»Le present est plein de l’avenir, et chargé du passé«

Vorträge des XI. Internationalen Leibniz-Kongresses

31. Juli – 4. August 2023, Leibniz Universität Hannover, Deutschland

Band 1

Herausgegeben von
Wenchao Li, Charlotte Wahl,
Sven Erdner, Bianca Carina Schwarze und Yue Dan

In Gedenken an

Prof. Dr. Dr. h.c. Wolfgang-Uwe Friedrich

* 17.07.1952   † 16.06.2023

Veranstalter

Gottfried-Wilhelm-Leibniz-Gesellschaft e. V.
Leibniz Universität Hannover
Gottfried Wilhelm Leibniz Bibliothek – Niedersächsische Landesbibliothek

in Verbindung mit

Leibniz Society of North America
Sociedad Española Leibniz
Societas Leibnitiana Japonica
Sodalitas Leibnitiana
Société d’études leibniziennes de langue française
Association Leibniz Israel
Red Iberoamericana Leibniz
Societatea Leibniz din România
Centre d’Études Leibniziennes
Sino-German Leibniz Research Centre

Gefördert durch

Die Deutsche Forschungsgemeinschaft (DFG) – Projektnummer 517991912
Mittel aus zukunft.niedersachsen
VGH Versicherungen

Für Sachspenden danken wir

Verlag Walter De Gruyter

Titelbild: https://commons.wikimedia.org/wiki/File:1703_Martin_Bernigeroth_Kupferstich_Gottfried_Wilhelm_Leibniz,_Porträtsammlung_der_Herzog_August_Bibliothek_Wolfenbüttel.jpg

Impressum

Gottfried-Wilhelm-Leibniz-Gesellschaft e.V.
c/o Gottfried Wilhelm Leibniz Bibliothek
Niedersächsische Landesbibliothek
Waterloostr. 8
30169 Hannover
Tel.: +49 511 1267331
Fax: +49 511 1267202
www.gottfried-wilhelm-leibniz-gesellschaft.de

INHALTSVERZEICHNIS (Band I)

Filippo Costantini (Venice)

LEIBNIZ'S NOTION OF PART

Introduction

When scholars speak of Leibniz's mereology, they usually refer to the Real Addition calculus (or Plus-Minus calculus), a calculus that Leibniz developed in a series of works between 1685–86.[1] However, this calculus does not directly regard the parthood relation, but a more general relation – the *inesse or containment* relation – and a composition operation known as Real Addition.

In these papers, Real Addition (symbolized with '$\oplus$') is an operation that allows us to add together any kind of things in order to obtain an *aggregate* of exactly those things. For example, "B $\oplus$ N = L means that B is (contained) in L or L contains B".[2] Here Leibniz considers Real Addition as a primitive operation and defines the containment relation (*inesse*) by means of it. So Leibniz's definition becomes: $Cxy \equiv \exists z(y \oplus z = x)$, where Cxy means that x contains y (or that y is in x). In *Specimen calculi coincidentium et inexistentium*, after stating Postulate 2 ("any plurality of terms, such as A and B, can be added to compose a single term, A $\oplus$ B or L"), Leibniz explains that "our general construction depends upon the second postulate, in which is contained the assumption that any term and any other term can be put together as components. Thus God, soul, body, point, and heat compose an aggregate of these five things".[3]

Leibniz defines the parthood relation as a restriction of *inesse* (or *containment*) via the homogeneity relation:

> (Parthood Relation) x is a part of y if and only if y *contains* x and x and y are *homogeneous*. In formulas: $Pxy \equiv Cyx \wedge Hxy,$ where Pxy must be read as "x is a part of y" and Hxy must be read as "x is homogenous to y".

