Simple Strategies in Multi-unit Assignment Problems

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Abstract

This paper examines strategic behavior in multi-unit assignment problems, employing simple manipulation strategies. Assuming responsive preferences and priorities, we demonstrate that dropping strategies are exhaustive in the immediate acceptance mechanism, while truncation strategies fall short. This finding clarifies the trade-offs among stability, simplicity, and manipulability in assignment mechanisms, with implications for real-world applications, such as course allocation.

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1. Introduction

We study multi-unit assignment mechanisms , which involves assigning objects (courses) to agents (students) based on their priorities and preferences (Sönmez and Ünver, 2010; Budish, 2011; Kojima, 2013), focusing on simple order-preserving manipulations.

It is well known that no mechanism is both stable and strategy-proof in multi-unit assignment problems . When preferences and priorities are responsive[1], Romero-

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[1] Responsiveness is a common assumption in this context: Budish and Cantillon (2012); Kojima (2013); Kojima and Ünver (2014); Doğan and Klaus (2018); Abdulkadiroğlu and Sönmez

Medina and Triossi (2024) shows that the Immediate Acceptance mechanism (IA) (Abdulkadiroğlu and Sönmez, 2003) implements the set of stable allocations in Nash equilibrium (NE).

Our main result shows that *dropping strategies* are exhaustive in the IA. That is, for any outcome achievable through arbitrary manipulation, there exists an equivalent strategy in which the agent drops (removes) acceptable options while preserving the original order on the remaining acceptable courses. This implies that although the IA is manipulable, its analysis can be confined to dropping strategies.

By contrast, truncation strategies —where agents declare all options below a certain threshold as unacceptable—fail to generate all equilibrium outcomes under the IA. We construct an example where it is impossible to achieve any stable allocations through truncation strategies. Conversely, truncation strategies are sufficient in many-to-one markets under the Deferred Acceptance (DA) (Roth and Vande Vate, 1991, Theorem 2).

The analysis of both dropping and truncation strategies has focused on stable mechanisms. In many-to-one matching problems under DA, Roth and Rothblum (1999) demonstrated that submitting a truncated version of true preferences can benefit agents. Ehlers (2008) extended this result. Jaramillo et al. (2013) proved that any stable matching is a NE outcome under the student-optimal stable mechanism, but not all NE outcomes are stable matchings.

For many-to-many matching markets, Jaramillo et al. (2014) showed that dropping strategies are exhaustive for groups of agents on the same market side under pairwise stable mechanisms. In any stable matching mechanism, Kojima and Pathak (2009) demonstrates that every profitable manipulation by a college can be replicated or improved upon using a dropping strategy.

In this paper, we demonstrate that dropping strategies provide a tractable approach to studying manipulation in the IA. Technically, our approach relates to the equilibrium implementation result of Romero-Medina and Triossi (2024), which

(2003)

shows that the IA implements all stable matchings in NE. We focus on the exhaustiveness of simple manipulation strategies. We show that all NE outcomes in the IA can be obtained using dropping strategies, and that truncating is weakly dominated by truth-telling.

Our findings suggest that IA-based allocation systems are more transparent and predictable than previously believed and can be better understood by focusing on dropping behavior.

2. The Model

There is a finite set of courses C and a finite set of students S, with $C \cap S = \emptyset$. Let $I = C \cup S$. Each course $c \in C$ has priorities, a linear order over students, $\succ_c$. Analogously, each student $s \in S$ has preferences, represented by a linear order over courses, $\succ_s$.

We consider responsive priorities and preferences. Let $c \in C$, let $\succ_c$ be a linear order over $S \cup \{\emptyset\}$ and let q_c be a quota.

Priorities P_c are **responsive** to $\succ_c$ with quota q_c if, for every $S', S'' \in 2^S$ such that $|S'| \leq q_c - 1$, and for every $s, s' \in S \setminus S'$, the following conditions hold:

i. $S' \cup \{s\} \; P_c \; S' \quad \Longleftrightarrow \quad s \succ_c \emptyset$;

ii. $S' \cup \{s\} \; P_c \; S' \cup \{s'\} \quad \Longleftrightarrow \quad s \succ_c s'$.

