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Research paper

Strategic priority-based course allocation

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ABSTRACT

We introduce the conditional acceptance mechanism for solving the course allocation problem under priorities. This mechanism implements the set of stable allocations in both Nash equilibrium and undominated Nash equilibrium when preferences and priorities are substitutable. We model a post-allocation adjustment mechanism using a repeated version of the conditional acceptance mechanism that mitigates the inefficiencies caused by deviating from equilibrium. Both mechanisms are straightforward to implement, simplify the elicitation of students' preferences, and share features with currently employed course allocation mechanisms.

1. Introduction

We address the course allocation problem, which involves assigning course schedules to students based on their preferences and course priorities (Sönmez and Ünver, 2010; Budish, 2011; Kojima, 2013). Inspired by existing allocation procedures, we aim to design natural mechanisms (Alcalde, 1996) that minimize the information required from students, mitigating the strategy space's complexity.

Allocating course schedules under priorities raises two crucial issues. First, the allocation must respect student preferences and course priorities. However, for multi-unit assignment problems, no stable and strategy-proof mechanism is available. The same holds for efficient and individually rational allocations (Sönmez, 1999). Second, eliciting student preferences regarding course schedules is challenging (Budish et al., 2017). The course allocation problem is a combinatorial assignment problem (see, for example, Budish, 2011) and requires students to express preferences over a large set of course schedules.

To overcome the impossibility of implementing stable allocations under dominant strategies and the challenge of eliciting student preferences, we relax the equilibrium concept and concentrate on Nash equilibrium (NE), and subgame perfect Nash equilibrium ($SPNE$) implementation. This approach tackles both problems: the resulting allocation is stable; therefore, there is no justified envy, and the students' strategic behavior alleviates the preference elicitation problem.

When eliminating justified envy is the designer's objective, a well-known tension between efficiency and stability (see Abdulkadiroğlu and Sönmez, 2003) arises. The interaction between preferences and priorities determines the conflict between efficiency and stability. The intensity of the conflict depends on the correlation between preferences and priorities (Che and Tercieux, 2019). Also, the priority formation process affects the importance agents attribute to the need to respect priorities (König et al., 2023). Eliminating the tension between stability and efficiency for all preference profiles imposes an essentially homogeneous priority structure (Kojima, 2013) and a serial dictatorship as an implementing mechanism.

We study three revelation mechanisms: the student-optimal stable (SO) mechanism (Gale and Shapley, 1962), the immediate acceptance (IA) mechanism (Abdulkadiroğlu and Sönmez, 2003), and the novel conditional acceptance (CA) mechanism. We show

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that these mechanisms implement the set of stable allocations in Nash equilibrium under substitutable preference in different priority domains. Our results extend the previous implementation findings by Alcalde (1996) and Ergin and Sönmez (2006) for the marriage and school admission problems, respectively.

More precisely, we prove that the SO mechanism implements the set of stable allocations in NE if and only if there is no conflict between efficiency and stability, which is if and only if the priority structure is essentially homogeneous.

Next, we prove that the IA mechanism implements the set of stable allocations in NE if preferences are substitutable and priorities are slot-specific.

Our main result is implementing the set of stable allocations in Nash equilibrium through the CA mechanism under substitutable preferences and priorities. Substitutable priorities accommodate requirements such as non-standard class sizes or an even number of students.

The CA mechanism involves students submitting ranked course schedules. In the first step, seats are allocated to the highest-priority students, who claim them as their first choice. Then, students who have been assigned no course go to a second step, where they claim the course schedule they have ranked second. The students assigned to courses cannot claim additional ones but can lose a tentatively allocated course if a higher-priority student claims it. Then, students who have been assigned no course go to a third step, and so on. Through iterative steps, the CA mechanism reassesses and adjusts allocations until all seats are filled by the higher-priority students or no more requests are made.

The CA mechanism is similar to the IA mechanism in that each student can receive courses in only one of the steps. It also resembles the SO mechanism in that courses are provisionally assigned and can be lost to higher-priority students.

Errors in course allocation can lead to significant costs and inefficiencies for students. Post-allocation adjustment mechanisms are commonly used to address this problem. However, the design of these mechanisms can impact the primary mechanism's strategic properties in specific scenarios. For example, in situations where the post-allocation mechanism is unstable or allows students to drop courses (see Examples 6 and 7).

To minimize the costs of errors while maintaining incentives, we propose using additional rounds of the CA mechanism to allocate any remaining seats. If students do not make errors, this procedure implements the set of stable allocations. Otherwise, it reduces their impact.[1] Our proposed mechanism, the extended conditional acceptance (ECA), implements the set of stable allocations in $SPNE$ when preferences and priorities are slot-specific.[2]

The CA mechanism uses students' information to alleviate the challenges of allocating courses. We find examples of course allocation procedures that use the same principle. For example, the course allocation process at the College of Arts and Sciences of the University of Pennsylvania (UPenn) has a course selection period that allows the students to gather the information necessary to produce an order list of courses. During this period, students can add and drop courses on Penn InTouch's app.[3] Students are encouraged to act strategically and prioritize their favorite courses during the course selection period.

Once the course selection period ends, the choices become final, and the courses are allocated. Following the allocation, a post-allocation adjustment round allows students to withdraw from courses and register for new ones. A similar mechanism is employed at Eötvös Loránd University in Hungary, as described in Rusznak et al. (2021).

In some situations, students can provide enough information for course allocation by expressing their preferences for individual courses rather than course schedules. For instance, when mandatory courses are offered in simultaneous sections or when referring to graduate seminars or elective courses with non-overlapping time slots, student preferences can be expressed using slot-specific preferences.

When student preferences are slot-specific, it is feasible to adapt the IA, CA, and ECA mechanisms to preferences for individual courses. We present a version of our mechanisms adapted to this simplified student strategy space that retains their properties.

For example, the Political Science Department at Aarhus University in Denmark uses a mechanism that resembles a simplified version of the ECA mechanism, with two rounds. Master's students submit a course list to enroll in their desired courses. If they are not allocated a seat in one or more of their desired courses, they must register again during the second registration period. Students registered for elective courses are bound by their choice in each round, as these courses are critical for their study program.[4]

1.1. Related literature

We study allocating course schedules to strategic students, respecting references and priorities as an implementation problem. Previously, Kara and Sönmez (1996) proves the implementability of the set of stable allocations in Nash equilibrium for one-to-one matching markets and Kara and Sönmez (1997) for many-to-one matching markets. Hatfield and Kominers (2017) extends this results to many-to-many matching with contracts. These papers apply results by Moore and Repullo (1990) (see also Maskin, 1999), whose mechanisms rely on integer games and large message spaces.

[1] The idea of allocating the remaining courses repeating the exact mechanism is not new. For example, it was proposed by Coles et al. (2010) for the National Resident Matching Program.

[2] Slot-Specific preferences and priorities include responsive preferences and priorities as particular cases.

[3] Source: https://www.college.upenn.edu/registration-tips, accessed on 04/29/2024.

[4] Source: https://studerende.au.dk/en/studies/subject-portals/political-science/teaching/registration-for-courses/registration-for-masters-courses, accessed on 04/29/2024.

Sotomayor (2004) and Echenique and Oviedo (2006) prove that natural mechanisms such as a take-it-or-leave-it offer mechanism implement the set of stable allocations in subgame perfect Nash equilibrium for many-to-many matching markets. Furthermore, Romero-Medina and Triossi (2023) extends these results to many-to-many matching markets with contracts, employing natural non-revelation mechanisms. Differently, the present paper analyzes revelation mechanisms based on features observed in real-world course allocation procedures.

Universities worldwide allocate course schedules to students at least once a year. Thus, the problem has received attention from the theoretical and practical perspectives. Traditional mechanisms, such as the first-come-first-served mechanism, are commonly used but face criticism for inducing a competitive race that can overload systems and lead to inequitable outcomes. Also, the mechanism does not respect students' priorities (Bichler and Merting, 2021; Aziz et al., 2019).

Another frequently employed method is the serial dictatorship (SD) mechanism, which, after setting a unique priority order based on specific criteria such as a random draw or the student's average grade (Pápai, 2002; Ehlers and Klaus, 2003), lets students choose course schedules in this order. The SD mechanism is group-strategy-proof and efficient. When priorities are involved, and students are ordered according to the unique priority order for all courses, the SD mechanism yields stable and efficient allocations.

Budish and Cantillon (2012) studies the course draft mechanism used at Harvard University, among others. The draft mechanism orders students randomly, enabling them to select their preferred courses in successive rounds. Students can choose only one course per round, and the selection order is reversed in each round. At Harvard, students behave strategically, and the outcome of the draft mechanism improves the overall welfare of the SD mechanism. However, the draft mechanism does not allow students to express preferences over course schedules, and courses have no priorities.

In some situations, allocation mechanisms rely on bidding systems where students allocate a certain amount of fake money to register for courses. In such cases, priorities may be given little importance and only come into play as tie-breakers.

In examining bidding mechanisms at business schools, Sönmez and Ünver (2010) introduces the Gale-Shapley Pareto-dominant market mechanism. This mechanism assigns course priorities to break ties based on students' bids and implements the SO mechanism. The outcome of the Gale-Shapley Pareto-dominant market mechanism can improve upon other bidding mechanisms.

Budish (2011) suggests using pseudo-markets to allocate course schedules without priorities and presents the approximate competitive equilibrium from equal incomes ($A-CEEI$) mechanism. The $A-CEEI$ mechanism is efficient and approximately strategy-proof in large markets. The $A-CEEI$ mechanism bounds absolute envy, indicating a weaker fairness concept. Kornbluth and Kushnir (2021) presents the Budget-Adjusted Pseudo-Market mechanism, which introduces priorities into the $A-CEEI$ mechanism. Both mechanisms consider weak forms of stability and efficiency. A weakly stable allocation can leave some students willing to fill empty seats. A weakly efficient allocation can be Pareto dominated by another allocation that fills more seats.

Implementing the $A-CEEI$ mechanism is complex and computationally demanding (Budish et al., 2017 and Budish and Kessler, 2022). These challenges tend to increase with the number of students and courses. Assuming that student utility functions are additive reduces the complexity of this issue. A version of the mechanism is used at Wharton School.