Since both the containment and the homogeneity relations are reflexive (everything contains itself and is homogenous to itself), so is the parthood relation: everything is a part of itself. Even though it is Leibniz himself who introduces the relation of 'part' in this precise way, when he speaks of 'part' he usually has in mind the notion of *proper part*, i.e. a notion that does not contemplate the case in which something

1 On Real Addition see, for instance, Chris Swoyer: "Leibniz's calculus of real addition", *Studia Leibnitiana* 26(1) (1994), pp. 1–30; Wolfgang Lenzen: "Guilielmi Pacidii Non plus ultra, oder: Eine Rekonstruktion des Leibnizschen Plus-Minus-Kalküls", *History of Philosophy and Logical Analysis* 3(1) (2000), pp. 71–118; and Massimo Mugnai: "Leibniz's mereology in the essays on logical calculus of 1686–1690", in V. De Risi (ed.): *Leibniz and the Structure of Sciences* (Cham: Springer, 2019), pp. 47–69.

2 A VI, 4A, 832.

3 A VI 4A 842.

is part of itself. In other words, a proper part (here symbolized as PP) is a part of a whole that is different from the whole $PPxy \equiv Pxy \wedge \neg(x = y)$.[4]

In this paper the word ‘mereology’ will refer to this stricter theory, and not to the Real Addition calculus. To understand the role that mereology plays within Leibniz’s mathematics and philosophy, it is enough to look at the notion of homogeneity. Homogeneity indicates quantitative comparability: if x and y are homogenous, then there is a ratio between them, i.e. they are quantitatively comparable.[5] Mereology thus works as a backbone theory for Leibniz’s theory of quantity. In different places, Leibniz characterizes quantity in the following way:

> We can define the quantity of a thing as the property of the whole insofar as it has all its parts.[6]

Such a characterization claims that ‘having a quantity’ implies ‘being a whole with parts’. This is presented as a general principle, which allows Leibniz to apply mereology in the estimation of quantities. The converse is also valid: ‘being a whole’ implies ‘having a quantity’. To see this, just suppose that W is a whole with parts. By definition of parthood, each of its parts are homogenous to W, which means that there is a ratio between W and each of its parts. So W has quantity.[7] We can thus summarize this principle in the following biconditional that I shall call Part-Quantity:

> (Part-Quantity) x has quantity if and only if x is a whole with parts

My aim in this paper is to focus on two very important features of Leibniz’s notion of part:

1. Leibniz’s definition of part implies a gunky conception of (proper) part (i.e. for any x and y, there is a z, such that if y is a proper part of x, then z is a proper part of y), which makes (proper) parthood a non-well-founded relation;

2. Given Leibniz’s rejection of infinite and/or maximal wholes, his mereology is junky too, i.e. for any whole (any object with parts) there is a strictly bigger whole.

4 In some writings probably dating back to 1700 (see Gottfried W. Leibniz: *Mathesis Universalis, écrits sur la mathématique universelle*, edited by David Rabouin (Paris: Mathesis Vrin, 2018), p. 164), Leibniz even introduced a further relation – the *intra* relation – which differs from the *inesse*/containment relation insofar as it is irreflexive: nothing is ‘intra’ itself. Then he proceeded to define x as a part of y if and only if x is *intra* y and is homogeneous to it. Clearly, this definition gives us the notion of proper part. From a logical point of view, the two ways of defining proper parthood are equivalent; from a historical point of view, the latter casts doubt on the fact that Leibniz has always considered mereology as a sub-part of the containment calculus. However, in *Initia rerum mathematicarum metaphysica* (GM VII 17–29) of 1714, Leibniz gives the standard definition of parthood based on the containment relation.

5 “There is ratio only between homogenous quantities, and this is clear by definition” (GM VII 34). “Among all the relations, the simplest is that which is called ratio or proportion, i.e. a relation between two homogenous quantities, which originates only from them, without assuming a third [quantity]” (GM VII 23 – 1714).

6 GM VII, 30/A VI 4a 418.

7 The characterization of quantity via the parthood relation is not new, but is already present in Aristotle’s *Metaphysics* 5, 13.

But before focusing on these two properties, it is worth accurately distinguishing the concept of *whole* from those of *aggregate* and *unity*.

