

Risk Classification in Insurance Markets with Risk and Preference Heterogeneity

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Abstract

This paper studies a competitive model of insurance markets in which consumers are privately informed about their risk and risk preferences. We provide a characterization of the equilibria, which depend non-trivially on consumers' type distribution, a desirable feature for policy analysis. The use of consumer characteristics for risk classification is modeled as the disclosure of a public informative signal. A novel property of signals, monotonicity, is shown to be necessary and sufficient for their release to be welfare improving for almost all consumer types. We also study the effect of changes to the risk distribution in the population as the result of demographic changes or policy interventions. We show that an increase in the risk distribution, according to the monotone likelihood ratio ordering of distribution, leads to lower utility for almost all consumer types. In contrast, the effect is ambiguous when considering the first order stochastic dominance ordering.

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1 Introduction

Risk classification is a natural consequence of profit maximization by insurance providers in competitive markets. It consists of using individual information to predict or assess the risk level of potential insurees, which may involve both medical information (e.g, pre-existing conditions or medical history) as well as demographic characteristics (e.g., income, age or gender). As highlighted in Handel et al. (2015) (henceforth HHW), allowing for more precise risk classification has the potential of reducing the information asymmetry between consumers and firms and hence alleviate the problem of adverse selection. But at the same time it implies that consumers with different observables may be offered very different premia for the same coverage. This dispersion in prices and, as a consequence, in consumption may have adverse effects on welfare. Hence, a trade-off arises between reducing adverse selection and price dispersion. Models commonly used in the insurance literature face limitations to address this issue as they either assume that agents' privately observed characteristics can be summarized by a one-dimensional variable (Rothschild and Stiglitz (1976)) or exogenously restrict the set of available contracts to have only one or two elements (Akerlof (1970), HHW). We analyze a parsimonious and tractable model with a rich set of contracts and consumer characteristics, in which changes to the distribution of risk and preferences — which may be the result of demographic changes or policy interventions such as risk classification — have non-trivial effects on both prices and welfare.

In our model, consumers are privately informed both about their risk level and risk preferences, the two main dimensions of heterogeneity emphasized in the empirical literature (see, for example, Cohen and Einav (2007)). More specifically, consumers have normally distributed losses and their preferences exhibit constant absolute risk aversion.¹ Consumers' private information concerns both the expected value of the loss they face (capturing their risk level), which has a continuous distribution, and their risk aversion, which has a binary distribution. The difference between the two levels of risk aversion is a key parameter of our model, which captures the amount of preference heterogeneity.

We follow Dubey and Geanakoplos (2002), Bisin and Gottardi (2006) and, more recently, Azevedo and Gottlieb (2017) – henceforth AG – in assuming that consumers and firms act as price takers and have consistent beliefs regarding the types of consumers trading any contract which may be offered. In equilibrium all contracts that are traded generate zero profits.

We differ from the previous literature on risk classification (in particular, HHW, Finkelstein et al. (2009) and Garcia and Tsur (2021)) by allowing for a continuum of different levels of coverage to be available for trade in equilibrium. In the presence of such a rich set of contracts, competitive equilibria in models with one-dimensional heterogeneity (only given by risk) are fully separating, which implies that the risk distribution only affects prices through its support (see Rothschild and Stiglitz (1976), Dubey and Geanakoplos (2002) and AG).² An irrelevance result then follows: for example, if the set of possible risk levels for both men and women in the population are the same, the introduction of gender-based pricing has zero effect. Conversely, in the presence of multiple dimensions of private information, consumers of different types may share the same willingness to pay for coverage at the margin and, as a consequence, equilibria may feature some pooling. As each insurance contract is priced on the basis of the average risk level of consumers choosing it, any change in the type distribution has a direct effect on prices since it affects the relative frequency of the different types purchasing the same contract. It also has an indirect effect, as price changes affect contract choices and hence which consumer types are pooled together. These effects allow us to obtain interesting results for the consequences of risk classification.

¹This is in line with the estimated model in Einav et al. (2013) and the calibrated model in Azevedo and Gottlieb (2017).

