TRANSCENDENTAL CURVES BY THE INVERSE TANGENT PROBLEM:

HISTORICAL AND DIDACTICAL INSIGHTS FOR CALCULUS

Davide CRIPPA*, Pietro MILICI **
*Ca'Foscari University, Malcanton Marcorà
Calle Contarini, Dorsoduro 3484/D, 30123 Venezia, and Institute of Philosophy, Czech Academy of Sciences, Jilská 1, 110 00, Prague.
**University of Insubria, Via O. Rossi 9, Varese, Italy
davide.crippa@unive.it
pietro.milici@uninsubria.it

ABSTRACT

In this paper, based on a workshop held at ESU 2022 (Salerno), aimed at secondary school and university teachers, as well as historians and didacticians, we suggest a new approach to calculus from a constructive geometric perspective. Specifically, we will provide a historical presentation of certain geometric instruments related to calculus, and we will introduce a new device for hands-on experiences. After that, we will describe the activities of the workshop presented at the latest ESU 9 in Salerno, in which we alternated the introduction of historical sources and laboratory activities.

1 *Introduction*

Mathematical instruments are an important focus in recent research in the didactics of mathematics (cf. Monaghan, Trouche & Borwein, 2016). In this regard, the history of instruments to trace curves (or "Zeicheninstrumente" in German) has also been recently studied by Bartolini Bussi & Maschietto (2007), Tournès (2009), and van Randenborgh (2015).

Historically, these instruments have played a key role in the constitution of analytic geometry and calculus. This role gradually waned: during the 18th century geometric instruments were marginalized, separated from "pure" mathematics except when they were used as a resource for teaching, and confined in some cases to treatises on instruments in general.[30] This process went

[30] Among these treatise, we can mention: Nicolas Bion (1652-1723), *Traité de la construction et des principaux usages des instrumens de mathématique*, Paris 1709 (cf. also Turner, 2014); Jakob Leupold (1664–1727), *Theatrum arithmetico-geometricum, Das ist:Schau-Platz der Re-*

alongside with the separation of finite and infinitesimal analysis from geometry to form an independent discipline, the so-called "algebraization of analysis" culminated in the 19th century with Bolzano, Cauchy and Weiestrass.

In this paper, we shall focus, on the basis of historical examples, on the epistemological implications of geometric instruments with respect to the understanding of fundamental concepts of analytic geometry and calculus.

Moreover, we shall explore the fruitfulness of a concrete approach to calculus for today's teaching by proposing learning experiences involving machines that implement historical ideas.

2 History

One of the explicit goals of the pioneers of calculus, such as Leibniz, Huygens, and the brothers Bernoulli was the construction and study of curves, particularly those that could not be associated with algebraic equations (Blåsjö, 2017). According to Leibniz and his friends, discovering new curves through already known instruments, or finding new artifacts to construct curves that had been previously observed in natural phenomena were still paramount.

The episode, recounted by Leibniz and represented in Fig. 1, of Perrault's posing of the problem of finding the curve traced by a watch attached to a chain and dragged along a tabletop is just an example of how the manipulation of an artifact, initially not devised for geometrical purposes, could lead to the discovery of new geometric objects and new insights into the solution of open problems (Bos, 1988; Tournès, 2009).

In addition to providing new materials for the geometer, the use of instruments in the early modern period also had more theoretical goals. As in Descartes (1637), instruments could frame the relations between symbolic computations in finite and infinitesimal analyses and the study of curves, as well as answer foundational questions, such as those about the admissibility of curves in geometry (Bos, 2001; Panza, 2011).

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chen- und Meßkunst, Leipzig 1727; Gianbattista Suardi (1711–1767), *Nuovi istrumenti per la descrizione die diverse curve antiche e moderne*, Brescia 1752; George Adams (1750–1795), *Geometrical and geographical essays containing a general description of the mathematical instruments used in geometry, civil and military surveying, leveling, and perspective*, London 1791. See Randenborgh (2015, p. 46).

Broadly speaking, a geometric instrument can be defined as an "artifact", that is a material object that, among other things, makes a mathematical idea concrete (Randenborgh, 2015, pp. 6ff., and Vollrath, 2003). For example, the Euclidean compass is an object employable in various ways and with different goals; however, when used by students to perform the constructions prescribed in Euclid's Elements, it encapsulates Euclid's third postulate, and embodies the real definition of a circle.

Figure 1. Perrault's problem as it was imagined by Giovanni Poleni around 50 years later (Poleni, 1729, table).