In the $A-CEEI$ mechanism, students must provide cardinal preferences over course schedules. Budish et al. (2017) develops a procedure to express preferences in a manner accessible to students. Experiments by Budish and Kessler (2022) observe that reporting cardinal preferences entails more errors than reporting ordinal ones.

Our methodology diverges from the $A-CEEI$ mechanism, particularly in handling priorities and preferences. The $A-CEEI$ mechanism depends on market mechanisms to resolve conflicts between demand and supply at the expense of simplifying preferences and priorities and allowing some instances of justified envy. The CA mechanism incorporates substitutable preferences and priorities explicitly addressing situations where course capacities and optimal class sizes may not align or special requirements for coursework exist. It also ensures that the allocation process closely aligns with institutional priorities and student needs. Moreover, the CA mechanism achieves stable allocations in equilibrium with minimal student input. In short, the CA mechanism addresses complex allocation scenarios beyond the capacity of mechanisms like SD, SO, IA, and $A-CEEI$.

The paper is organized as follows. Section 2 introduces the model and notation. Section 3 presents our results with the one-shot mechanism. Section 4 presents our proposal for post-allocation adjustment. Section 5 presents simplified versions of the mechanisms, and Section 6 concludes. The proofs are in the Appendix.

2. The model

There is a finite set of courses C and a finite set of students S, with $C \cap S = \emptyset$. Each course $c \in C$ has priorities, which is a linear order over the set of subsets of students, 2^S. The weak order associated with P_c is denoted by R_c. For each $S' \subseteq S$ and each $c \in C$, $Ch_c\left(S', P_c\right)$ is the choice set of course c from S'. Formally, $Ch_c\left(S', P_c\right) = \max_{P_c} 2^{S'}$. When there is no ambiguity about P_c, we write $Ch_c\left(S'\right)$ instead of $Ch_c\left(S', P_c\right)$. A non-empty set of students $S' \subseteq S$ is acceptable to c if $S' P_c \emptyset$; otherwise, it is unacceptable. We represent priorities as ordered lists of acceptable sets of students. For example, $P_c : \left\{s_1\right\}, \left\{s_1, s_2\right\}, \left\{s_2\right\}$ means that the course gives higher priority to having student s_1 enrolled than having both students $\left\{s_1, s_2\right\}$ enrolled, which, in turn, receives higher priority than enrolling student s_2 alone. All other subsets of students are unacceptable.

We say that P_c is substitutable if for all $S' \subseteq S$, $s, s' \in S \backslash S'$ and $s \in Ch_c\left(S' \cup \left\{s, s'\right\}\right)$, then $s \in Ch_s\left(S' \cup \{s\}\right)$. In other words, P_c is substitutable if, whenever course c selects a student from a given set of students, it selects her also from smaller subsets of students. Let $\mathcal{P}$ be the set of substitutable priorities on 2^S.

A particular class of substitutable priorities is the class of **slot-specific priorities** introduced by Kominers and Sönmez (2016). Under slot-specific priorities, each course $c \in C$ has a finite set of slots, Σ_c, with generic element σ. Each slot σ has a priority order $\succ_\sigma$, a strict, complete, and transitive binary relation over $S \cup \{\emptyset\}$, in which $\{\emptyset\}$ represents the possibility of maintaining the slot empty. The higher a student is ranked under $\succ_\sigma$, the stronger her claim for slot σ in course c. If $\emptyset \succ_\sigma s$, student s is not acceptable for slot σ. Student s is unacceptable to c if she is unacceptable to any of c's slots. Otherwise, she is acceptable to c. $A_c\left(P_c\right)$ denotes the set of acceptable students for c. The total **supply** of course c is $q_c = \left|\Sigma_c\right|$. Let us define q as the vector of supplies for the various courses, $q = \left(q_c\right)_{c\in C}$. We assume the slots are numbered according to a linear **order of precedence** $\rhd_c$. Given two slots $\sigma, \sigma' \in \Sigma_c$, $\sigma \rhd_c \sigma'$ means that slot σ is to be filled before the slot σ' whenever possible. For each course c, we assume that slots in Σ_c are ordered in such a way that $\sigma^1 \rhd_c \sigma^2 \rhd_c \cdots \rhd_c \sigma^{q_c}$. Let $S' \subseteq S$. The choice of school c from S', denoted by $Ch_c\left(S'\right)$, is obtained as follows: slots at school c are filled one at a time following the order of precedence. The highest-priority acceptable student in S' under $\succ_{\sigma^1}$, for example, student s^1, is chosen for slot σ^1 of school c; the highest-priority acceptable student in $S' \backslash \left\{s^1\right\}$ under $\succ_{\sigma^1}$, for example, student s^2, is chosen for slot σ^2 of school c, and so on. Formally, $Ch_c\left(S'\right) = \bigcup_{i=1,\ldots,q_c} s_i^*$, in which $s_1^* = \max_{\succ_{\sigma^1}} S'$, and $s_i^* = \max_{\succ_{\sigma^i}} S' \backslash \bigcup_{j=1}^{i-1}\left\{s_j^*\right\}$ for $i = 2, 3, \ldots, q_c$. The choice function Ch_c satisfies substitutability (Chambers and Yenmez, 2017) and the irrelevance of rejected students condition.[5] It follows that Ch_c is rationalizable (see Theorem 1 in Alva, 2018) by a substitutable priority P_c (which is $Ch_c\left(S'\right) = Ch_c\left(S', P_c\right)$ for all $S' \subseteq S$). A slot-specific priority is denoted by a tuple $\left(\Sigma_c, \left(\succ_\sigma\right)_{\sigma\in\Sigma_c}, \rhd_c\right)_{c\in C}$. Let $\mathcal{L}$ be the set of slot-specific priorities on 2^S.

Slot-specific priorities such that all slots have an identical priority order, i.e., $\succ_{\sigma^1} = \succ_{\sigma^2}, \ldots, = \succ_{\sigma^{q_c}}$ are called **responsive.**[6] Let $\mathcal{R}$ be the set of responsive priorities on 2^S.

Each student $s \in S$ has preferences, a linear order over the subsets of curses, or course schedules, 2^C. The properties of the preferences are analogous to the properties of the priorities with identical notation. Let $P_S = \left(P_s\right)_{s\in S}$ be a preference profile. Analogously to the case of priorities and abusing notation, we denote by $\mathcal{P}$ the set of substitutable preferences on 2^C, by $\mathcal{L}$ be the set of slot-specific preferences on 2^C, and by $\mathcal{R}$ be the set of responsive preferences on 2^C. Let $s \in S$ and let r be an integer such that $1 \le r \le 2^{|C|}$, and let $C_{P_s}^r$ be the rth ranked acceptable course schedule according to P_s, if one exists. Let $C_{P_s}^r$ be empty otherwise. Formally, $C_{P_s}^1 = \max_{P_s} 2^C$ and, for all r, $1 \le r \le 2^{|S|} - 1$, $C_{P_s}^{r+1} = \max_{P_s}\left(2^C \backslash \bigcup_{i\le r}\left\{C_{P_s}^i\right\}\right) \cup \{\emptyset\}$.[7]

For each $S' \subseteq S$, set $P_{S'} = \left(P_s\right)_{s\in S'}$. For each $s \in S$, set $P_{-s} = P_{S\backslash\{s\}}$. Given a preference relation P on 2^C, the restriction of P to $C' \subseteq C$, denoted by $P_{|C'}$, is a preference that ranks all subsets in $2^{C'}$ as P does and ranks all other subsets of courses as not acceptable. Formally, $P_{|C'}$ is such that, for all $Q, T \subseteq C'$, $QP_{|C'}T$ if and only if QPT and for all $Q \nsubseteq C'$, $\emptyset P_{|C'} Q$.

An **allocation** is a function $\mu : C \cup S \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all allocations is denoted by $\mathcal{M}$. Allocation μ is **individually rational** for $x \in C \cup S$ if $Ch(\mu(x)) = \mu(x)$. Allocation μ is **blocked** by a pair $(c, s) \in C \times S$ if $s \notin \mu(c)$, $c \in Ch_s(\mu(s) \cup \{c\})$, and $s \in Ch_c(\mu(c) \cup \{s\})$. Finally, an allocation μ is **stable** for $\left(S, C, P_S, P_C\right)$ if it is individually rational for all $x \in C \cup S$ and no pair is blocking it. If P_S and P_C are substitutable, then a stable allocation exists (Echenique and Oviedo, 2006). The set of stable allocations is denoted by $\mathcal{S}\left(P_S\right)$, if there is no ambiguity about P_C.

Let $\mathcal{D} \subseteq \mathcal{P}$ be a set of preferences on 2^C. A (revelation) **mechanism** is a function φ that associates an allocation to every preference profile for students, $P_S = \left(P_s\right)_{s\in S} \in \mathcal{D}^{|S|}$, $\varphi : \mathcal{D}^{|S|} \to \mathcal{M}$. A mechanism is **stable** if $\varphi\left(P_S\right)$ is a stable allocation for each P_S. A mechanism is **strategy-proof** if $\varphi\left(P_S\right) R_s \varphi\left(P_s', P_{-s}\right)$ for each $P_S \in \mathcal{D}^{|S|}$, $s \in S$, and $P_s' \in \mathcal{D}$, in which R_s denotes the weak preferences associated to P_s. Given a priority profile P_C and a preference profile $P_S \in \mathcal{D}^{|S|}$, a mechanism φ induces a normal form game $\mathcal{G}\left(P_S\right) = \left(S, \mathcal{D}^{|S|}, \varphi, P_S\right)$, in which S is the set of players, $\mathcal{D}^{|S|}$ is the Cartesian product of students' strategy spaces, φ is the outcome function and P is the profile of student preferences. Let $\Phi : \mathcal{D}^{|S|} \rightrightarrows \mathcal{M}$ be a correspondence. We say that φ **implements** Φ in **Nash equilibrium** if, for each $P_S \in \mathcal{D}^{|S|}$, the set of Nash equilibria of $\mathcal{G}\left(P_S\right) = \left(S, \mathcal{D}^{|S|}, \varphi, P_S\right)$ coincides with $\Phi\left(P_S\right)$. We say that φ **implements Φ** in **undominated Nash equilibrium** (UNE) if, for each $P_S \in \mathcal{D}^{|S|}$, the set of undominated NE outcomes of $\mathcal{G}\left(P_S\right) = \left(S, \mathcal{D}^{|S|}, \varphi, P_S\right)$ coincides with $\Phi\left(P_S\right)$.

