δ and ε are large enough, for example, bigger than $\lambda(T)$, the limitation is not effective because all coalitions are allowed in both objections and counterobjections. On the other hand, the next lemma shows that if $\varepsilon \leq \lambda(T)$ it is possible to reduce the size of any counterobjection as much as we want while preserving the type of agents represented in the counterobjection.

Lemma 4.10. *Let x be an allocation and $(B, y) \in Ob^*(x)$. If (C, z) is a counterobjection* to (B, y), then for every $\varepsilon \in (0, \lambda(C))$ there exists $C^\varepsilon \subseteq C$ such that:*

(i) $\lambda(C^\varepsilon \cap K_n) = \frac{\varepsilon}{\lambda(T)} \lambda(C \cap K_n)$ *for every n, and*

(ii) $(C^\varepsilon, z) \in Cob^*_{\varepsilon,x}(B, y)$.

Proof. Let (C, z) be a counterobjection* to (B, y) and denote by $C^B = C \cap B$ and $C^{\neg B} = C \setminus B$. We can partition C^B into two sets, $C^B = P^B \cup R^B$, where

$$P^B = \{t \in C^B : z(t) \succ_t y(t)\} \quad \text{and} \quad R^B = C^B \setminus P^B.$$

For every n, consider the (possibly null) sets $P^B_n = P^B \cap K_n$; $R^B_n = R^B \cap K_n$ and $C^{\neg B}_n = C^{\neg B} \cap K_n$ consisting of the agents of type n respectively in P^B, R^B and $C^{\neg B}$.

By construction, C is the disjoint union $\left(\bigcup_n P^B_n\right) \cup \left(\bigcup_n R^B_n\right) \cup \left(\bigcup_n C^{\neg B}_n\right)$ and $\lambda(C) = \sum_n \left[\lambda\left(P^B_n\right) + \lambda\left(R^B_n\right) + \lambda\left(C^{\neg B}_n\right)\right]$.

In particular, since $\lambda(P^B) > 0$ and $\lambda(C \cap B) > 0$, there must be a n' for which $\lambda(P^B_{n'}) > 0$ and hence $\lambda(P^B_{n'} \cup R^B_{n'}) > 0$.

Consider now the atomless measure $\eta \colon \Sigma \to \mathbb{R}^{m+1}$ that assigns to each $S \in \Sigma_C$ the vector:

$$\eta(S) = \left(\int_S (z - \omega)\, d\lambda, \lambda(S)\right).$$

For any type of agents n, we apply Lyapunov convexity theorem to Σ_{P^B}, Σ_{R^B} and $\Sigma_{C^{\neg B}}$ and find, for every $\varepsilon \in (0, \lambda(C))$, three coalitions $S^B_n \subseteq P^B_n$, $I^B_n \subseteq R^B_n$ and $D^{\neg B}_n \subseteq C^{\neg B}_n$ such that $\eta\left(S^B_n\right) = \frac{\varepsilon}{\lambda(T)} \eta\left(P^B_n\right)$, $\eta\left(I^B_n\right) = \frac{\varepsilon}{\lambda(T)} \eta\left(R^B_n\right)$ and $\eta\left(D^{\neg B}_n\right) = \frac{\varepsilon}{\lambda(T)} \eta\left(C^{\neg B}_n\right)$.

For every n, let us put $C_n = S^B_n \cup I^B_n \cup D^{\neg B}_n$. This means that:

$$\begin{aligned} \lambda(C_n) &= \lambda(S^B_n) + \lambda(I^B_n) + \lambda(D^{\neg B}_n) \\ &= \frac{\varepsilon}{\lambda(T)} \lambda\left(P^B_n\right) + \frac{\varepsilon}{\lambda(T)} \lambda\left(R^B_n\right) + \frac{\varepsilon}{\lambda(T)} \lambda\left(C^{\neg B}_n\right) \\ &= \frac{\varepsilon}{\lambda(T)} \lambda(C \cap K_n) \quad \text{for every } n, \end{aligned} \tag{6}$$

with

$$\lambda(S^B_{n'}) = \frac{\varepsilon}{\lambda(T)} \lambda(P^B_{n'}) > 0, \tag{7}$$

$$\lambda(S^B_{n'} \cup I^B_{n'}) = \frac{\varepsilon}{\lambda(T)} \lambda(P^B_{n'}) + \frac{\varepsilon}{\lambda(T)} \lambda(R^B_{n'}) = \frac{\varepsilon}{\lambda(T)} \lambda\left(P^B_{n'} \cup R^B_{n'}\right) > 0, \tag{8}$$

and that, for every n,

$$\begin{aligned}\int_{C_n}(z-\omega)d\lambda &= \int_{S_n^B}(z-\omega)d\lambda+\int_{I_n^B}(z-\omega)d\lambda+\int_{D_n^{\neg B}}(z-\omega)d\lambda \\ &= \frac{\varepsilon}{\lambda(T)}\left[\int_{P_n^B}(z-\omega)d\lambda+\int_{R_n^B}(z-\omega)d\lambda+\int_{C_n^{\neg B}}(z-\omega)d\lambda\right] \\ &= \frac{\varepsilon}{\lambda(T)}\int_{C\cap K_n}(z-\omega)d\lambda. \qquad (9)\end{aligned}$$