However, one may doubt whether a watch dragged upon a table should be counted as a geometric instrument at the same level as the compass. What mathematical idea does this object realize? What is the nature of the curve described by the trajectory of the watch case? Perrault posed this question as an explicit challenge. During a controversy that saw the involvement of Huygens, Johann and Jakob Bernoulli, Leibniz declared himself to be the first to have answered correctly. Leibniz identified the curve traced by the moving watch with a "tractrix" or "tractoria" and described it as the curve having constant tangent-length (Leibniz, 1693, also Bos, 1988).

This was not a small discovery, since it shows that Perrault's instrument realizes a solution to a fundamental problem for the development of analysis: to find a curve whose tangents have constant length. More generally, problems that involve the determination of a curve starting from given properties of their tangents, or relations among tangents and other segments, are called "inverse-tangent problems".

Christian Huygens, who had also studied Perrault's curve, was less prone to accept the system formed simply by a watch moved on a table as a geometric instrument (Bos, 1988, p. 29ff.; Tournès 2009, pp. 15-16). According to him, a "dragged watch" should fulfill certain criteria to function as a proper curve-tracing instrument, which are not trivially met. First, Huygens demanded that the plane of the dragging be horizontal to avoid the effects of gravity. Second, the dragging of the weight from one position to the next should be reversible so that the purported instrument can change from position A to B and vice versa. Third, the motion should be sufficiently slow to prevent inertia. Due to the existing friction, the direction of the motion, made "real" by the dragged chain, would always be tangent to the curve. Only in this way, and because the length of the chain is invariant during the motion, the traced curve would have tangents of constant length.

Without suitable modifications to the physical configuration, manipulating the watch could easily get out of hand and fail to produce the desired curve. For this reason, Huygens proposed models of alternative machines that ensured the fulfillment of the three conditions described above, and thus would be fully geometrical (cf. Bos, 1988, p. 30; Tournès, 2009, p. 16; Blåsjö 2017). Unfortunately, Huygens never published his results on this topic. Leibniz, who nevertheless addressed the construction of transcendental curves in his published works and, at greater length, in the unpublished ones, eschewed the problem of constructing concrete machines. For him, the tractrix was a legitimate curve once the possibility of its construction via its tangent properties was conceded. It did not matter much whether a device that ensured such motion could be constructed. The issue was still open during the first half of the 18th century when the British mathematician John Perks and the Italians Giovanni Poleni and Gianbattista Suardi designed, and in some cases produced, artefacts which could trace the tractrix and the logarithmic. Thanks to them, during the first half of the century, new machines to trace transcendental curves entered mathematics.[31]

These efforts show that, for these authors, ideal machines were just not enough; mathematical ideas ought to be embodied into physical objects.

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[31] For a short introduction and the construction of a new machine cf. (Crippa & Milici, 2019)

Poleni’s machines are described in all their technical details in a letter to Hermann published in 1729 in Padua and reprinted in the *Fasciculus epistolarum mathematicarum* (1729), together with other letters for eminent Italian mathematicians. The machines described in the letter are also mentioned in a printed catalogue of mathematical and physical instruments that belonged to Poleni's “Cabinet of physics” (Talas, 2013, p. 58), demonstrating that they were also constructed and used. The explicit goal behind Poleni’s project was to improve known methods to construct the tractrix and the logarithmic, such that these curves could be generated in a way not more complex than those employed for the conic sections, and thus fulfill the demands for constructability laid by previous authors, *in primis* Huygens himself.

The construction of the tractrix depicted in Fig. 2 (left) exemplifies this task (cf. Tournès 2009, pp. 72ff). In the instrument, a solid bar replaces the original chain in Perrault’s experience, and a toothed wheel (“rotula signatoria”) orthogonal to the plane of the curve replaces the weight. It is the wheel, and not the weight, that traces the curve on the plane with the correct property of tangents (“cujus rotatione curva signatur”, Poleni, 1729, § 26), acting through a single continuous motion.

The fundamental difference with respect to previous models and descriptions is the introduction of a wheel to guide the tangent to the curve. The wheel replaced Perrault’s original watch and served as a fundamental component to obtain a precise mechanical device to guide the tangent. With respect to the dragged point of Perrault’s construction, a wheel (working in a similar way as when we turn the front wheel of a bike) also greatly enhances the precision. In this way, the wheel maintains the bar always tangent to the curve, so that a tractrix can be traced.[32]

[32] This technical innovation is not merely related to the accuracy of the output. As highlighted in (Dawson, Milici & Plantevin, 2021), the principle underlying the wheel and the dragged point are mathematically different, even though they coincide in the simplest cases.