An **extensive form mechanism** is $\Gamma = (H, M, g)$ where H is a finite set of histories, $M = \prod_{s\in S} M_s$ and $M_s = \prod_{h\in H} M_s^h$ for all $s \in S$.[8] An element of $M^h = \prod_{s\in S} M_s^h$ is a message vector at h and $m_s^h \in M_s^h$ is student s's message at h. Histories and messages are connected by the following property $M^h = \left\{m^h \mid \left(h, m^h\right) \in H\right\}$. There is an initial history $\emptyset \in H$ and each history $h^k \in H$ is represented by a finite sequence $\left(\emptyset, h^1, \ldots, h^{k-1}\right)$. If $h^{k+1} = \left(h^k, s\right)$, then history h^{k+1} proceeds history h^k. Since H is finite, there is a non-empty set of terminal histories $\overline{H} \subseteq H$ such that $\overline{H} = \left\{h \in H \mid \text{there is } h' \in H \text{ proceeding } h\right\}$. An element of M_s, m_s is student s's pure strategy, which specifies s's choices at each non-terminal history. Any strategy profile defines a unique terminal history, given the initial history. Sometimes, we call a terminal history a path. The outcome function $g : M \to \mathcal{M}$ specifies an outcome for each terminal history and, thus, for each strategy profile. Given a profile of preferences P_S, $\left(\Gamma, P_S\right)$ is an extensive form game with simultaneous moves.

We say that an extensive form mechanism **implements Φ** in **subgame perfect Nash equilibrium** if, for each $P_S \in \mathcal{D}^{|S|}$, the set of $SPNE$ outcomes of $\left(\Gamma, P_S\right)$ coincides with $\Phi\left(P_S\right)$.

5 Choice function Ch_c satisfies the irrelevance of rejected students condition if, for all $S' \subseteq S$ and all $s \in S \backslash S'$, $s \notin Ch_c\left(S' \cup \{s\}\right) \Rightarrow Ch_c\left(S' \cup \{s\}\right) = Ch_c\left(S'\right)$. This condition has been previously analyzed by Aygün and Sönmez (2013) in matching with contracts and called irrelevance of rejected contracts condition. It is a necessary condition for rationalizing a choice function.

6 Responsive preferences are a common assumption in course allocation (Budish and Cantillon, 2012; Kojima, 2013; Kojima and Ünver, 2014; Doğan and Klaus, 2018). Responsive priorities are a common assumption in course allocation and school choice (Kojima, 2013 and Abdulkadiroğlu and Sönmez, 2003 among others).

7 In the definition of $C_{P_s}^{r+1}$, the empty set prevents the selection of not acceptable course schedules.

8 We adapt the definition in Vartiainen (2007).

3. One-shot mechanisms

This section characterizes preference and priority domains where the SO and IA mechanisms implement stable allocations in the course allocation problem. Additionally, we introduce the CA mechanism, which extends the range of priority domains within which we implement the set of stable allocations.

3.1. The student-optimal stable mechanism

The SO mechanism is widely used for many-to-one assignment problems (Abdulkadiroğlu and Andersson, 2023; Roth and Peranson, 1999). However, NE outcomes of the SO mechanism can be unstable (Roth and Sotomayor, 1990; Haeringer and Klijn, 2009).

We present a version of the SO algorithm adapted from Kojima (2013). Given a priority profile $\left(P_c\right)_{c\in C}$ and a preference profile $\left(P_s\right)_{s\in S}$, the following procedure describes the SO **mechanism**.[9]

Step 1: For every $c\in C$ let $S_c^1=\{s\in S\mid c\in C_{P_s}^1\}$. Set $\mu^1(c)=Ch_c\left(S_c^1\right)$. For every $s\in S$ let $\mu^1(s)=\{c\in C\mid s\in\mu^1(c)\}$. Let $H^1=\{s\in S\mid C_{P_s}^1=\emptyset\}$.

Step r+1 (for $1\le r$): Let $T^{r+1}=S\backslash H^r$. For every $s\in T^{r+1}$ set $\widetilde{C}_s^{r+1}=\max_{P_s}\{C'\subseteq C\mid\mu^r(s)\subseteq C'\}$. For every $c\in C$ let $S_c^{r+1}=\{s\in S\mid c\in\widetilde{C}_s^{r+1}\}$. Set $\mu^{r+1}(c)=Ch_c\left(S_c^{r+1}\right)$. For every $s\in S$ let $\mu^{r+1}(s)=\{c\in C\mid s\in\mu^{r+1}(c)\}$. Let $H^{r+1}=\{s\in S\mid\mu^{r+1}(s)\,P_sC'\Rightarrow\emptyset P_sC'\}$.

Given substitutable priorities, the SO mechanism assigns to each substitutable profile of preferences $P_S=\left(P_s\right)_{s\in S}$ the stable allocation μ that is optimal for all students. Specifically, μ satisfies the condition $Ch_s(\mu(s)\cup\nu(s))=\mu(s)$ for all $s\in S$ and all stable allocations ν (Blair, 1988). We denote the student-optimal stable allocation under preferences P as $SO(P)$. However, in the case of multi-unit assignments, the SO mechanism is not strategy-proof. It may result in unstable allocations as Nash equilibrium outcomes.

Example 1. Let $S=\{s_1,s_2\}$ and $C=\{c_1,c_2\}$. Let preferences and priorities be as follows:

$P_{s_1}:\{c_1,c_2\},\{c_2\},\{c_1\};$ $\quad$ $P_{c_1}:\{s_1\},\{s_2\};$

$P_{s_2}:\{c_1\},\{c_2\};$ $\quad$ $P_{c_2}:\{s_2\},\{s_1\}.$

There exists a unique stable allocation is μ in which $\mu\left(s_1\right)=\left\{c_1\right\}$, $\mu\left(s_2\right)=\left\{c_2\right\}$.

Let $P'_{s_1}:\left\{c_2\right\},\left\{c_1\right\}$; $P'_{s_2}:\left\{c_1\right\},\left\{c_2\right\}$. Strategy profile $P'=\left(P'_{s_i}\right)_{i=1,2}$ is a NE of the game induced by the student-optimal stable allocation yielding allocation ν, in which $\nu\left(s_1\right)=\left\{c_2\right\}$, $\nu\left(s_2\right)=\left\{c_1\right\}$, which is unstable since it is blocked by $\left(c_1,s_1\right)$.

In Example 1, the NE outcome μ is Pareto optimal, and Pareto dominates the student-optimal stable allocation. This is not always the case, as shown in Example 2.

Example 2. Let $S=\{s_1,s_2,s_3,s_4\}$ and $C=\{c_1,c_2,c_3,c_4\}$. Let preferences and priorities be as follows:

$P_{s_1}:\{c_1,c_2\},\{c_2\},\{c_1\};$ $\quad$ $P_{c_1}:\{s_1\},\{s_2\};$

$P_{s_2}:\{c_1\},\{c_2\};$ $\quad$ $P_{c_2}:\{s_2\},\{s_1\};$

$P_{s_3}:\{c_4\},\{c_3\};$ $\quad$ $P_{c_3}:\{s_3\},\{s_4\};$

$P_{s_4}:\{c_3\},\{c_4\};$ $\quad$ $P_{c_4}:\{s_4\},\{s_3\}.$

There are two stable allocations, μ and ρ in which

$$\mu\left(s_1\right)=\left\{c_1\right\},\mu\left(s_2\right)=\left\{c_2\right\},\mu\left(s_3\right)=\left\{c_4\right\},\mu\left(s_4\right)=\left\{c_3\right\},$$

$$\rho\left(s_1\right)=\left\{c_1\right\},\rho\left(s_2\right)=\left\{c_2\right\},\rho\left(s_3\right)=\left\{c_3\right\},\rho\left(s_4\right)=\left\{c_4\right\}.$$

Let $P'_{s_1}:\left\{c_2\right\},\left\{c_1\right\}$; $P'_{s_2}:\left\{c_1\right\},\left\{c_2\right\}$; $P'_{s_3}:\left\{c_3\right\}$; $P'_{s_4}:\left\{c_4\right\}$. Strategy profile $P'=\left(P'_{s_i}\right)_{i=1,2,3}$ is a NE of the game induced by the student-optimal stable allocation yielding allocation ν, in which

$$\nu\left(s_1\right)=\left\{c_2\right\},\nu\left(s_2\right)=\left\{c_1\right\},\nu\left(s_3\right)=\left\{c_3\right\},\nu\left(s_4\right)=\left\{c_4\right\},$$

which is unstable since it is blocked by $\left(c_1,s_1\right)$. It is also not Pareto optimal since it is dominated by τ, in which

$$\tau\left(s_1\right)=\left\{c_2\right\},\ \tau\left(s_2\right)=\left\{c_1\right\},\tau\left(s_3\right)=\left\{c_3\right\},\tau\left(s_4\right)=\left\{c_4\right\}.$$

In addition, it does not Pareto dominate stable allocation μ.

[9] See Gale and Shapley (1962), for the original definition for the single-unit demand case, see Roth and Sotomayor (1990), for extensions to the multi-unit demand case. Alternative algorithms yielding the student-optimal stable allocation are presented, among others, in Echenique and Oviedo (2006) (the T algorithm) and Romero-Medina and Triossi (2023) (the HO and the SHO algorithms.).

Examples 1 and 2 exhibit a cycle $\{s_1\} P_{c_1} \{s_2\} P_{c_2} \{s_1\}$ that supports the NE outcome v. This cycle creates a situation where if student s_1 were to rank $\{c_1, c_2\}$ as her top choice, she would block the admission of student s_2 to course c_1. In turn, student s_2, having lost course c_1, would block the admission of student s_1 to course c_2. Consequently, such a deviation would not be profitable for s_1.

In the context of multi-unit assignment models, cycles involving the lowest-ranked students eligible for admission to two courses can arise. However, the occurrence of such cycles can be prevented if the priorities satisfy the condition of essential homogeneity (Kojima, 2013).