2. Aggregates, Wholes and Unities

The word 'aggregate' is used by Leibniz to denote any multiplicity of objects whose unity is merely mental, i.e. the objects are unified by an act of thought or perception. A heap of stone or a flock of sheep are examples of aggregates.[8] Other examples are the body, and the created universe. Bodies are said to be multitudes in *The Principles of Grace* § 1 ("les composés ou les corps sont des Multitudes"), while in *Monadology* § 2, insofar as they are composite, they are described as clusters or aggregates of substances ("un amas ou *aggregatum* des simples"). The universe is conceived as the aggregate of all created substances or the aggregate of all bodies.[9] Even though Leibniz uses different terms (aggregate, cluster, multitude, etc.), in what follows I shall only use the word 'aggregate' to refer to this notion. Notice that under 'aggregates' Leibniz classifies both scattered things such as a heap of stones or a flock of sheep, and compounded things such as a body.

The Real Addition calculus is the calculus that expounds the logic of aggregates. Some of the examples of aggregates given above (the body, the universe, etc.) have infinitely many constituents, showing that the Real Addition operation can be applied to both finitely and infinitely many things.

Wholes are a particular kind of aggregates. They are aggregates where the constituents are homogenous to the aggregate itself. In this case, the constituents are called the parts. Since a whole is a specific kind of aggregate, every whole has parts; and vice versa, if something has part, then it is a whole.[10] This is explicitly claimed in *Specimen Geometriae Luciferae:*

> It is also clear that a part inheres in the whole, or, in other words, that as soon as a whole is given, a part is immediately given as well […].[11]

The part-whole relation is thus a unique relation: we cannot have wholes without parts, and vice versa. Insofar as wholes have parts, they can, at least ideally, be divided into their parts: indeed, the process of estimation of quantity just consists in the division of a magnitude into parts congruent to a given measure. This clearly shows that neither aggregates nor wholes are metaphysical unities (or substantial unities) since the latter are indivisible and simple, i.e. they do not have parts. Therefore, monads are not wholes or aggregates.

8 On the metaphysics of aggregates see Filippo Costantini: "Composition as Identity and the Logical Roots of Leibniz's Nominalism", *Global Philosophy*, 33(23) (2023). https://doi.org/10.1007/s10516-023-09665-3.

9 "The world is the aggregate of all bodies" (Corporum omnium aggregatum dicuntur Mundus): A VI 4 1509 – circa 1684–86.

10 This is true also for aggregates, simply in virtue of the logical definitions of such notions provided above.

11 GM VII, 274.

Leibniz is not always so careful when distinguishing aggregates from wholes. For example, in a famous passage dated by the Akademie to 1685, he writes:

> If when several things are posited, by that very fact some unity is immediately understood to be posited, then the former are called parts [*partes*], the latter a whole [*totum*]. Nor is it even necessary that they exist at the same time, or at the same place: it suffices that they be considered at the same time. This from all the Roman emperors together, we construct one aggregate [*unum aggregatum*].[12]

Here we see that he explicitly conflates the two notions. In other contexts, however, it is clear that this way of expressing is not very rigorous:

> A whole is that of which many things constituting it come together, which are said parts. In a more rigorous way, we take a whole to be that which parts are homogeneous.[13]

Even though both aggregates and wholes have some kind of unity, they are not unities (*unum per se*). Metaphysical unities, i.e. substances or monads, are indivisible and simple, and so they have no parts. In virtue of Part-Quantity, substances, insofar as they are immaterial entities, have no quantity. Therefore, the magnitude of a corporal substance is given by the magnitude of its own body.[14]

3. A Junky Mereology

Leibniz's famous argument against infinite number is an argument against infinite wholes. The notion of infinite whole contradicts the Part-Whole axiom, according to which the whole is always strictly bigger than each of its (proper) parts. If we subtract a finite part from an infinite whole (for example, from an infinite area under a curve we subtract a finite portion of it) we obtain as a remainder a part of the original whole equal to it. Therefore, no infinite whole exists. By Part-Quantity, this is equivalent to saying that there is no infinite quantity or magnitude: infinite lines, areas or volumes are simply *fictions* of the mind.