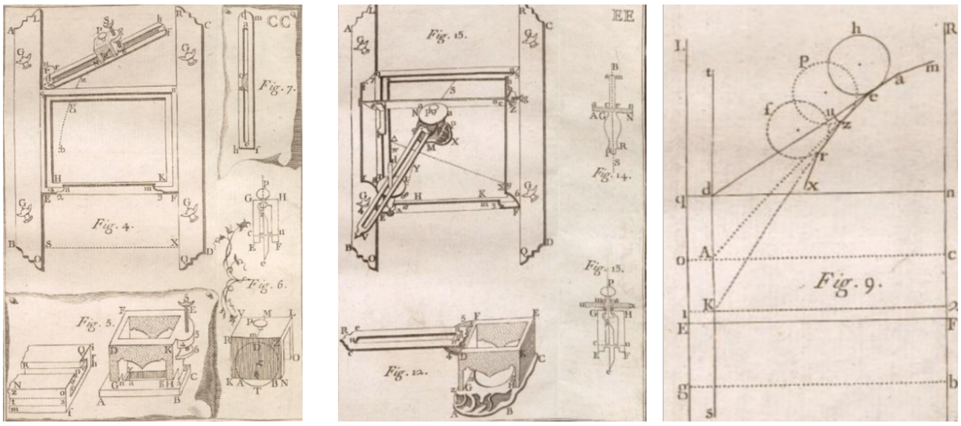

Figure 2. Poleni's drawings from his "Fasciculus". Left: a machine to trace the tractrix (the original chain in Perrault's watch has been replaced by a bar and the watch-case by a wheel). Center: a machine to trace the logarithmic curve. Right: the wheel marks infinitesimal segments corresponding to the direction of the tangents at each point.

Poleni employed a similar procedure to construct the logarithmic curve. As shown in Leibniz (1684), the logarithmic (or exponential) is a curve whose subtangents have constant length. The machine for the logarithmic, depicted in the center of Fig. 2, is then a variation of the instrument for the tractrix. The bar representing the tangent to the curve has a variable length, whereas a horizontal bar, namely the side of a moving rectangle in the figure, ensures that the subtangent to the curve is constant for any of its points. After a period of oblivion, geometric methods to solve the inverse tangent problem were independently rediscovered in the second half of the 19th century (Tournès, 2009, pp. 271ff.). At that time, the mechanical resolution of the inverse tangent problem was adopted not to legitimate or introduce specific curves, but to perform transformations related to the resolution of differential equations. Such machines are called "integraphs". They are devices that integrate functions introduced as geometrical curves. The main aim of these devices was to solve practical problems (e.g. finding solutions to differential equations that do not allow symbolic resolution). In at least two periods, during the 18th, and 20th centuries, machines for the construction of curves became part of cabinets with a pedagogical function. We find these in the 18th century "Cabinet of experimental philosophy" of Giovanni Poleni in Padua (Talas, 2013) and in the 20th century "Cabinet of Differential Calculus" of Ernesto Pascal in Naples (Tournès, 2009, p. 271ff.). It is significant that both scholars designed and constructed new devices. Even though there is no evidence that Poleni used them in his lectures, it is attested that he employed instruments in his classes of experimental philosophy (Talas, 2013, pp. 52ff.). Furthermore, machines in brass to trace the logarithmic and the tractix are mentioned in a printed list of Poleni's machines, and Poleni himself had samples built in order to circulate them among his colleagues.[33] Therefore, we cannot even exclude the possibility that Poleni might have used his machines to enhance students' understanding of the fundamental concepts of calculus. Manipulating curves with a machine would not justify the principles of calculus, but it may have helped pupils accept operations involving counter-intuitive objects such as infinitesimal segments and familiarize themselves with abstract relations and definitions, such as the new definition of tangent as the line connecting two infinitely close points. Thanks to the action of the wheel, in particular, these machines may have made concrete and visible a fundamental principle of the Leibnizian calculus: "a curve line can be considered composed by infinitely many small lines, or elements, infinitely small … which include angles, from which the curvature is generated" (Poleni, 1729, §44).[34] As Poleni himself noticed, the wheel marks the infinitesimal segments ae, ez ... (Fig. 2, on the right), corresponding to the directions of the tangents at each point of the curve, so that these "infinitely small lines are described by the motion of our instrument" (Poleni, 1729, §49). This example shows that, by making abstract