We claim that $C^\varepsilon = \bigcup_n C_n$ is the desired coalition. Condition (i) follows from Equation (6), given that $C^\varepsilon \cap K_n = C_n$ for every n. We focus on (ii) and show that $(C^\varepsilon, z) \in Cob^*_{\varepsilon,x}(B, y)$. To prove that C^ε attains z, use Equation (9) and that C attains z to write:

$$\begin{aligned}\int_{C^\varepsilon}(z-\omega)\,d\lambda &= \sum_n \int_{C_n}(z-\omega)\,d\lambda \\ &= \sum_n \frac{\varepsilon}{\lambda(T)}\int_{C\cap K_n}(z-\omega)\,d\lambda \\ &= \frac{\varepsilon}{\lambda(T)}\int_C (z-\omega)\,d\lambda \\ &\le 0.\end{aligned}$$

From the inclusion $C^\varepsilon \subseteq C$ and the fact that (C, z) is itself a counterobjection* to (B, y) we obtain that $z \in \mathcal{M}_{C^\varepsilon}$, $z(t) \succeq_t y(t)$ for a.e. $t \in C^\varepsilon \cap B$ and $z(t) \succeq_t x(t)$ for a.e. $t \in C^\varepsilon \setminus B$. Furthermore, since $S^B_{n'} \subseteq \{t \in C^\varepsilon \cap B : z(t) \succ_t y(t)\}$, from (7) it follows that $\lambda(\{t \in C^\varepsilon \cap B : z(t) \succ_t y(t)\}) > 0$. Observe that $\lambda(C^\varepsilon \cap B) > 0$ because of (8) and $S^B_{n'} \cup I^B_{n'} \subseteq C^\varepsilon \cap B$; whereas by (6), we have that $\lambda(C^\varepsilon) = \sum_n \lambda(C_n) = \sum_n \frac{\varepsilon}{\lambda(T)}\lambda(C \cap K_n) = \varepsilon\frac{\lambda(C)}{\lambda(T)} \le \varepsilon$. Finally, for a.e. $t \in C^\varepsilon$ and $s \in B \setminus C^\varepsilon$ of the same type of t we have two possibilities: if $s \notin C$ then $z(t) \succeq_t y(s)$ because (C, z) is a counterobjection* to (B, y); if $s \in (B \cap C) \setminus C^\varepsilon$ then $z(t) \succeq_t z(s) \succeq_t y(s)$ because (C, z) is a counterobjection* to (B, y) and $z \in \mathcal{M}_C$. We conclude that (C^ε, z) is a ε-counterobjection* to (B, y). □

The next lemma shows that a similar result can be proved for objections* (see Proposition 4.3 in [7]).

Lemma 4.11. *Let x be an allocation and $(B, y) \in Ob^*(x)$. Then, for every $\delta \in (0, \lambda(B))$ there exists $B^\delta \subseteq B$ such that:*

- (i) *$\lambda(B^\delta \cap K_n) = \frac{\delta}{\lambda(T)}\lambda(B \cap K_n)$ for every n, and*
- (ii) *$(B^\delta, y) \in Ob^*_\delta(x)$.*

Proof. See Proposition 4.3 in [7]. □

We are now ready to show that, contrary to the local bargaining set defined by [18] (see Theorem 3 in [19]), our bargaining set is such that for all $\varepsilon, \delta > 0$, $BS^*_{\varepsilon\delta} = BS^*$. The inclusion $BS^* \subseteq BS^*_{\varepsilon\delta}$ is a direct consequence of Lemma 4.10, whereas the converse inclusion $BS^*_{\varepsilon\delta} \subseteq BS^*$ rests on the equitability condition we impose in the blocking mechanism. Indeed, the proof consists in applying Lemma 4.11 to an objection* (B, y) and get a smaller objection* (B', y') with $\lambda(B') = \delta$, the equitabiliy condition imposed among agents of the same type ensures that any counterobjection* to (B', y') counterobjects* (B, y) too. This also holds with the global bargaining set of [23] and fails with the notion of [18].