This fact imposes some constraints on the composition of objects into wholes.[15] Given some objects, it is not always possible to compose them into a whole. In particular, if infinitely many objects are given, we cannot compose them into a whole on pain of contradiction. A way to formalize this idea is to claim that only finitely many objects can be summed to compose a whole. Equivalently, it is enough

12 A VI 4, 627, LLC 271. The English translation comes from Gottfried W. Leibniz: *The Labyrinth of the Continuum. Writings on the Continuum Problem, 1672–1686*. Translated and edited by Richard T.W. Arthur (New Haven: Yale University Press, 2001), p. 271. References to it will be abbreviated as LLC.

13 C 476 – circa 1700–02.

14 I am using quantity and magnitude as synonyms.

15 Notice that this argument cannot be used to argue that the Real Addition operation must also be restricted to the case in which we sum together only finitely many objects. The reason is that this argument requires the principle that the whole is always bigger than each of its parts, and so it requires the parthood relation. But parthood is only a restriction of *inesse*, meaning that the argument is ineffective for the general Real Addition calculus.

to claim that any pair of (homogeneous) things composes a whole. I shall call this principle the Finite Composition Principle (FCP):

(Finite Composition Principle) $\forall x \forall y \exists z (Sxy = z)$

where 'Sxy' will denote the notion of mereological sum (or fusion) of x and y. The resulting mereology is therefore strictly weaker than Classical Extensional Mereology;[16] in particular, not only are there no infinite wholes, but also there is no whole comprising everything, i.e. the universe is not a whole:

> Yet M. Descartes and his followers, in making the world out to be indefinite so that we cannot conceive of any end to it, have said that matter has no limits. They have some reason for replacing the term 'infinite' with 'indefinite', for there is never an infinite whole in the world, though there are always wholes greater than others *ad infinitum*. As I have shown elsewhere, the universe itself cannot be considered to be a whole.[17]

In other words, given a whole, we can always find a bigger whole, i.e. Leibniz has a junky conception of wholes and parts.

This passage requires some further explanation. The universe is infinite, in the sense that it is constituted by infinitely many substances. Therefore, it cannot be conceived as a whole in which each substance is a part, otherwise it would be an infinite whole. Since no infinite whole can exist on pain of contradicting the Part-Whole Principle, the universe is merely an infinite *aggregate*, not a whole. As an infinite aggregate, the universe constitutes an *actual* infinite: there actually are infinitely many substances. The actuality here is just a consequence of the fact that the creation of the world is the creation of *all* substances together.[18]

The actual infinity of the universe is what guarantees that "there are always wholes greater than other *ad infinitum*". The idea is that the Finite Composition Principle allows us to add together more and more things into bigger and bigger wholes. What does "bigger" mean here? In many places, Leibniz defines "smaller than" (and symmetrically "bigger than") in the following way:

(Smaller than) A is smaller than B if and only if A is equal to a proper part of B.

Suppose that A and B are two different things, possibly overlapping, but such that one is not contained in the other.[19] Let us compose them via FCP into a whole C. Then A and B will both be proper parts of C: in virtue of the definition of "smaller than", since each of them are proper parts of C, they are both smaller than C. If we now compose C with a further thing D, we will obtain a bigger whole E. It seems that in this way, by applying FCP repeatedly, we can get bigger and bigger wholes.

16 Classical Extensional Mereology (CEM) is the mereological system (first developed by Leśniewski) that has been endorsed by Goodman, Quine, and David Lewis. On CEM see Aaron J. Cotnoir and Achille Varzi: *Mereology* (Oxford: Oxford University Press, 2021).

17 *New Essays,* II, XIII, § 21; A VI 6, 150–1.

18 "Created things are actually infinite. For any body whatever is actually divided into several parts, since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. So bodies are actually infinite, i.e. more bodies can be found than there are unities in any given number" (A VI iv 1393/LLC 235).

19 Two things overlap if they share at least a part in common; on the contrary, if they have no part in common, they are disjoint.

However, this fact obtains only if we restrict composition to things that possibly overlap, but such that one is not part of the other. For example, suppose that you compose my left hand and my left arm. The result of the composition operation is just my left arm (since my hand is already part of it). The whole is not therefore bigger than each of the two components, being equal to one of them. Or again, suppose you compose my left hand with itself: the result will simply be my left hand (mereological composition is idempotent).[20] Again, the resulting whole is not bigger than each of the starting components. In general, we do not have any guarantee that the application of FCP gives us bigger and bigger wholes.