The weak order induced by P_c is denoted by R_c. Responsive preferences for students are defined in a similar way using corresponding notation. Each student $s \in S$ has responsive preferences P_s over subsets of courses, 2^C. The set of responsive priorities or preferences for agent $i \in I$ is denoted by $\mathcal{P}_i$. For all $I' \subseteq I$ let $\mathcal{P}_{I'} = \prod_{i \in I'} \mathcal{P}_i$. We denote $P = P_S \cup P_C$. A student is **acceptable** to course c if $s \succ_c \emptyset$. Let $A(\succ_c)$ be the set of students acceptable to course c.

We represent priorities as ordered lists with quotas. For example, $\succ_c$: s_1, s_2 with $q_c = 2$ implies that $P_c : \{s_1, s_2\}, s_1, s_2$. Student preferences are represented analogously.

A choice function derived from the choice set of course c from S' based on the responsive priority profile P_c is denoted by $Ch_c(S', P_c)$. Formally, $Ch_c(S', P_c) =$

$\max_{P_c} 2^{S'}$. When there is no ambiguity regarding P_c, we write $Ch_c(S')$ instead of $Ch_c(S', P_c)$. Similarly $Ch_s(C')$ and $Ch_s(C', P_s)$ are defined in an analogous manner.

An **allocation** is a function $\mu : C \cup S \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all allocations is denoted by $\mathcal{M}$. Allocation μ is **individually rational** for $x \in C \cup S$ if $Ch_x(\mu(x)) = \mu(x)$. Allocation μ is **blocked** by a pair $(c, s) \in C \times S$ if $s \notin \mu(c)$, $c \in Ch_s(\mu(s) \cup \{c\})$, and $s \in Ch_c(\mu(c) \cup \{s\})$. Finally, an allocation μ is **stable** for (C, S, P) if it is individually rational for all $x \in C \cup S$ and no pair is blocking it. The set of stable allocations is denoted by $\mathcal{S}(P_S)$ if there is no ambiguity about P_C.

A strategy is an ordered preference list of a subset of students. More precisely, for each student s, $\triangleright_s$ is the set of strategies, and $\triangleright \equiv \prod_{s\in S} \triangleright_s$ is the set of strategy profiles.

A student's **dropping strategy** removes acceptable courses while preserving the order over the remaining ones (Kojima and Pathak, 2009). Formally, for $\succ$ over $C \cup \{\emptyset\}$ and $Z \subseteq A(\succ)$, the dropping strategy $\succ^Z$ satisfies:

i. $c \succ^Z \emptyset \iff c \in Z$;
ii. For $c, c' \in Z$, $c \succ^Z c' \iff c \succ c'$.

A student's **truncation** at c keeps only courses at least as good as c (Roth and Vande Vate, 1991). Formally, for $\succ$ over $C \cup \{\emptyset\}$ and $c \in A(\succ)$, the truncation strategy $\succ_{|c}$ satisfies:

i. $c' \succeq_{|c} \emptyset \iff c' \succeq c$;
ii. $c'' \succ_{|c} c''' \iff c'' \succ c'''$.

Let $\mathcal{D}_s \subseteq \triangleright_s$ for every $s \in S$ and let $\mathcal{D} = \prod_{s\in S} \mathcal{D}_s$. A (revelation) **mechanism** is a function φ that associates an allocation to every preference profile for students, $\succ_S = (\succ_s)_{s\in S} \in \mathcal{D}$, $\varphi : \mathcal{D} \to \mathcal{M}$. A mechanism is **stable** if $\varphi(\succ_S)$ is a stable allocation for each $\succ_S$. Given a priority profile P_C and a preference profile P_S, a mechanism φ induces a normal form game $\mathcal{G}(P_S) = (S, \mathcal{D}, \varphi, P_S)$, in which S is the set of players, $\mathcal{D}$ is the Cartesian product of students' strategy spaces, φ is the outcome function and P_S is the profile of student preferences. Let $\widehat{\mathcal{P}} \subseteq \mathcal{P}_S$ and let $\Phi : \widehat{\mathcal{D}} \rightrightarrows \mathcal{M}$ be a correspondence. We say that φ **implements** Φ in ***NE*** if, for each $P_S \in \widehat{\mathcal{P}}$, the set of NE outcomes of $\mathcal{G}(P_S) = (S, \mathcal{D}, \varphi, P_S)$ coincides with $\Phi(P_S)$.