Theorem 4.12. *For all* $\varepsilon, \delta > 0$, $BS^*_{\varepsilon\delta} = BS^*$.

Proof. As already noted, only the cases when $\varepsilon, \delta \in (0, \lambda(T))$ are relevant. Let $x \in BS^*$ and assume to the contrary that $x \notin BS^*_{\varepsilon\delta}$ for certain given $\varepsilon, \delta \in (0, \lambda(T))$. This means that there exists an objection* (B, y) to x such that $\lambda(B) \leq \delta$ with no ε-counterobjection*. Since $x \in BS^*$ and $(B, y) \in Ob^*(x)$, there exists $(C, z) \in Cob^*_x(B, y)$. Lemma 4.10 ensures the existence of a ε-counterobjection* to (B, y) which is a contradiction. For the converse, let $x \in BS^*_{\varepsilon\delta}$ and assume to the contrary that $x \notin BS^*$. This means that there exists an objection* (B, y) to x with no counterobjection*. Lemma 4.11 implies the existence of $(B^\delta, y) \in Ob^*_\delta(x)$, where B^δ contains the same types of agents of B (see Lemma 4.11 (i)) and such that $\lambda(B^\delta) \leq \delta$. Being x in $BS^*_{\varepsilon\delta}$, there is a ε-counterobjection* (C, z) to (B^δ, y). We conclude the proof by showing that (C, z) is also a counterobjection* to (B, y). Clearly, since $B^\delta \subseteq B$, the unique check regards agents in $B \setminus B^\delta$ and, in particular, conditions $(C2^*)$ and $(C5^*)$ of Definition 4.2. Note that for any $t \in (C \cap B) \setminus B^\delta$, since B and B^δ contain the same types of agents, there is $s \in B^\delta$ such that $s \sim t$. If $s \in C \cap B^\delta$, since $y \in \mathcal{M}_B$ and $z \in \mathcal{M}_C$, we have that $z(t) \succeq_t z(s) \succeq_t y(s) \succeq_t y(t)$. If, instead, $s \in B^\delta \setminus C$, then being y in $\mathcal{M}_B$, we get that $z(t) \succeq_t y(s) \succeq_t y(t)$. Thus, $z(t) \succeq_t y(t)$ for a.e. $t \in C \cap B$, and from the inclusion $C \cap B^\delta \subseteq C \cap B$ we also have that $\lambda(\{t \in C \cap B : z(t) \succ_t y(t)\}) > 0$. Thus $(C2^*)$ is satisfied and, with similar arguments, one proves also condition $(C5^*)$ that is, for all $t \in C$, $z(t) \succeq_t y(s)$ if $s \in B \setminus C$ and $s \sim t$. $\square$

5. **Final remarks.** This paper studies different notions of bargaining sets for atomless economies and proposes two sets of results. The first is concerned with the characterizations of justified objections of all different types. The second proposes a definition of bargaining set that meets three fundamental requirements for cooperative solution concepts in exchange economies.

The variations in the definitions of bargaining set depend on the different types of strategic behavior that agents adopt when trying to improve upon an objection with a valid counterobjection. To capture the differences between the strategic interactions that describe each notion of bargaining set, we associate to any objection (B, y) an auxiliary economy in which the problem of understanding whether (B, y) is justified translates into understanding whether y is in the core of the associated economy. On this ground, one could explore the properties of these auxiliary economies and propose new characterizations. For example, consider a non-justified local objection (B, y) to an allocation x and consider the auxiliary production economy $\mathcal{E}^B_x$. Proposition 3.10 gives that y, seen as an allocation in $\mathcal{E}^B_x$, is not in the core, and so it can be objected (in $\mathcal{E}^B_x$). A natural question would be whether one can take such objection to y arbitrarily large in size, that is, whether Vind's Theorem applies to the economy $\mathcal{E}^B_x$. This is equivalent to the following question: if (B, y) is not justified (and hence unstable in some sense), is it there a local counterobjection that involves arbitrarily many agents in B, or would a significant number of agents in B prefer the objection y to any other agreement?

The second part of the paper focuses on the bargaining set* and shows that, in atomless economies, it is non-empty, it characterizes the set of competitive allocations, and it remains unaltered if one allows only small coalitions to form. It remains unclear how this blocking mechanism changes if one puts a lower bound on the size (or diameter) of the coalitions that are allowed to raise objections and counterobjections, in the spirit of Vind's and Grodal's characterizations of the core. A last observation is that it seems unlikely that one could provide core-like characterizations of the objection process that defines the bargaining set* similar to those in Proposition 3.9 and Proposition 3.10. The reason is that the notion of objection* requires a combination of global and local considerations that can hardly be reproduced in a workable restricted economy.

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