When dealing with measuring objects by the repetition of a certain fixed object assumed as a measure, Leibniz explicitly interprets the repetition of the measure as the mereological sum of *disjoint* copies of the measure. If I want to know how many times a line-segment assumed as a measure is contained in a longer line-segment, I need to consider disjoint copies of the measure that together comprise the line-segment whose length I am trying to establish. This may suggest that a similar move should be made in the present context. If we restricted FCP to disjoint objects, then each application of it would produce a whole bigger than the parts that have been composed. However, this strategy has at least two problems. First, it is too drastic since it excludes the case in which A and B overlap but neither A nor B is a part of the other: in this case, the application of FCP to A and B gives us a whole bigger than each of A and B. Second, this seems a quite *ad hoc* move, which should be avoided if a better option is available. And it is here that the infinity of the universe comes into play. Because if we have infinitely many things at our disposal, then given a certain thing A, there is always a distinct thing B (not contained in A) such that the composition of A and B results in a whole bigger than each of them. In other words, it is the fact that the universe is infinite that guarantees that the application of FCP can always result in bigger and bigger wholes. The picture is thus as follows: the universe is an actual infinity of substances, but not a whole. There are only finite wholes, and given an arbitrary whole, there is always a bigger whole that can be obtained via FCP. From an algebraic point of view, this means that Leibniz's mereology has no top element, i.e. there is no element such that everything is part of it.

A possible objection regards the fact that, in the quotation above, Leibniz seems to say that the world itself is indefinite rather than infinite, contrary to what I have just argued. However, as shown by a more attentive reading, Leibniz does not claim that the world is indefinite, but rather simply that there are some good reasons that justify Descartes (and his followers) in claiming that the world is indefinite. These reasons are linked to the fact that the world is not a whole. By looking at the part-whole relation, we must conclude that there is no biggest whole; rather, the world is junky, i.e. there is an indefinite sequence of bigger and bigger wholes. But, contrary to Descartes, Leibniz can also appeal to the more general *inesse* relation, and the cognate notion of aggregate. The world can thus be an actual infinite aggregate, despite not being a whole. In other words, the distinction between inesse/aggregate

20 Mereological composition or sum is idempotent, meaning that for all As, $A + A = A$.

and part/whole allows Leibniz to accept the reasons that give us an indefinite sequence of bigger and bigger wholes, and at the same time to view the universe as constituting an actually infinite aggregate (and not merely something indefinite).

Before moving to the next topic, it is worth clarifying that FCP is compatible with a finite whole being decomposed into infinitely many parts. In other words, FCP is fully compatible with a body being divided into infinitely many parts. What we need to guarantee is that there is always at least one finite decomposition[21] of the whole. If there is such a decomposition, then the whole can be obtained by applying FCP starting from it. Therefore, we need to show that a finite decomposition can always be found. Here we must distinguish the case of physical bodies from that of geometrical figures. Concerning the former, even though bodies are divided into infinitely many parts, we can always find a finite number of parts that taken together equal the whole. This is guaranteed by the structure of these parts. Bodies are divided by their internal motions, that is, different parts are individuated by different motions. But each part is a body on its own, and so it is further divided into smaller sub-parts by some smaller motions. These sub-parts are also bodies which are divided into even smaller parts by some further motions. The division thus happens in different levels, with smaller and smaller parts. Since there is no last level of division, we obtain that there are infinitely many parts. In other words, it is the lack of a last level of division that allows Leibniz to conclude that the parts of each body are infinitely many. This means that there is no reason to claim that at any level we have infinitely many parts. Indeed, the picture strongly suggests that at each level we have only finitely many parts. But this exactly gives us a finite decomposition of the whole that was what we were looking for. Contrary to physical bodies, geometrical figures are not already divided into parts, but they are given as wholes that *can be* divided in different ways. In this case it is always possible to divide a geometrical figure into a finite number of parts. For example, I can bisect a line, a square or any other figure. The conclusion is that, given any whole, no matter whether it presents decompositions into infinitely many parts, there is always a decomposition into finitely many parts that allows the application of FCP.