Let φ be a mechanism, and consider an agent s with a preference $\succ_s$ over $C \cup \{\emptyset\}$. A set of strategies $\mathcal{S}_s \subseteq \mathcal{D}_s$ is **exhaustive** if, for any alternative report $\tilde{\succ}_s \in \mathcal{D}_s$ and outcome $\mu = \varphi(\tilde{\succ}_s, \succ_{-s})$, there exists a strategy $\succ'_s \in \mathcal{S}_s$ such that $\varphi(\succ'_s, \succ_{-s}) R_s \mu$.

In words, an exhaustive set of strategies ensures that the agent can always find a strategy within the set that yields an outcome at least as good as any other outcome they could achieve.

3. Results

We study the many-to-many version of the IA (Romero-Medina and Triossi, 2024), extending the school assignment mechanism of Abdulkadiroğlu and Sönmez (2003).

In the IA, each student initially submits their preferences over individual courses. In the first step, we consider each student's preferred acceptable set of courses [2]. Within this initial step, among the students who choose a specific course, those with the highest priorities for that course are assigned to it. In the r^{th} step, we only consider the r^{th} choice on the preference lists of the remaining students. At the end of each step, a student assigned at least one course is eliminated from further consideration. This iterative process continues until no students remain. The assignments made in each step are deemed final.

We assume that the quota vector $(q_s)_{s \in S}$, is public information. Let $s \in S$. We denote that a message of student s by $m_s = \succ_s$, in which $\succ_s$ is a strict order on $C \cup \{\emptyset\}$. M_s denotes the set of messages for student s. Given a message m_s and q_s, let $P_s = P_s(m_s)$ be responsive preferences that rationalize m_s (see Alva, 2018

[2] Notice that for each preference over courses $\succ_s$ there might be more than one preference over the set of courses P_s that are responsive to $\succ_s$. However, this will not affect the best response outcome nor the equilibrium outcome of the game.

Theorem 1).[3] Let $C^1_{P_s}$ be the preferred set of courses according to P_s, and, for every $r \in \mathbb{N}$, $0 \leq r \leq \left|2^C\right| - 1$, let $C^{r+1}_{P_s}$ be the $(r+1)^{th}$ ranked set of courses according to P_s. Formally $C^1_{P_s} = Ch_s\left(2^C, P_s\right)$, $C^{r+1}_{P_s} = Ch_s\left(\left(\left(2^C \setminus \bigcup_{1 \leq t \leq r} C^t_{P_s}\right) \cup \{\emptyset\}\right), P_s\right)$.

Let $(P_c)_{c \in C}$ be a responsive priority profile. Let $(m_s)_{s \in S} = (\succ_s)_{s \in S} \in \triangleright$ be a message profile and let $(P_s)_{s \in S} = \left(P_s\,(m_s)_{s \in S}\right)$ be the corresponding preference profile. The following procedure describes the ***IA***.

Step 1: For every $c \in C$ let $S^1_c = \{s \in S \mid c \in C^1_{P_s}\}$. Set $\mu^1(c) = Ch_c\left(S^1_c, P_c\right)$ and let $S^1 = S \setminus \left(\{s \in S \mid C^1_{P_s} = \emptyset\} \cup \bigcup_{c \in C} \mu^1(c)\right)$.

Step r+1: For every $c \in C$ let $S^{r+1}_c = \{s \in S^r \mid c \in C^{r+1}_{P_s}\}$. Set $\mu^{r+1}(c) = \max_{P_c}\{S' \mid \mu^r(c) \subseteq S' \subseteq \mu^r(c) \cup S^{r+1}_c\}$ and let
$S^{r+1} = S \setminus \left(\{s \in S \mid C^{r+1}_{P_s} = \emptyset\} \cup \bigcup_{c \in C} \mu^{r+1}(c)\right)$.

Let $r^* = \min\{r \geq 1 \mid C^r = \emptyset \text{ or } S^r = \emptyset\}$ and set $IA(\succ) = \mu^{r^*}$. Such a r^* exists because C and S are finite.