Definition 1. Priorities $(P_c)_{c\in C}$ satisfy essential homogeneity if there is no $c_1, c_2 \in C$ such that:

i. $\{s_1\} P_{c_1} \{s_2\} P_{c_1} \emptyset$ and $\{s_2\} P_{c_2} \{s_1\} P_{c_2} \emptyset$;
ii. there exist $S_{c_1}, S_{c_2} \subseteq S \setminus \{s_1, s_2\}$ such that $|S_{c_1}| = q_{c_1} - 1$, $|S_{c_2}| = q_{c_2} - 1$, $\{s\} P_{c_1} \{s_2\}$ for each $s \in S_{c_1}$, and $\{s\} P_{c_2} \{s_1\}$ for each $s \in S_{c_2}$.

Essential homogeneity allows variations in priority among the top q_c students for course c. These students are guaranteed admission to course c regardless of their application timing; hence, their relative ranking does impact the NE outcome.

All stable allocations are a NE outcome of the game induced by the SO mechanism. Indeed, it is easy to check that if μ is the student-optimal stable allocation for preferences $(P_s)_{s\in S}$, then $(P_{|\mu(s)})_{s\in S}$ is a NE of the game induced by the SO mechanism yielding μ as outcome. However, if priorities are not essentially homogeneous, the set of stable allocations is generally a strict subset of the set of NE outcomes.

Proposition 1. *Let preferences be substitutable. Let $(P_c)_{c\in C} \in \mathcal{R}$. The SO mechanism implements the set of stable allocations in NE if and only if $(P_c)_{c\in C}$ satisfies essentially homogeneity.*

In other words, if priorities are responsive and preferences are substitutable, the set of NE outcomes of the game induced by SO coincides with $S(P_S)$ for all $P_S \in \mathcal{P}^{|S|}$ if and only if $(P_c)_{c\in C}$ satisfies essentially homogeneity. The proof of Proposition 1 is based on the fact that under essential homogeneity, the student-optimal stable allocation results from a serial dictatorship (Kojima, 2013, Theorem 3). Kojima (2013) also shows that the strategy-proofness of the SO mechanism is equivalent to the essential homogeneity of course priorities and the existence of a stable and efficient mechanism. By combining Theorem 1 from Kojima (2013) with our Proposition 1, we can deduce that maintaining stability in NE outcomes of the SO mechanism is the same as enforcing strategy-proofness, or equivalently, efficiency for all profiles of substitutable preferences.

3.2. The immediate acceptance mechanism

We define a many-to-many version of the IA mechanism, which extends the many-to-one version used in school assignment problems (Abdulkadiroğlu and Sönmez, 2003) to the multi-unit case. We prove that the IA mechanism implements the set of stable allocations in Nash equilibrium under slot-specific priorities.

Slot-specific priorities model situations where certain groups of students are given priority for a portion of the seats, which are otherwise assigned based on a given criterion. This approach facilitates diversity in the classroom (Dur et al., 2018, 2020 for applications to school choice). Slot-specific priorities also encompass other approaches, such as majority quotas defined by Kojima (2012) and minority reserves introduced by Hafalir et al. (2013).

Let us introduce our many-to-many version of the IA mechanism. Initially, each student submits their preferences. In the first step, we consider each student's favorite acceptable set of courses. Within this initial step, among the students who choose a specific course, those with the highest priorities for that course are assigned to it. In the rth step, we only consider the rth choice in the preference list of the remaining students. At the end of each step, a student assigned at least one course is eliminated from further consideration. This iterative process continues until no students are left. The assignments made in each step are considered final.

Given a priority profile $(P_c)_{c\in C}$ and a preference profile $(P_s)_{s\in S}$, the following procedure describes the IA **mechanism**.

Step 1: For every $c \in C$ let $S_c^1 = \left\{ s \in S \mid c \in C_{P_s}^1 \right\}$. Set $\mu^1(c) = Ch_c\left(S_c^1\right)$.
Let $H^1 = \bigcup_{c\in C} \{\mu^1(c)\} \cup \left\{ s \in S \mid C_{P_s}^1 = \emptyset \right\}$.

Step r+1 (for $1 \le r$): Let $T^{r+1} = S \setminus H^r$. For every $c \in C$ let $S_c^{r+1} = \left\{ s \in T^{r+1} \mid c \in C_{P_s}^{r+1} \right\}$.
Set $\mu^{r+1}(c) = \max_{P_c} \{\mu^r(c) \cup S' \mid S' \subseteq S_c^{r+1}\}$.
Let $H^{r+1} = \bigcup_{c\in C} \{\mu^{r+1}(c)\} \cup \left\{ s \in S \mid C_{P_s}^{r'} = \emptyset \text{ for some } r' \le r+1 \right\}$.

Let $r^* = \min\{r \ge 1 \mid H^r = S\}$ and set $IA(P) = \mu^{r^*}$. Such a r^* exists because C and S are finite.

In the IA mechanism, all students assigned at least one course at any step are removed. The procedure continues until all students have been removed. Students never lose a seat at a course they have been assigned at any step of the mechanism. However, if the priorities are specific to each slot, they may be moved to seats with different precedence as the mechanism progresses.

The following example illustrates the operation of the IA mechanism.

Example 3. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3, c_4\}$. Let preferences and priorities be as follows:

$$P_{s_1} : \{c_1, c_3\}, \{c_1\}, \{c_3\}; \qquad P_{c_1} : \{s_4\}, \{s_3\}, \{s_2\}, \{s_1\};$$
$$P_{s_2} : \{c_4\}, \{c_1\}, \{c_2\}; \qquad P_{c_2} : \{s_4\}, \{s_3\}, \{s_2\}, \{s_1\};$$
$$P_{s_3} : \{c_4\}, \{c_3\}, \{c_1\}, \{c_2\}; \qquad P_{c_3} : \{s_1\}, \{s_2\}, \{s_3\};$$
$$P_{s_4} : \{c_4\}; \qquad P_{c_4} : \{s_1, s_4\}, \{s_4\}, \{s_1\}.$$

The IA mechanism proceeds as in Table 1.

Table 1
The IA mechanism steps from Example 3.

	s_1	s_2	s_3	s_4	T^i
Step 1	$\{c_1, c_3\}$	$\{c_4\}$	$\{c_4\}$	$\{c_4\}$	
μ^1	$\{c_1, c_3\}$	$\emptyset$	$\emptyset$	$\{c_4\}$	$\{s_2, s_3\}$
Step 2		$\{c_1\}$	$\{c_3\}$		
μ^2	$\{c_1, c_3\}$	$\emptyset$	$\emptyset$	$\{c_4\}$	$\{s_2, s_3\}$
Step 3		$\{c_2\}$	$\{c_1\}$		
μ^3	$\{c_1, c_3\}$	$\{c_2\}$	$\emptyset$	$\{c_4\}$	$\{s_3\}$
Step 4			$\{c_2\}$		
$\mu^4 = IA(P)$	$\{c_1, c_3\}$	$\{c_2\}$	$\emptyset$	$\{c_4\}$	$\emptyset$

Notice that μ^4 is not stable. It is indeed blocked by (c_1, s_3).

The unique stable allocation is ρ in which

$$\rho(s_1) = \{c_3\}, \rho(s_2) = \{c_2\}, \rho(s_3) = \{c_1\}, \rho(s_4) = \{c_4\}.$$

Having each student s_i ranking $\rho(s_i)$ as the unique acceptable course is a NE of the game induced by IA mechanism and yields ρ as the outcome.

Theorem 1. *The IA mechanism implements the set of stable allocation in NE if preferences are substitutable and priorities are slot-specific.*

In other words, if priorities are slot-specific and preferences are substitutable, the set of Nash equilibria of the game induced by the IA mechanism coincides with $\mathcal{S}(P_S)$, for all $P_S \in \mathcal{P}^{|S|}$.

To prove the claim, we first show that each student can obtain any course schedule that can be an outcome of the mechanism, ceteris paribus, by ranking it first (Lemma 1 in the Appendix). This result helps in proving that the students' strategic behavior contributes to eliminating unstable allocations. If an allocation μ is unstable, a student s, who is part of a blocking pair (c, s), can profitably deviate by ranking $Ch_s(\mu(s) \cup \{c\})$ at the top of her preference list. Then, we prove that, given a stable allocation μ, the strategy profile in which each student $s \in S$ ranks $\mu(s)$ at the top of her preferences, $P_{s|\mu(s)}$ is a Nash equilibrium of the game induced by the IA mechanism yielding μ as outcome.

The equilibrium strategies defined in part (ii) of the proof of Theorem 1 are undominated. Thus, we obtain Corollary 1.

Corollary 1. *The IA mechanism implements the set of stable allocation in UNE if preferences are substitutable and priorities are slot-specific.*

Under these assumptions, the NE outcomes are stable. However, they are vulnerable to coalitional strategic behavior. More precisely, a strong NE only exists if the student-optimal stable allocation is efficient. It follows from the same argument employed in the proof of Theorem 3 in Sotomayor (2004). In this case, the unique strong NE outcome is the student-optimal stable allocation. Thus, our mechanisms implement the student-optimal stable allocation in strong NE if and only if the priority structure is essentially homogeneous (see Kojima, 2013). The same argument applies to all mechanisms implementing the set of stable allocation in NE.[10]

Under substitutable preferences and priorities, all stable allocations are Nash equilibrium outcomes of the IA mechanism. However, not all Nash equilibrium outcomes are stable allocations, and this is because not all outcomes of the mechanism are individually rational for courses, as shown in Example 4.

Example 4. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2\}$. Each student wants to enroll in exactly one course. The maximum number of students c_1 can enroll is three, but the ideal number is two. Let preferences and priorities be as follows:

$$P_{s_1} : \{c_2\}, \{c_1\}; \qquad P_{c_1} : \{s_1, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3\}, \{s_1, s_2\}, \{s_1\}, \{s_3\}, \{s_2\};$$
$$P_{s_2} : \{c_1\};$$
$$P_{s_3} : \{c_1\}; \qquad P_{c_2} : \{s_4\}, \{s_1\}, \{s_2\}, \{s_3\}.$$
$$P_{s_4} : \{c_2\};$$

10 We thank an anonymous reviewer for suggesting that we explore the implications of coalitional strategic behavior.

Notice that priorities are substitutable. Truth-telling results in allocation μ, in which $\mu(c_1) = \{s_1, s_2, s_3\}$ and $\mu(c_2) = \{s_4\}$, which is not individually rational because $Ch_{c_1}(\mu(c_1)) \neq \mu(c_1)$. However, truth-telling is a Nash equilibrium of the IA mechanism because any student but s_1 is assigned to her preferred course, and s_1 has no profitable deviations.