[33] Poleni sent exemplars of his machines to the mathematicians Gabriele Manfredi and Jacopo Riccati, and to his friend Antonio Conti (Tournès, 2009, p. 79). Although an exemplar of an alleged machine to draw transcendental curves is preserved in the Museum of History of Physics in Padua, its function and design are presently unknown. We thus have to conclude that none of Poleni's geometrical machines has survived until today in its original form. Following Poleni's tables and descriptions, models of the machines for the tractrix and the exponential have been recently reconstructed (Milici & Plantevin, 2022). A video of such a reconstruction is available online at the link www.youtube.com/watch?v=LIsQkML2Tis

[34] Poleni's definition ("lineam curvam … mex conſiderari posse ceu compositam ex infinitis lineolis (sive elementis) rectis, infinite parvis, ma, ae, ez, zr, rx comprehendentibus inter se angulos, ex quibus lineæ curvature progignitur") is actually a postulate in L'Hopital treatise: "We suppose that a curved line may be considered as an assemblage of infinitely many straight lines, each one being infinitely small, or (what amounts to the same thing) as a polygon with an infinite number of sides, each being infinitely small, which determine the curvature of the line by the angles formed amongst themselves." (Bradley, Petrilli & Sandifer, 2015, p. 3).

notions of differential calculus concrete and more manipulable than the original devices using weights and strings, Poleni's machines can represent a suitable basis for didactical experiences.

3 *The material artifact*

With the idea that the historical machines discussed above can be possible vectors of "didactical ideas" (van Radenborgh, 2015, p. 6), in the workshop we merged a presentation of selected excerpts from the historical sources surveyed above with related laboratory activities involving material artifacts. Specifically, we proposed a "kit for calculus" (patented, invented and constructed by the second author) integrating the 18th and 20th centuries resolutions of inverse tangent problems, with a specific focus on the simplicity of the design. The machine, that can be called a "T-sliding integraph" (because there is a T-shaped rod with two perpendicular guides), is introduced in www.machines4math.com (also with some related videos); it is built by digital construction tools, and the source files (together with assembling instructions) are freely available at www.thingiverse.com/thing:5532958. A previous version of the kit is described in more details in the same volume (Maschietto and Milici, 2023).

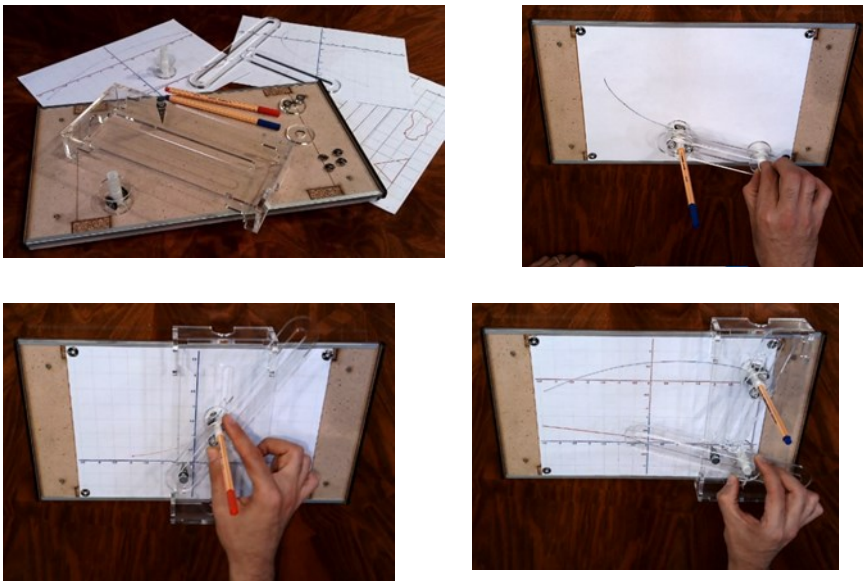

Figure 3. The proposed kit for calculus: its components and various uses.

The components of the proposed device are shown at the top left of Fig. 3. There are two pointers (one with a couple of wheels) that can be used to follow an already traced curve or trace a curve-to-be using a marker. There is also a wooden base (where paper sheets can be attached) and a transparent plastic case that can slide on it. Finally, there are two rods that can be joined to make a “T": these rods can be used as guides for the pointers. These components can be assembled in several ways as required for various possible activities.

4 The workshop

In the workshop, we alternated the introduction of historical sources (with guided discussions) and laboratory activities (participants were divided into small groups with one kit per each group).