In the IA, all students who, at any stage, are assigned to at least one course or have not asked for any course are removed. The process continues until all students have been eliminated. Students never lose their place in a course to which they have been assigned at any point in the mechanism.

In Lemma 1, we examine the effectiveness of dropping strategies in achieving the outcomes of the IA.

Lemma 1. *Let* $\succ = (\succ_s)_{s \in S}$ *and let* $\mu = IA(\succ)$*, and* $\mu(s) = IA(\succ)(s)$*. For each* $s \in S$ *and* $C' \subseteq \mu(s)$*,* $C' = IA\left(\widetilde{\succ}^{C'}_s, \succ_{-s}\right)(s)$*, for every* $\widetilde{\succ}$ *such that* $C' \subseteq A\left(\widetilde{\succ}\right)$*.*

Proof. Let $s \in S$ and let $c \in C' \subseteq \mu(s)$. In the first stage of the algorithm, each $c \in C'$ has at least one vacant seat. Since s is acceptable to every $c \in C$ and priorities are responsive, $C' = IA\left(\widetilde{\succ}^{C'}_s, \succ_{-s}\right)$, because the IA is individually rational. ■

Lemma 1 implies that, given the strategies of the other players, any player can attain any achievable, individually rational set of alternatives by listing the al-

[3] Note that the same method is used to define the mechanism called "simplified immediate acceptance" in Romero-Medina and Triossi, 2024

ternatives included in any order. Specifically, if all alternatives are individually rational, she can achieve this set through a dropping strategy.

In the context of the IA, any deviation from a non-equilibrium strategy can be obtained through a dropping strategy, and there is a best response in dropping strategies. Furthermore, the result implies that any allocation can be achieved by using a preference profile where all students adopt dropping strategies. It follows from Lemma 1 that dropping strategies are exhaustive. We enunciate this result in Corollary 1.

Corollary 1. *Dropping strategies are exhaustive in the IA mechanism.*

Theorem 1 demonstrates that the IA successfully implements the set of stable allocations in NE employing dropping strategies.

Theorem 1. *The IA implements the set of stable allocations in* NE *when preferences and priorities are responsive. For each allocation μ that is stable in market (C, S, P), there exists a* NE *of the IA that results in μ, where all agents employ a dropping strategy.*

Proof. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations, and then we prove that any stable allocation is an NE outcome of the game induced by the IA, where all agents employ a dropping strategy.

i. Let $\succ_S^*$ be a NE of $\left(S, \mathcal{P}^{|S|}, IA, P_S\right)$ and let $\mu = IA\left(\succ_S^*\right)$. As observed, μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume $Ch_s\left(\mu\left(s\right)\right) \neq \mu\left(s\right)$ for some $s \in S$. Let $\succ_s' = \succ_s^{ch_s(\mu(s))}]$, by Lemma 1: $IA\left(\succ_s', \succ_{-s}^*\right)(s) = Ch_s\left(\mu\left(s\right)\right)$. Thus, the deviation is profitable to s, which yields a contradiction. We prove by contradiction that no course-student pair blocks μ. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $\succ' = \succ_s^{ch_s(\mu(s) \cup \{c\})}$. Because $s \in Ch_c\left(\mu\left(c\right) \cup \{s\}\right)$, the deviation is profitable to s, which yields a contradiction. Thus, allocation μ is stable.

ii. Let μ be a stable allocation. For each s, let $\succ_s^* = \succ_s^{\mu(s)}$. Set $\succ_S^* = \left(\succ_s^*\right)_{s \in S}$. We have $IA\left(\succ_S^*\right) = \mu$. We prove by contradiction that $\succ_S^*$ is a NE. Assume that $s \in S$ has a profitable deviation, $\succ_s'$, and let $\mu' = IA\left(\succ_s', \succ_{-s}^*\right)$. Let $c \in$

$Ch_s(\mu(s) \cup \mu'(s)) \setminus \mu(s)$. Because P_s is responsive, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $\succ''_s = \succ_s^{ch_s(\mu(s) \cup \{c\})}$, then $IA\left(\succ''_s, \succ^*_{-s}\right)(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c, s) blocks μ, which yields a contradiction.