The instability of NE allocations under the IA mechanism arises from the definitive nature of acceptances. In Example 4, when student s_1 applies, the priorities of the course c_1 dictate rejecting the application. However, the IA mechanism does not allow such rejection. When priorities are slot-specific, the individual rationality of the mechanism's outcome is not an issue. This is because adding acceptable students until reaching the course's capacity always results in an individually rational allocation if priorities are slot-specific.

3.3. The conditional acceptance mechanism

Some course allocation problems require a priority structure that is broader than slot-specific. Substitutable priorities can be helpful when the desired class size is smaller than the course capacity, such as when students need to work in pairs, but the number of available seats is odd. However, using the NE and IA mechanisms may result in unstable allocations when substitutable priorities are in place (see Example 4).

The CA mechanism expands our results to the domain of substitutable priorities. It employs the structure of the IA mechanism to encourage students to acquire and use information regarding course priorities and their peers' preferences. Additionally, it ensures that students' allocation to a specific course remains individually rational.

In the CA mechanism, each student's message space consists of a preference profile for course schedules. In the first step, we only consider the schedule each student presents as their top choice among the students requesting a specific course, denoted as c, and the group with the highest priority is selected. Once the first step is complete, all students assigned to at least one course and those not requesting any course are removed.

During the rth step, we only consider the rth choice in the preference list of the remaining students. Each course considers the students already assigned to it, along with the new students requesting a seat at this step. The course evaluates the set of students chosen in the previous step, combined with the new applicants, and allocates seats to the subset with the highest priority. All students assigned at least one course during this step, and those who did not request any course are subsequently removed. The mechanism continues until all students have been eliminated.

Given a priority profile $(P_c)_{c\in C}$ and a preference profile for students $(P_s)_{s\in S}$, the following procedure describes the CA **mechanism**.

Step 1: For every $c \in C$ let $S_c^1 = \left\{ s \in S \mid c \in C^1_{P_s} \right\}$.
Set $\mu^1(c) = Ch_c(S_c^1)$.
Let $H^1 = \bigcup_{c\in C} \{\mu^1(c)\} \cup \left\{ s \in S \mid C^1_{P_s} = \emptyset \right\}$.

Step r+1 (for $1 \le r$**)**: Let $T^{r+1} = S \backslash H^r$. For every $c \in C$ let $S_c^{r+1} = \mu^r(c) \cup \left\{ s \in T^{r+1} \mid c \in C^{r+1}_{P_s} \right\}$.
Set $\mu^{r+1}(c) = Ch_c(S_c^{r+1})$.
Let $H^{r+1} = \bigcup_{c\in C} \{\mu^{r+1}(c)\} \cup \left\{ s \in S \mid C^{r'}_{P_s} = \emptyset \text{ for some } r' \le r+1 \right\}$.

Let $r^* = \min\{r \ge 1 \mid H^r = S\}$ and set $CA(P) = \mu^{r^*}$. Such a r^* exists because C and S are finite.

For each course c, the CA mechanism considers the students who have applied to that course at this step, denoted as S_c^{r+1}, along with the previously accepted students. From this pool, the course selects the students with highest priority.

Students who have obtained at least one course in the current step or any previous step and those who have not ranked any acceptable course in the current step are removed from consideration. The procedure continues until all students have been removed.

The CA mechanism has characteristics of both the IA and SO mechanisms. Like the IA mechanism, courses accept students at most once. Additionally, like in the SO mechanisms, courses can replace previously accepted students with new ones.

Example 5. Consider the preferences and the priorities of Example 3.

The CA mechanism proceeds as in Table 2.

Table 2
The CA mechanism steps from Example 5.

	s_1	s_2	s_3	s_4	T^i
Step 1	$\{c_1, c_3\}$	$\{c_4\}$	$\{c_4\}$	$\{c_4\}$	
μ^1	$\{c_1, c_3\}$	$\emptyset$	$\emptyset$	$\{c_4\}$	$\{s_2, s_3\}$
Step 2		$\{c_1\}$	$\{c_3\}$		
μ^2	$\{c_3\}$	$\{c_1\}$	$\emptyset$	$\{c_4\}$	$\{s_3\}$
Step 3			$\{c_1\}$		
μ^3	$\{c_3\}$	$\emptyset$	$\{c_1\}$	$\{c_4\}$	$\emptyset$
$\mu^3 = CA(P)$	$\{c_3\}$	$\emptyset$	$\{c_1\}$	$\{c_4\}$	$\emptyset$

Notice that μ^3 is not stable. It is wasteful and blocked by (c_2, s_2).

Having each student s_i declaring $P'_{s_i} : \mu\left(s_i\right)$ is a NE of the game induce by CA mechanism and yields μ, the unique stable allocation, as outcome (see Example 3).

We present our main result in Theorem 2.

Theorem 2. *The CA mechanism implements the set of stable allocations in NE if preferences and priorities are substitutable.*

In other words, if priorities are substitutable and preferences are substitutable, the set of NE outcomes of the game induced by CA mechanism coincides with $\mathcal{S}\left(P_S\right)$, for all $P_S \in \mathcal{P}^{|S|}$.

The strategy of proof of Theorem 2 is analogous to the strategy of proof of Theorem 1. In the CA mechanism, the strategic behavior of the students helps eliminate unstable allocations. Indeed, from Lemma 2, in the Appendix, it follows that if a pair (c, s) can block an outcome allocation μ, then μ, $P_{s|Ch_s(\mu(s)\cup\{c\})}$ represents a profitable deviation for student s. Finally, if μ is a stable allocation, the strategy profile in which each student s ranks $\mu(s)$ first is a NE of the game induced by the CA mechanism.

The equilibrium strategies defined in part (ii) of the proof of Theorem 2 are undominated. Therefore, the following result holds.

Corollary 2. *The CA mechanism implements the set of stable allocation in UNE if preferences and priorities are substitutable.*

Proposition 1 and Theorems 1, and 2 describe a clear picture of our flexibility in terms of priorities design if we want to implement stable allocations under substitutable preferences. This relationship is represented in Fig. 1.

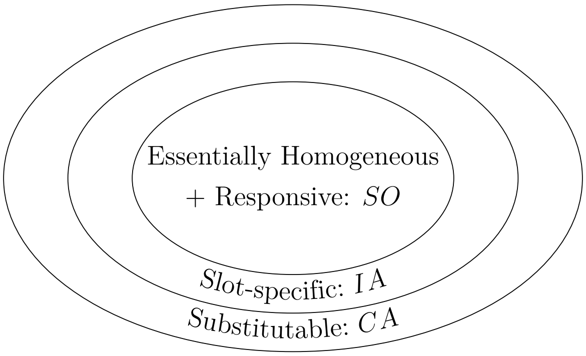

Fig. 1. Nash implementability of stable allocations under priorities.

4. The extended conditional acceptance mechanism

Out-of-equilibrium play in the CA mechanism can result in student losses. When students lose a course along the mechanism, they cannot replace it. Usually, the issue of students with incomplete course schedules is addressed through post-allocation adjustments, either in the form of an administrative allocation or by running a new allocation procedure. This stage is often referred to as post-allocation adjustment. In post-allocation adjustments, students can usually drop courses and register for new ones. However, it is worth noting that including a post-allocation adjustment can introduce distortions in implementing stable allocations, as proved in Example 6.

Example 6. Let $S = \{s_1, s_2\}$ and $C = \{c_1, c_2, c_3\}$. Let preferences and priorities be as follows:

$P_{s_1} : \{c_1, c_2\}, \{c_2\}, \{c_1\};$

$P_{s_2} : \{c_3\}, \{c_2\};$

$P_{c_1} : \{s_1\};$

$P_{c_2} : \{s_2\}, \{s_1\};$

$P_{c_3} : \{s_2\}.$

Assume that the seats are assigned employing the CA mechanism and that there is a post-allocation adjustment in which the students can drop courses and register new ones with empty seats. Those empty seats are also assigned in the following stage, employing the CA mechanism.

The game has a $SPNE$, which yields an unstable allocation. At the first stage students play $P'_{s_1} = P_{s_1}$, $P'_{s_2} : \left\{c_2, c_3\right\}, \left\{c_3\right\}, \left\{c_2\right\}$. In the second stage, students play their best responses. Student s_2 drops course c_2 but student s_1 cannot register any course. The outcome allocation is μ in which $\mu\left(s_1\right) = \left\{c_1\right\}$, $\mu\left(s_2\right) = \left\{c_3\right\}$, which is wasteful and thus unstable.

Allocation μ is an equilibrium outcome also if the CA mechanism repeats multiple times and students are allowed to drop courses and register only for empty seats. It is indeed easy to construct an $SPNE$ in which students play as above in the first stage and in which student s_2 drops course c_2 only at the last stage, preventing s_1 from registering for any course.

Even the introduction of a waiting list cannot prevent the possibility of a post-allocation adjustment generating instabilities. Example 7 shows that the waiting lists procedure limits post-allocation trades but induces coordination problems, resulting in unstable and inefficient allocations.

Example 7. Let $S = \{s_1, s_2, s_3, s_4\}$, and $C = \{c_1, c_2, c_3, c_4, c_5\}$. Let preferences and priorities be as follows:

$$P_{s_1} : \{c_2, c_4\}, \{c_1, c_4\}, \{c_1, c_2\}, \{c_4\}, \{c_2\}, \{c_1\};$$
$$P_{s_2} : \{c_5\}, \{c_1\}, \{c_2\};$$
$$P_{s_3} : \{c_3\}, \{c_2\};$$
$$P_{s_4} : \{c_1, c_2\}, \{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}, \{c_3, c_4\}, \{c_3, c_5\}, \{c_4, c_5\}, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\};$$

$$P_{c_1} : \{s_1\}, \{s_2\}, \{s_4\};$$
$$P_{c_2} : \{s_2\}, \{s_3\}, \{s_1\}, \{s_4\};$$
$$P_{c_3} : \{s_3\}, \{s_1\};$$
$$P_{c_4} : \{s_4\}, \{s_1\}, \{s_2\};$$
$$P_{c_5} : \{s_4\}, \{s_2\}.$$

Assume that the seats are assigned employing the CA mechanism and that there is a post-allocation adjustment in which the students can drop courses and submit a preference order for the courses they have not registered for yet. Those empty seats and dropped courses are assigned employing the CA mechanism.