4. A Gunky Notion of Part

The notion of homogeneity has an important consequence for the notion of part: if x and y are homogeneous, and x has proper parts, so does y. The principle tells us that the (proper) parthood relation is hereditary:

Part-Hereditary: $\forall x \forall y (H(x, y) \land \exists w (PPwx \rightarrow \exists z PPzy))$.

Why is this principle true? To answer this question, and so to justify Part-Hereditary, we must look deeper at the homogeneity relation.

> Two entities are homogeneous to which two other entities can be assigned which are equal to them and similar to each other. Given A and B; if L is taken equal to A, and M equal to B, and L and M are similar, we call A and B homogeneous. Hence I usually say also that homogeneous

21 A finite decomposition is a decomposition into *finitely* many parts.

> entities are those which can be made similar to each other by means of transformation, like curves and straight lines. That is if A is transformed into its equal L, it can be made similar to B, or to its equal M into which B is assumed to have been transformed.[22]

In short, we have the following definition:

> Homogeneity: A and B are homogenous if and only if either A and B are similar or they can be transformed into C and D respectively, such that C and D are similar.

Similar things share all their qualitative properties, like two line-segments, two squares or two balls. Therefore, "similar are those things that can be discerned only by their magnitude".[23] This clearly implies that we can always determine the ratio between two similar things: for example, by dividing one into parts congruent to the other.[24] But a curve C and a straight line L are not similar, and so they cannot be directly compared. What we need to do is to transform the curve C into a straight line C' and compare this straight line C' with L. Clearly, this works only if the transformation preserves equality, in our example the transformation must preserve the length.[25] Equality means identity of quantity: two line-segments are equal if they have the same length; two squares are equal if they have the same areas, etc. Once we have transformed C into C', we can compare C' and L since they are similar, and we can thus establish the ratio between them. Then, we can substitute C' with C *salva quantitate* (preserving quantity) since C and C' are equal. In this way we are able to determine the ratio between the curve C and the straight line L.

Now that the notion of homogeneity is on the table, we can return to Part-Hereditary. Let us start by considering the special case in which two homogeneous objects – say a and b – are homogeneous in virtue of the fact that they are similar. For simplicity, we can imagine that they are two line-segments. Having (proper) parts is a qualitative property, not a quantitative one (compare 'having parts' with 'having n-parts', i.e. a determined number of parts, which is a quantitative property). Then it directly follows that if one between a and b has parts, so the other. This is because similar things share all qualitative properties, and having parts is a qualitative property. Therefore, if x is part of y, and x and y are homogenous in virtue of them being similar, then x has parts as well. This establishes Part-Hereditary in the case in which the two objects are similar.

Let us now consider the most general case, where a and b are homogeneous (but not similar), a is transformed into v, b is transformed into w, and v and w are similar. We want to show that if a has parts, so does b. To do that we shall exploit Part-Quantity, according to which 'having quantity' is equivalent to 'being a whole with parts'. With this idea, we can justify Part-Hereditary with the general case of homogeneity as follows: suppose that a has parts. Since a is transformed into v, a and v are equals, i.e. they have the same quantity. In particular, v has quantity, and so, by Part-Quantity, it is a whole with parts. v is similar to w, which also implies

22 GM VII 19. The English translation comes from L 667.

23 A VI 4a 418.

24 Leibniz's theory of measure consists in a generalization of Euclid's algorithm known also as antyphairesis. Such a theory directly applies to similar things and can be extended to non-similar things via the definition of homogeneity.

25 In this case, the transformation is simply a rectification of the curve.

that w is a whole with parts, as argued in the previous case. But w is equal to b, so they have the same quantity, which implies that b has a quantity, and so it is a whole with parts. This establishes the more general form of Part-Hereditary: if two homogenous objects are such that one has parts, so then does the other.