■

It turns out that truncation strategies fail to replicate all outcomes for the IA. The claim of Theorem 1 does not hold for truncation strategies, as shown in Example 1.

Example 1. *Let $S = \{s_1, s_2\}$ and $C = \{c_1, c_2\}$. Let $q_{s_1} = 2, q_{s_2} = 1$, and $q_{c_i} = 1$. Preferences and priorities are as follows:*

$\succ_{s_1}: \{c_2\}, \{c_1\};$ $\qquad\qquad$ $\succ_{c_1}: \{s_1\}, \{s_2\};$

$\succ_{s_2}: \{c_1\}, \{c_2\};$ $\qquad\qquad$ $\succ_{c_2}: \{s_2\}, \{s_1\};$

The only stable allocation in this market is $\mu = \{(s_1, c_1), (s_2, c_2)\}$. The preferences of student s_1 over sets of courses are $P_{s_1}: \{c_2, c_1\}, \{c_2\}, \{c_1\}$, s_2 can obtain c_2 through the IA by declaring $\succ_{s_2}$: $\{c_2\}$ or $\succ_{s_2}$: $\{c_2\}; \{c_1\}$. None of those strategies is a truncation of her original preference $\succ_{s_2}$: $\{c_1\}, \{c_2\}$. However, $\succ_{s_2}$: $\{c_2\}$ is a dropping strategy of her original preference. Theorem 2 in Jaramillo et al. (2014) demonstrates that in stable matching mechanisms, truncation strategies are exhaustive for each agent with a quota of 1. Example 1 shows that this result does not extend to unstable mechanisms such as the IA.

These features are not merely an artifact of considering an allocation for every profile of linear orders, $\succ_S$, rather than profiles over sets of courses P_S. Although the IA is not defined in this paper to handle preferences over sets of alternatives, thinking for analogy if we examine $P_{s_1}: \{c_1, c_2\}, \{c_2\}, \{c_1\}$ no truncation strategy profile can produce a stable allocation. In particular $P_{s_1}: \{c_1, c_2\}, \{c_2\}$; $P_{s_2}: \{c_1, \}, \{c_2\}$, results in the unstable matching $\mu = \{(s_1, c_1), (s_1, c_2)\}$ and also $P_{s_1}: \{c_1, c_2\}$; $P_{s_2}: \{c_1, \}, \{c_2\}$, as well as $P_{s_1}: \{c_1, c_2\}$; $P_{s_2}: \{c_1, \}$.

In Example 1, students achieve the same outcome by either stating their true preferences or presenting a truncated preference profile. This case can be generalized as follows.

Proposition 1. *In the IA, every truncation strategy is weakly dominated by truth-telling.*

Proof. We prove that a truncation strategy yields a weakly worse outcome than the truth-telling for any agent in the IA. Let $s \in S$ and let $\bar{c} \in C$ be the lowest ranked course in $IA(\succ_s, \succ_{-s})$. If $\bar{c} \succeq c$ the definition of the IA implies that $IA\left(\succ_{s\ |_c}, \succ_{-s}\right)(s) = IA(\succ_s, \succ_{-s})(s)$.

If $c \succ \bar{c}$ then the definition of the IA implies that, for all $c' \in IA\left(\succ_{s\ |_c}, \succ_{-s}\right)(s)$, $c' \in IA\left(\succ_s, \succ_{-s}\right)(s)$ and $IA\left(\succ_{s\ |_c}, \succ_{-s}\right)(s) \subsetneq IA\left(\succ_s, \succ_{-s}\right)(s)$. By responsiveness it follows that $IA\left(\succ_s, \succ_{-s}\right)(s)\, P_s IA\left(\succ_{s\ |_c}, \succ_{-s}\right)(s)$, which completes the proof of the claim. ■

4. Conclusion

This paper analyzes the strategic behavior of agents under the IA. Assuming responsive preferences and priorities, we establish that dropping strategies are exhaustive. This contrasts with truncation strategies, which do not generate all equilibrium outcomes in many-to-many markets under the IA. We demonstrate that dropping strategies serve as a tool for agents to achieve stable allocations under the IA, regardless of the students' quota.

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