Consider the following first-stage strategies for the students

$$P'_{s_1} : \{c_2\}, P'_{s_2} : \{c_1\}, P'_{s_3} : \{c_2, c_3\}, \{c_3\}, \{c_2\}, P'_{s_4} : \{c_4, c_5\}, \{c_4\}, \{c_5\}.$$

The first stage outcome is allocation μ^1 such that

$$\mu^1(s_1) = \emptyset, \mu^1(s_2) = \{c_1\}, \mu^1(s_3) = \{c_2, c_3\}, \mu^1(s_4) = \{c_4, c_5\}.$$

Consider the following second-stage strategies for the students. Student s_3 drops course c_2, the other students drop no course. They declare preferences $P''_{s_1} : \{c_2\}$; $P''_{s_2} : \emptyset$; $P''_{s_3} : \emptyset$, $P''_{s_4} : \emptyset$. The outcome is allocation μ such that

$$\mu(s_1) = \{c_2\}, \mu(s_2) = \{c_1\}, \mu(s_3) = \{c_3\}, \mu(s_4) = \{c_4, c_5\}.$$

Allocation μ is unstable and blocked by (s_1, c_1).

The previous messages can be sustained as a $SPNE$ equilibrium. Only student s_1 could try to deviate profitably. However, ranking course c_1 as acceptable in the second stage is not profitable because, at the second stage, student s_2 would get course c_2 and student s_1 would end with, at most, course c_1. By ranking c_1 as acceptable in the first stage, student s_1 would indeed obtain course c_1. However, in all second-stage subgames following this kind of deviation, student s_2 would get c_2. Indeed, an optimal strategy for student s_2 is to set $P^2_{s_2} : \{c_2\}$ in the sub-game induced by any deviation such that student s_1 obtains c_1 at the first stage.

Allocation μ is also inefficient. It is indeed Pareto dominated by allocation ν such that

$$\nu(s_1) = \{c_4\}, \nu(s_2) = \{c_5\}, \nu(s_3) = \{c_3\}, \nu(s_4) = \{c_1, c_2\}.$$

We introduce the extended conditional acceptance mechanism or ECA mechanism, a repeated version of the CA mechanism. The ECA mechanism incorporates a post-allocation adjustment phase by iteratively applying the CA mechanism to students who still need to register for courses and courses with vacant seats. At each stage of the ECA mechanism, the assignment is final, and students are not allowed to drop any courses they have been allocated. The ECA mechanism ensures that all students can complete their course registrations while maintaining the definitive nature of the assignment at each stage.

Given a priority profile $(P_c)_{c\in C}$ and a preference profile for students $(P_s)_{s\in S}$, the following procedure describes the ECA **mechanism**.

Stage 1: Let $\mathcal{P}^0_s$ be the set of admissible preferences for student $s \in S$. Students submit a preference profile $P^1 = (P^1_s)_{s\in S} \in \prod_{s\in S} \mathcal{P}^0_s$. Set $\mu^1(P^1) = CA(P^1)$. For each $s \in S$, set $\mathcal{P}^1_s = \left\{ P_s \mid \left|\mu^1(P^1)(c)\right| = q_c \Rightarrow c \notin A_s(P_s) \right\}$.

Stage r+1 (for $1 \le r$): Students submit a preference profile $P^{r+1} = (P^{r+1}_s)_{s\in S} \in \prod_{s\in S} \mathcal{P}^r_s$.
Define priorities P^{r+1}_c on S as follows. For $S', S'' \subseteq S\backslash\mu^r(c)$, $S' P^{r+1}_c S''$ if and only if $\mu^r(c)\cup S' P_c \mu^r(c)\cup S''$. Define $CA^{r+1}(P^{r+1})$ as the result of the conditional acceptance mechanism with priorities $(P^{r+1}_c)_{c\in C}$ under profile of preferences P^{r+1}.
Set $\mu^{r+1}(P^{r+1}) = \mu^r \cup CA^{r+1}(P^{r+1})$.
For each $s \in S$, set $\mathcal{P}^{r+1}_s = \{P_s \in \mathcal{P}^r_s \mid |\mu^r(P^r)(c)| = q_c \Rightarrow c \notin A_s(P_s)\}$.

Let $r^* = \min\{r : \mu^{r+1} = \mu^r\}$. Such a r^* exists because $\mathcal{P}^{r+1}_s \subseteq \mathcal{P}^r_s$ and C and S are finite. Set $ECA = \mu^{r^*}$.

In the first stage of the mechanism, students submit a preference profile, P^1, and each student is definitively assigned according to the CA mechanism, resulting in allocation $\mu^1 = CA(P^1)$. In stage $r + 1$, students submit a ranking P^{r+1} of schedules of courses with available seats. Then, additional courses are assigned using the conditional acceptance mechanism, in which the allocation from the previous stage determines the course priorities, μ^r, resulting in allocation $\mu^{r+1} = \mu^r \cup CA(P^{r+1})$. The procedure continues until no student submits a new ranking.

Example 8 shows how the ECA mechanisms can help correct naive play.

Example 8. Consider the same problem as in Examples 3 and 5.

Assuming students play sincerely, at the first stage, student s_2 is tentatively assigned with the seat in course c_1. However, it loses the seat to student s_3, resulting in an empty seat in course c_2 (see Example 5). This occurs even when s_2 has ranked course c_2 as acceptable and is also acceptable to course c_2. It follows from the fact that student s_2, by listing c_1 (and c_4) as acceptable, was aiming too high.

In the second stage of the ECA mechanism, students can only rank courses with empty seats. Thus, if student s_2 plays sincerely, she can correct the 'mistake' in stage 1 and enroll in course c_2. The resulting allocation is the unique stable allocation, μ.

Theorem 3 shows that the ECA mechanism implements the set of stable allocation when both preferences and priorities are slot-specific.[11]

Theorem 3. *The ECA mechanism implements the set of stable allocation in $SPNE$ if preferences and priorities are slot-specific.*

In other words, if preferences and priorities are slot-specific, the set of $SPNE$ outcomes of the game induced by ECA mechanism coincides with $\mathcal{S}\left(P_S\right)$, for all $P_S \in \mathcal{L}^{|S|}$.

To prove the claim, we first show that a student can obtain any possible outcomes at the first stage by ranking them in the first place in the first stage message (Lemma 3 in the Appendix). This result is instrumental in proving that the students' strategic behavior helps eliminate unstable allocations. If an allocation μ is unstable, a student s, who is part of a blocking pair (c, s), can profitably deviate by ranking $Ch_s(\mu(s) \cup \{c\})$ at the top of her preference list in the first stage message. Then, we prove that, given a stable allocation μ, there is a $SPNE$ in which each student $s \in S$ ranks $\mu(s)$ at the top of her preferences, $P_{s|\mu(s)}$ in the first stage message is a and yields μ as outcome.

The ECA mechanism preserves the strategic properties of the CA mechanism. If a student deviates from an equilibrium strategy, the penalty is mitigated through participation in a post-allocation adjustment. Including this adjustment does not compromise the strategic properties of the CA mechanism, provided that preferences and priorities are slot-specific.

5. Simpler environments

The complexity of the strategy space may hinder the practical implementation of the mechanisms. In this section, we focus on slot-specific preferences, which extend the assumption of responsive preferences commonly used in the course allocation literature (Sönmez and Ünver, 2010; Budish and Cantillon, 2012; Aziz et al., 2019).

We introduce two mechanisms in which students' preferences can be expressed as an ordered list of individual courses. Let $s \in S$. We assume that the message for student s is denoted as $m_s = \left(\Sigma_s, (\succ_\sigma)_{\sigma\in\Sigma_s}, \rhd_s\right)$. Here, M_s represents the set of messages for student s. Given a message m_s, we define $P_s = P_s\left(m_s\right)$. For each $s \in S$, in which $m_s \in M_s$ and $P_s = P_s\left(m_s\right)$ are preferences that rationalize m_s (see Alva, 2018 Theorem 1).

Given a priority profile $\left(Ch_c\right)_{c\in C}$ and a profile of messages $\left(m_s\right) s \in S$, we define the simplified CA mechanism (SCA) with the outcome function $SCA\left(\left(m_s\right)_{s\in S}\right) = CA\left(\left(P_s\left(m_s\right)\right)_{s\in S}\right)$. In the SCA mechanism, students play the game induced by the corresponding mechanism with preferences that rationalize the message of each student.[12]

Proposition 2. *Assume that the preferences are slot-specific and priorities are substitutable. The SCA mechanism implements the set of stable allocations in NE.*

The proof of Proposition 2 follows from Theorem 2. After constructing the set of slot-specific priorities from the preference profile expressed by the student. The absence of course complementarity in this simplified environment is the only reason for restricting student preferences.

We can also define a simplified version of the IA mechanism. Given a priority profile $\left(Ch_c\right)_{c\in C}$ and $\left(\geq_s, q_s\right)_{s\in S}$, the simplified IA mechanism is defined by the following outcome function $SIA\left(\left(m_s\right)_{s\in S}\right) = IA\left(\left(P_s\left(m_s\right)\right)_{s\in S}\right)$.

Proposition 3. *Assume that preferences and priorities are slot-specific. The simplified IA mechanism implements the set of stable allocations in NE.*

The proof of Proposition 3 follows from Proposition 2.

Finally, we can include a post-allocation adjustment defining a simplified version of the ECA mechanism, which we call the simplified ECA mechanism, naturally after defining the SCA mechanism. Proposition 4 is a direct consequence of Theorem 3.

Proposition 4. *Assume that the preferences and priorities are slot-specific. The simplified ECA mechanism implements the set of stable allocations in $SPNE$.*

[11] The definition of slot-specific preferences is analogous to the definition of slot-specific priorities.

[12] If the preferences are responsive, the message can be simplified to $(q_s, >_s)$, in which q_s represents the demand for courses of student s, and $>_s$ is the common ranking of individual courses.