4.1 First activity: ***the tangent***. After a historical introduction to organic geometry, we considered the historical problem of justifying the existence of transcendental curves in an organic way, that is, through an instrument. To demonstrate how the limits set by Descartes can be overcome, we proposed the following laboratory activity. After tracing an arbitrary curve on a sheet of paper, the audience was invited to move the pointers on the traced curve. The aim of this activity is to focus on the difference of use between the “smooth pointer” (the one whose bottom is smooth) and the “wheeled pointer” (the one whose bottom has two parallel wheels). The introduction of the wheels imposes the restriction that, to move the wheeled pointer on a curve, the piece must be rotated so that its direction is tangent to the traced curve. The audience then experienced the passage from direct to inverse tangent problems, by linking the direction of the wheels to a rod and moving the other extremity of the rod along a line (cf. top right of Fig. 3). After a brief description of the experience of Perrault’s watch and some historical references to Huygens and Leibniz, the audience was guided to reenact the construction of the tractrix in the manner of early modern geometers and link the mechanical components to the possibility of tracing a curve given its tangent properties.

4.2 Second activity: ***the exponential*** (cf. bottom left of Fig. 3). In this activity, the audience was invited to assemble the kit for calculus in a way that imposes the constant subtangent. After presenting Poleni’s letter to Hermann and the tables reproduced in fig. 2, the audience realized that they were handling a modern version of Poleni’s machine for the logarithmic curve. The guided analysis of the machine resulted in the construction of an exponential curve without introducing notions of limits or summations (from an analytic perspective, the machine geometrically solves a differential equation). As a variation, if one imposes the direction of the wheel not as the one passing through the peg fixed on the sliding case but perpendicular to it, the device can trace parabolas (cf. Maschietto, Milici & Tournès, 2019). In this case, with a suitable choice of reference frame, the differential equation is easily converted into an integral.

Third activity: ***derivatives and antiderivatives*** (cf. bottom right of Fig. 3). In this case, a sheet with two reference frames, one for each pointer, was adopted (cf. Maschietto & Milici, 2023, §2). This activity ideally follows the first one, in which one can show that, when the wheeled pointer follows a curve, the direction of the wheels must be the tangent to the curve. Furthermore, the T-shaped rod imposes the restriction that the direction of the tangent must be parallel to the line passing through the peg fixed on the case and the other pointer. With a suitable reference frame, the ordinate of the smooth pointer can be shown to represent the slope of the tangent to the curve followed by the wheeled pointer. This geometric configuration corresponds to the definition of the derivative. In contrast to the previous case, if the smooth pointer is moved, the wheeled pointer traces an antiderivative. This activity is in the direction suggested by Blum (1982), according to which integraphs may be used to make students discover the fundamental theorem of calculus by themselves (e.g., by approximating piecewise the function to be integrated by many constant functions). The third activity was integrated with historical notes on integraphs.

5 Conclusions

Geometric instruments for curve tracing can play an important role in didactics, because they are artifacts that embody multiple ideas: historical, mathematical, technical, and pedagogical. In this article, we presented a possible convergence of historical considerations and hands-on activities. We began with a case study to introduce historical machines for the construction of certain transcendental curves. These machines embody a mathematical idea, namely the concept of a curve known through the property of its tangents (for a modern mathematical setting of these machines see Milici, 2020). Then, we proposed a series of activities presented during a workshop at the latest ESU-9 in Salerno, based on the suitable use of a machine to present some key concepts of differential and integral calculus. Even though these experiences do not directly reproduce historical episodes, they are devised in the same spirit: to use concrete artifacts to visualize and anchor abstract concepts, such as the notions of tangent and slope, or epistemologically difficult ones, such as that of an infinitesimal segment. The workshop highlighted the fact that most participants, who are teachers or researchers in history and didactics, were sympathetic with the rationale behind this workshop, and with the importance of developing more concrete, engaging and intuitive approaches to calculus than the standard ones. They also agreed that the activities presented at the workshop could be fruitfully applied in secondary education. However, this step is far from obvious, as several questions during the discussion following our presentation have pointed out. A major critical aspect concerns the use of the calculus kit as an exploratory device and can be summarized as follows. To enhance students' understanding of calculus, didactical activities should be carefully designed so that students are guided in the process of discovery and each step of their activity is duly motivated in terms of what the questions they should pose and the conclusions they should achieve through the manipulation of the device. Therefore, we also plan to find new collaborations to improve these underdeveloped aspects of our proposal and to bring this historical and tangible approach to calculus in high schools.

Acknowledgements: This article was made possible thanks to the support of the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement
No 101024431-LEGITIMATH.

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