The fact that the notion of part is defined by means of homogeneity implies that a part is always homogeneous to the whole of which it is a part. If the whole is a one-dimensional object (a line with a length), then each of its proper parts will be one-dimensional lines; if the whole is a material extended body, then each of its parts will be material extended bodies. Once Part-Hereditary has been established, it is easy to derive another interesting consequence that we shall label PP-Non-Well-Foundedness, namely the claim that if *something* is a (proper) part of a whole, then *it* is composed of further parts. Clearly this implies that the parts of the parts will be composed of further parts, and so on without end. The upshot is that with regard to the part-whole relation, we never reach a point where we find things without proper parts. In other words, Leibniz's mereology is a gunky one:

PP-Non-Well-Foundedness $\forall x \forall y, \exists w (PPyx \rightarrow PPwy)$

This principle amounts to saying that the PP-relation is not well-founded: given a part, we can always find an infinitely descending chain of sub-parts. One ought to notice that we have formulated the principle with regard to *PP* (proper parthood) and not to *P* (parthood): since the latter is reflexive, the existence of an infinitely descending chain is trivial (just consider the chain starting from a whose links are always a: a is a part of a, which is a part of a, and so on).[26]

Such a principle derives straightforwardly from Part-Hereditary, as shown by the following Natural deduction proof:

1	$PPba$	Assumption
2	$\forall x \forall y (H(x,y) \wedge \exists w (PPwx) \rightarrow \exists z (PPzy))$	Part-Hereditary
3	$Hab \wedge \exists w PPwa \rightarrow \exists z PPzb$	2, ∀-Elim x2
4	$Cab \wedge Hab \wedge \neg(a = b)$	1, def. of $PPba$
5	$\exists w PPwa$	1, ∃-Intr.
6	Hab	4, ∧-Elim.
7	$Hab \wedge \exists w PPwa$	5, 6, ∧-Intr.
8	$\exists z PPzb$	3, 7, →-Elim.
9	$PPba \rightarrow \exists z PPzb$	1, 8, →-Intr.
10	$\forall x \forall y (PPyx \rightarrow \exists z PPzy)$	9, ∀-Intr. x2

Natural Deduction derivation of PP-Non-Well-Founded (line 10)

26 We do not agree with Heinrich Burkhardt and Wolfgang Degen: "Mereology in Leibniz's Logic and Philosophy", *Topoi* 9(1) (1990), pp. 3–13, when they impose a principle of well-foundedness on the PP-relation to account for the idea that monads are simple entities in composite aggregates. Their mistake is not technical, but rather philosophical: they misinterpret the relation between monads and composite as a case of part-whole relation, while it is clear that monads are contained in bodies but are not a part of them.

It is important to stress that Leibniz explicitly recognizes this feature of the notion of part. In particular, focusing on the case of *similar objects*, he writes that “because in the line [recta] the part is similar to the whole, it is evident that in each part there will again be another part, and so given any line we can take a smaller line, since the part is smaller and it is still a line” (GM V 206, my translation). This is a property nowadays known as self-similarity, i.e. a property that obtains when any part of a whole is similar to the whole. But, as shown by the above derivation, this principle is valid for the more general case where the part is homogenous (but not similar) to the whole. Leibniz’s mereology is thus a gunky one, which means that, form an algebraic point of view, there is no bottom element, i.e. there is no element which is part of all other elements.

5. Why Junky and Gunky?

Part-quantity establishes a direct link between mereology and *Mathesis Universalis* conceived as the theory of (the estimation of) quantity.[27] Leibniz’s proper mereology (and not the Real Addition calculus) is therefore a theory at the basis of the notion of quantity and mathematics more generally. With this in mind, it should not be surprising that the parthood relation turns out to be both junky and gunky. Indeed, if objects endorsed with a quantitative aspect are wholes with parts, and “given a magnitude, it is possible to find a bigger or a smaller magnitude” (GM V 206), then the same should be true for parts and wholes. Indeed, the fact that the parthood relation is junky and gunky grounds the fact that, given a magnitude or quantity, there are always bigger and smaller magnitudes:

> Let us take a straight line, and extend it to double its original length. It is clear that the second line, being perfectly similar to the first, can be doubled in its turn to yield a third line which is also similar to the preceding ones; and since the same principle is always applicable, it is impossible that we should ever be brought to a halt; and so the line can be lengthened to infinity.[28]

Finally, notice that the claim that the parthood relation is gunky should not be conflated with the claim that a body is divided into infinitely many parts, since each part of a body is a body in its own right with further parts. The latter claim only regards bodies and depends on the general conception of body that Leibniz inherited from Descartes. This view is compatible with the idea that there are other cases of the parthood relation where the relation behaves differently (for example, admitting atoms, i.e. parts that do not have further parts). The claim that parthood is gunky does not regard only the body, but generalizes to all objects with parts. It is a claim directly derived from Leibniz’s definition of the part-whole relation.

27 On *mathesis universalisis* in Leibniz, see Gottfried W. Leibniz: *Mathesis Universalis, écrits sur la mathématique universelle*, edited by David Rabouin (Paris: Mathesis Vrin, 2018).

28 *New Essays,* II, XVII, § 3; A VI,6, 158.

6. What about Atoms?

The PP-Non-Well-Foundedness principle that we have derived above is a conditional statement to the effect that if something has proper parts, then its proper parts have proper parts as well. As such, it is a weaker principle than what in standard contemporary mereology is usually considered the axiom that states that there are no atoms. This is the following:

Atomless $\forall x \exists z PPzx$

This states that everything has proper parts, and so it is incompatible with the claim that there are atoms, i.e. things with no proper parts. The notion of atom can be defined as follows (where the subscript 'p' denotes that we are defining atoms with regard to the (proper) parthood relation):

Definition of Atom $A_p(x) \equiv \neg\exists z PPzx$

Contra *Atomless*, PP-Non-Well-Foundedness does not prohibit the existence of atoms. In fact, if we suppose that there are atoms, and we instantiate the variable x in PP-Non-Well-Foundedness with an atom a, then the antecedent becomes false (it would claim that the atom a has some proper parts), and therefore the whole conditional becomes true. This shows that Leibniz's mereology (and in particular PP-Non-Well-Foundedness) is compatible with the existence of atoms. Moreover, we can also show that atoms cannot be part of composite objects. Let us suppose an atom a to be a proper part of something. By PP-Non-Well-Foundedness, a must have (proper) parts as well, which contradicts the definition of atoms. Therefore, no atom is a proper part of a composite object. This latter result depends on PP-Non-Well-Foundedness, which is a direct consequence of PP-Hereditary, which in turn depends on the notion of homogeneity. As such, an atom cannot be a proper part of a composite object, because it is not homogeneous to it. As a consequence, mereology does not apply to them, in the precise sense that any model of Leibniz's mereology will not have any atom.

These two features of atoms (their lacking proper parts and the fact that they cannot be part of wholes) exactly correspond to the two main features of substances in Leibniz's metaphysics. A substance (or monad) is a simple entity in the precise sense that it does not have any (proper) part, and it is not a part of a composite object, like a corporeal body, since it is not homogeneous to it. The relation between substances and corporeal bodies requires us to resume the Real Addition calculus: substances are contained in bodies but are not part of them. And there is nothing that prevents us from saying that mereological atoms can be *contained in* (*inesse*) composite objects without being part of them.

Finally, it is notable that in the context of the Real Addition calculus, we find a definition of 'atom' that we may formalize as follows: $A_c(x) \equiv \neg\exists z Cxz$ (there is nothing that x contains; $A_c(x)$ indicates the property of being an atom with regard to the containment, and not the parthood relation). Examples of atoms with regard to the containment relation are points in space and instants in time.[29] Clearly, $A_c(x)$ implies $A_p(x)$, but the inverse is not true: something might be simple with regard to the parthood relation (so it will not have parts and will not be divisible), but there might be something contained in it.

29 Here is the text: "I call atom [*ultimum inesistens*] what is in [something] so that nothing is in it, or if L is an atom, and we assume that A+B=L, then A=B=L. For example, the point is in space, the instant in time" (A VI 4a, 822). Thanks to Wolfgang Lenzen for bringing this passage to my attention.