6. Conclusion

This paper presents the CA mechanism for allocating courses to students based on their preferences and priorities. Under the assumption of substitutable preferences and priorities, the CA mechanism implements the set of stable allocations in Nash equilibrium and undominated Nash equilibrium, and it is simple to compute in practice. The CA mechanism builds upon the IA mechanism but allows courses to accept tentatively and reject students to preserve individual rationality.

Deviating from equilibrium strategies in the CA mechanism might be costly. To address this issue, we propose the ECA mechanism. This mechanism repeats the CA mechanism and implements the set of stable allocations in subgame perfect Nash equilibrium under slot-specific preferences and priorities. Our findings indicate that the post-allocation adjustment mechanism should not allow students to drop courses.

We conclude our analysis by examining the design possibilities in markets with less complex preferences. Our results prove the feasibility of designing a mechanism that motivates students to strategically acquire and employ information to overcome the inherent challenges in the course allocation problem. Our findings support the design features found in practical mechanisms while highlighting others' weaknesses.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

Proof of the results in Section 3.1

Proof of Proposition 1.

(a) We prove that if priorities are not essentially homogeneous; there is a NE of the game induced by the SO mechanism yielding an unstable allocation as outcome. Assume priorities are not homogeneous, let c_1, c_2, s_1, s_2, S_{c_1}, and S_{c_2} like in Definition 1. Let preferences be responsive and as follows: P_{s_1} : $\{c_1, c_2\}, \{c_2\}, \left\{c_1\right\}$; P_{s_2} : $\{c_1\}, \{c_2\}$. For all $s \in S_{c_1}$, let $c_1 P_s c P_s \emptyset$ for all $c \in C \setminus \left\{c_1\right\}$. For all $s \in S \in S_{c_1} \cap S_{c_2}$, let $c_1 P_s c P_s \emptyset$ for all $c \in C \setminus \left\{c_1, c_2\right\}$. For all $s \in C \in S_{c_2} \setminus S_{c_1}$, let $c_2 P_s c P_s \emptyset$ for all $c \in C \setminus \left\{c_2\right\}$. For all $s \in \left(S_{c_1} \setminus S_{c_2}\right) \cup \left(S_{c_2} \setminus S_{c_1}\right)$ let $q_s = 1$. For all $s \in S \in S_{c_1} \cap S_{c_2}$, let $q_s = 2$. Let the preferences of the other students be arbitrary responsive preferences. In the student-optimal stable allocation, μ, we have $\mu\left(s_1\right) = \left\{c_1\right\}$ and $\mu\left(s_2\right) = \left\{c_2\right\}$. Let allocation ν defined as follows $\nu\left(s_1\right) = \left\{c_2\right\}$, $\nu\left(s_2\right) = \left\{c_1\right\}$, and $\nu(s) = \mu(s)$ for all $s \in S \setminus \left\{s_1, s_2\right\}$. Allocation ν is unstable because it is blocked by $\left(c_1, s_1\right)$. Let $P_s^* = P_{|\nu(s)}$ for all $s \in S$. Strategy profile $P_S^* = \left(P*_s\right)_{s \in S}$ is a NE of the game induces by the SO mechanism which yields ν.

(b) First, we prove that all NE outcomes are stable allocations, and then we prove that any stable allocation is a NE outcome of the game induced by the SO mechanism.

(i) From Theorem 3 in Kojima (2013), the student-optimal stable allocation is the result of a serial dictatorship because priorities are essentially homogeneous. Let P_S^* be a NE of the game induced by the SO when the preference profile is $P_S = \left(P_s\right)_{s \in S}$ and let $\mu = SO\left(P_S^*\right)$. We prove that, in the serial dictatorship yielding the student-optimal stable allocation, each student s, playing P_s^* obtains the same set of course that would have obtained by playing P_s. By contradiction, let $k \geq 1$ be the minimal integer where the student choosing at step $k \geq 1$ of the serial dictatorship obtains a set of students different from what she would have obtained by playing P_s. Since the SD mechanism is strategy-proof, she would make a profitable deviation playing P_s, which yields a contradiction. It follows that μ coincides with the student-optimal stable allocation according to P, and thus it is stable.

(ii) Let μ be a stable allocation. Consider the following strategy profile: $\left(P_{s|\mu(s)}\right)_{s\in S}$. The stability of μ implies that the strategy profile is a NE. ■

Proof of the results in Section 3.2

In Lemma 1, we show that each student can obtain any course schedule that can be an outcome of the mechanism, ceteris paribus, by ranking it first.

Lemma 1. *Let $P = \left(P_s\right)_{s\in S}$ be a preference profile for students and let $\mu = IA(P)$. For each $s \in S$ and $C' \subseteq \mu(s)$ $C' = IA\left(P_{s|C'}, P_{-s}\right)$.*

Proof. Let $s \in C$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the IA mechanism at which c has been assigned to s, $r(c) = \{r \in \mathbb{N} \mid s \in \mu^r(c)\}$. Notice $r(c) = r\left(c'\right)$ for all $c, c' \in C'$. Let σ be the seat to which s is assigned at step $r(c)$. Thus, student s is the highest priority student for seat σ among the ones in $\mu^r(c)$ who are not assigned to a seat preceding σ. Formally, for each $c \in C'$, $r \le r(c)$, if $s' \in \mu^r(c)$ and $s' \succ_\sigma s$, there exists a seat $\sigma' \in \Sigma_c$, $\sigma' \rhd_c \sigma$ such that $s' \succ_{\sigma'} s$. Thus, $C' = IA\left(P_{s|C'}, P_{-s}\right)$. ■

This result allows us to prove Theorem 1.

Proof of Theorem 1. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations, and then we prove that any stable allocation is a NE outcome of the game induced by the IA mechanism.

(i) Let P_S^* be a NE of $\left(S, \mathcal{P}^{|S|}, IA, P\right)$ and let $\mu = IA\left(P_S^*\right)$. As observed, μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Let $P_s' = P_{s|Ch_s(\mu(s))}$, by Lemma 1: $IA\left(P_s', P_{-s}^*\right)(s) = Ch_s(\mu(s))$. Thus, the deviation is profitable to s, which yields a contradiction. We prove by contradiction that no course-student pair blocks μ. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s)\cup\{c\})}$. Because $s \in Ch_c(\mu(c) \cup \{s\})$, the deviation is profitable to s, which yields a contradiction. Thus, allocation μ is stable.

(ii) Let μ be a stable allocation. For each s, let $P_s^* = P_{s|\mu(s)}$. Set $P_S^* = \left(P_s^*\right)_{s\in S}$. We have $IA\left(P_S^*\right) = \mu$. We prove by contradiction that P_S^* is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P_s', and let $\mu' = IA\left(P_s', P_{-s}^*\right)$. Let $c \in Ch_s\left(\mu(s) \cup \mu'(s)\right) \setminus \mu(s)$. Because P_s is substitutable, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $P_s'' = P_{s|Ch_s(\mu(s)\cup\{c\})}$, then $IA\left(P_s'', P_{-s}^*\right)(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c, s) blocks μ, which yields a contradiction. ■

Proof of the results in Section 3.3

Lemma 2 illustrates that in the CA mechanism, a student can obtain any possible outcomes by ranking them in the first place.

Lemma 2. *Let $P = \left(P_s\right)_{s\in S}$ be a preference profile for students and let $\mu = CA(P)$. If the priorities are substitutable, for each $s \in S$ and $C' \subseteq \mu(s)$, $C' = CA\left(P_{s|C'}, P_{-s}\right)(s)$.*

Proof. Let $s \in S$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the CA mechanism at which c has been assigned to s along the mechanism. Formally, $r(c) = \min\{r \in \mathbb{N} \mid s \in \mu^r(c)\}$. Notice that $r(c) = r\left(c'\right)$ for all $c, c' \in \mu(s)$ and that $\mu^r(s) = \emptyset$ for all $r < r(c)$. The substitutability of Ch_c implies that $C_{P_s}^{r(c)} P_s C'$; otherwise, $s \in \mu^r(c)$ for some $r < r(c)$. For all $i \le r(c)$, let $P_s^{r(c)}$ be a preference profile over 2^C such that $C_{P_s^{r(c)}}^{r(c)} = C'$, and for $j \neq r(c)$: $C_{P_s^{r(c)}}^{r(c)} = C_{P_s}^j$ if $C_{P_s}^j \neq C'$ and $C_{P_s^{r(c)}}^j = C_{P_s}^{r(c)}$ if $C_{P_s}^j = C'$. Notice that $CA\left(P_s^{r(c)}, P_{-s}\right)(s) = C'$. For all i, $i < r(c)$, let P_s^i be a preference over 2^C such that $C_{P_s^i}^i = C'$, and for $j \neq i$: $C_{P_s^j}^j = C_{P_s}^{j+1}$ if $C_{P_s^{j+1}}^{j+1} \neq C'$ and $C_{P_s^j}^j = C_{P_s^{j+1}}^{j+1}$ if $C_{P_s^{j+1}}^{j+1} = C'$. Intuitively, each P_s^j lifts C' to place j in the preference of s without changing the ranking above the jth place.

We prove by contradiction that $CA\left(P_s^{i-1}, P_{-s}\right)(s) = CA\left(P_s^i, P_{-s}\right)(s) = C'$ for all i, $1 \le i < r(c)$. For every preference on 2^C, Q_s, let $\mu_{Q_s}^j$ be the outcome at the step j of the CA mechanism when preferences are $\left(Q_s, P_{-s}\right)$. Notice that $\mu_{P_s}^i = \mu_{P_s^j}^i$ for all i, j, $2 \le i < j \le r(c)$. Thus, to prove that $CA\left(P_s^{i-1}, P_{-s}\right)(s) = CA\left(P_s^i, P_{-s}\right)(s)$ for all $i < r(c)$, it suffices to show that $s \in Ch_c\left(\mu_{P_s}^{i-1}(c) \cup \left\{s \in S \mid c \in \bigcup_{s'\neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right)$ for all i, $2 \le i \le r(c)$. By contradiction, assume that it is not the case, and let j be the maximum integer such that $s \notin Ch_c\left(\mu_{P_s}^{j-1}(c) \cup \left\{s \in S \mid c \in \bigcup_{s'\neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right)$ and $s \in Ch_c\left(\mu_{P_s}^{j}(c) \cup \left\{s \in S \mid c \in \bigcup_{s'\neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right)$.

Because P_c is substitutable, $s \in Ch_c\left(\mu_{P_s}^j(c) \cup \{s\}\right)$. The jth step of the mechanism when preferences are $\left(P_s^j, P_{-s}\right)$ yields $\mu_{P_s^j}^j(c)$ to course c. We have

$s \notin Ch_c\left(\mu_{P_s}^{j-1}(c) \cup \left\{s \in S \mid c \in \bigcup_{s'\neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right) = \mu_{P_s^j}^j(c)$. The irrelevance of rejected students condition implies that $Ch_c\left(\mu_{P_s}^{j-1}(c) \cup \left\{s \in S \mid c \in \bigcup_{s'\neq s} C_{P_{s'}}^{i-1}\right\} \cup \{s\}\right) = Ch_c\left(\mu_{P_s}^j(c) \cup \{s\}\right) = \mu_{P_s}^j(c)$. In particular, $s \notin Ch_c\left(\mu_{P_s}^j(c) \cup \{s\}\right)$, which yields a contradiction. Thus $CA\left(P_s^1, P_{-s}\right)(s) = C'$. It follows that $CA\left(P_{s|C'}, P_{-s}\right)(s) = C'$, which concludes the proof. ■

Proof of Theorem 2. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations, and then we prove that any stable allocation is a NE outcome of the game induced by the CA mechanism. Fix preferences $P = (P_s)_{s\in S}$.

(i) Let P^*_S be a NE of the game induced by the CA mechanism and let $\mu = CA(P^*_S)$. Allocation μ is individually rational for each course by definition. We prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Let $P'_s = P_{s|Ch_s(\mu(s))}$. Because P_s is substitutable, P'_s is substitutable as well. By Lemma 2: $CA(P'_s, P^*_{-s})(s) = Ch_s(\mu(s))$. Thus, the deviation is profitable to s, which yields a contradiction. Assume that there exists a pair blocking μ, $(c,s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s)\cup\{c\})}$. Because $s \in Ch_c(\mu(c) \cup \{s\})$, the deviation is profitable to s, which yields a contradiction. It follows that allocation μ is individually rational and cannot be blocked by any course-student pair; thus, it is stable.

(ii) Let μ be a stable allocation. For each s, let $P^*_s = P_{s|\mu(s)}$. Set $P^*_S = (P^*_s)_{s\in S}$. We have $CA(P^*_S) = \mu$. We prove by contradiction that P^*_S is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P'_s, and let $\mu' = CA(P'_s, P^*_{-s})$. Let $c \in Ch_s(\mu(s) \cup \mu'(s)) \setminus \mu(s)$. Because P_s is substitutable and from Lemma 2, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $P''_s = P_{s|Ch_s(\mu(s)\cup\{c\})}$, then $CA(P''_s, P^*_{-s})(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c,s) blocks μ, which yields a contradiction. ■

Proof of the results in Section 4

Lemma 3 proves that in the ECA mechanism, a student can obtain any possible outcomes at the first stage by ranking them in the first place.

Lemma 3. *Let P be a strategy profile for students. Let $s \in S$ and let $C' \subseteq ECA(P)(s)$. If the priorities are slot-specific $C' = ECA(P')(s)$, in which $P'^h_s = P_{|C'}$ for all first stage histories h, $P'^h_s : \emptyset$ for all other histories, and $P'^h_{s'} = P^h_{s'}$ for all h, for all $s' \neq s$.*

Proof. If students employ strategy profile P, all seats assigned to s in stage $r \geq 2$ were not assigned in stage 1. Thus, from Lemma 2 and because stage 1 assignments are definitive, $C' = ECA(P')(s)$. ■

Proof of Theorem 3. The proof of the claim is in two parts. First, we prove that all $SPNE$ outcomes are stable allocations, and then we prove that any stable allocation is a $SPNE$ outcome of the game induced by the ECA mechanism. Let $P_S = (P_s)_{s\in S}$ be a profile of slot-specific preferences.

(i) Let P^*_S be a $SPNE$ of the game induced by the ECA mechanism and let $\mu = ECA(P^*_S)$. The definition of ECA shows that allocation μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Since $Ch_s(\mu(s)) \subseteq \mu(s)$, by Lemma 3 there exists a strategy profile which yields $Ch_s(\mu(s))$ to student s. It follows that student s has a profitable deviation from equilibrium strategy P^*_S, which yields a contradiction.
Next, we prove by contradiction that μ has no blocking pairs. Assume that there exists a pair blocking μ, $(c,s) \in C \times S$. It must be the case that student s has never ranked course c as acceptable along the equilibrium path. Let $P'^h_s = P_{s|Ch_s(\mu(s)\cup\{c\})}$ for all first stage histories. Let $P'^h_s : \emptyset$ for all other histories. Let $P'_s = (P'^h_s)^{h\in H}$. If students play P^*_S, all seats assigned to s in a stage $r \geq 2$ had not been assigned in stage 1. Thus, since stage 1 assignments are definitive, $ECA(P'_s, P^*_{-s})(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that P'_s is a profitable deviation for s, which yields a contradiction.

(ii) Let μ be a stable allocation. Let $s \in S$ and let $h \in H$ be a first-stage history. For each s, let $P^{*h}_s = P_{s|\mu(s)}$. A history, h belonging to stage $r \geq 2$, is characterized by the intermediate allocation $\mu^r = \mu^{rh}$ determined in the stage $r-1$ history that h proceeds.[13] Let $C^r = C^{rh} = \{c \in C : |\mu^r(c)| < q_c\}$, C^r is the set of courses with empty seats at the end of the stage $r-1$. Let $(P^r_c)_{c\in C} = (P^{rh}_c)_{c\in C}$ be the corresponding "stage-priorities" at stage r as defined while introducing the ECA mechanism. For all s in S, let $P^r_s = P^{rh}_s$ a strict order defined on 2^C such that, for all $c \in C$, $C', C'' \subseteq C$: (a) if $c \notin C^r_{P_s} \cup \{\emptyset\}$ and $c \in C'$, $\emptyset P^r_s C'$; (b) if $C, C'' \subseteq C^r_{P_s} \setminus \mu^r(s)$ and $C' P^h_s C''$ if and only $(\mu^r(s) \cup C') \, P_s \, (\mu^r(s) \cup C')$. Since P_s is slot-specific, then P^r_s is slot-specific as well. Let v^r be a stable allocation for (C, S, P^r) in which $P^r = (P^r_s, P^r_c)_{s\in S, c\in C}$. For each history $h \in H$, let $P^{*h}_s = P^r_{s|v^h(s)}$ for all $s \in S$. The stability of $\mu^r \cup v^r$ implies that, $\left[(P^{*h}_s)^{h\in H}\right]_{s\in S}$ is an $SPNE$ yielding μ as outcome. ■

Proof of the results in Section 5

Proof of Proposition 2. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations, and then we prove that any stable allocation is a NE outcome of the game induced by the SCA mechanism.

(i) Let $m^* = \left(\Sigma^*_s, (\succ^*_\sigma)_{\sigma\in\Sigma^*_s}, \rhd^*_s\right)_{s\in S}$ be a NE of the game induced by the SCA mechanism when student preferences are given by $(P_s)_{s\in S}$ and let $\mu = SCA(m^*)$. Allocation μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume that μ is not individually rational for student $s \in S$. Let $P'_s = P_{|Ch_s(\mu(s),P_s)}$. Preferences P'_s are slot-specific as well. Let $m'_s = \left(\Sigma'_s, (\succ'_\sigma)_{\sigma\in\Sigma^*_s}, \rhd_s\right)_{s\in S}$ in which Σ'_s is the set of slots for student s under P'_s, $\rhd_s$ is the

[13] In this case, to reduce the notational burden, we refer to stage r and not to the history when there is no ambiguity.

order of precedence of the slot in Σ_s' according to P_s' and for all $\sigma \in \Sigma_s$, $\succ_\sigma$ is the order induced by P_s' for slot σ, for every $\sigma \in \Sigma_s$. By Lemma 2, m_s' is a profitable deviation for student s, which yields a contradiction. We next prove by contradiction that any pair does not block μ. Assume that there exists a pair blocking μ, $(c,s) \in C \times S$. Let $m_s' = \left(\Sigma_s', (\succ_\sigma')_{\sigma\in\Sigma_s^*}, \rhd_s\right)_{s\in S}$ obtained as above from the restriction of P_s to the individual courses in $Ch_s\left(\mu(s)\cup\{c\}, P_s\right)$. Because $s \in Ch_c\left(\mu(s)\cup\{c\}, P_s\right)$, the deviation is profitable to s, which yields a contradiction.

(ii) Let μ be a stable allocation. For each s, Let $q = \max_{P_s'}\left\{|C'| \mid C' \subseteq C, C' P_s \emptyset\right\}$. Let $m_s = \left(\Sigma_s, (\succ_\sigma)_{\sigma\in\Sigma_s^*}, \rhd_s\right)_{s\in S}$ be derived from the restriction of P_s to the individual courses in $\mu(s)$. Notice that $\left(m_s\right)_{s\in S}$ yields μ as outcome. We prove by contradiction that $\left(m_s\right)_{s\in S}$ is a Nash equilibrium. Assume that student s has a profitable deviation, $\left(m_s'\right)$, and let μ' be the outcome of such a deviation. Let $c \in Ch_s\left(\mu(s)\cup\mu'(s), P_s\left(m_s\right)\right)\setminus\mu(s)$. Because $P_s\left(m_s\right)$ is slot-specific, $c \in Ch_s\left(\mu(s)\cup\{c\}, P_s\left(m_s\right)\right)$. Let m_s'' be derived from the restriction of P_s to the individual courses of $Ch_s\left(\mu(s)\cup\{c\}, P_s\left(m_s\right)\right)$. Then, $\left(m_s''\right)$ is a profitable deviation as well, yielding $Ch_s\left(\mu(s)\cup\{c\}, P_s\right)$. Thus, the pair (c,s) blocks allocation μ, which yields a contradiction. ■

The proof of Proposition 3 is similar to the proof of Proposition 2 and thus omitted.

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