²The same outcome is obtained as a directed search equilibrium in the presence of search frictions (Gale (1992), Guerrieri et al. (2010)); as a free-entry equilibrium in Rothschild and Stiglitz (1976), or as the unique pure-strategy possible equilibrium outcome in a game-theoretic version of that model (Farinha Luz (2017)). While a competitive or a search equilibrium always exists, free-entry equilibria and pure strategy equilibria fail to exist for some risk distributions.

Our main results are the following. First, we show in Section 3 that equilibria always exist where a convex set of contracts is traded and exhibit the following pattern of trades. Contracts with very high or low levels of coverage feature separation, with each contract in these two regions purchased by a single type, respectively, with high risk and high risk aversion, or with low risk and low risk aversion. Contracts with intermediate levels of coverage are purchased instead by two types (an outcome we refer to as discrete pooling), who have different levels of risk but have the same marginal willingness to pay for coverage. Finally, at the boundary between each of the two separating regions and the discrete pooling region there is (at most) one coverage level featuring continuous pooling, i.e., a contract that is chosen by a positive mass of types. We show the equilibrium is not unique. In fact, a continuum of equilibria of this kind exists, a common feature when some pooling occurs.

Second, in Section 4 we focus on the case where preference heterogeneity is small relative to risk heterogeneity and provide a characterization of how equilibrium prices change with preference heterogeneity. In particular, we show that a well defined Taylor approximation of equilibrium prices exists, is independent of the equilibrium selection used and depends on the type distribution in a tractable way. This approximation allows us to characterize how equilibrium prices and allocation vary with the type distribution and can be used as a basis for meaningful comparative static analysis that is independent of equilibrium selection. In this section we also present a numerical approach to study competitive equilibria in our model, which allows us to investigate the robustness of our results to larger levels of preference heterogeneity.

Third, in Section 5 we analyze the consequences of risk classification by considering the price and welfare effects of the disclosure of a public signal that provides partial information to firms regarding consumers' risk level. For example, the signal may correspond to some demographic characteristics of consumers, which are correlated with their risk. The availability of this signal - or simply the possibility to use it in pricing - leads to market segmentation, since firms will treat consumers differently according to their public signal realization. We analyze the welfare impact of the signal release from an interim perspective, by comparing the utility of any consumer's risk-and-preference type in equilibrium without and with public signal, taking for the latter the expectation across signal realizations.

Our main result identifies a novel property of signals, monotonicity, that we show is necessary and sufficient for its release to be interim Pareto welfare improving for any type distribution. Monotonicity is expressed in terms of the Kullback-Leibler (KL) divergence measure, or relative entropy. Consider two arbitrary types A and B . Monotonicity requires that, as the risk dimension of type B becomes more distant from that of type A , the signal distributions associated with both types become more distinct, in the sense of the KL measure. Signal monotonicity is implied by the standard monotone likelihood ratio property of signals used in contract theory, but is weaker in general.

Intuitively, the disclosure of an informative signal has a first, direct effect on prices: keeping the equilibrium allocation fixed, the disclosure of a signal leads to a mean-preserving spread in the price of each coverage, as contracts are priced on the basis of the expected risk level of the pool of consumers purchasing that coverage. This expectation is computed conditionally on the information available to firms which changes when a signal becomes available. However, the disclosure of a signal has also an indirect equilibrium effect by modifying the set of types that are pooled in each contract. We show that, if the signal is monotonic, this indirect effect due to changes in the equilibrium allocation amplifies price reductions and dampens price increases resulting from different signal realizations. As a consequence, signal disclosure leads to an expected price *reduction*. This expected price reduction is shown to generate an expected utility gain for almost all consumer types, a surprising result given the presence of risk aversion in our model. This result is established analytically for low levels of preference heterogeneity. In Subsection 5.5 we use our numerical approach to study the robustness of this finding. We show its validity extends to larger levels of preference heterogeneity, and more so when the levels of

risk aversion in the population are not too high. We also show it extends to richer preference heterogeneity, with more than two levels of risk aversion.

Finally, Section 6 analyses the effects on equilibrium allocations and welfare of changes in the type distribution in the population. We focus on changes in the type distribution entailing a reduction in risk in the population while maintaining the same support (that is, with smaller mass on riskier types). In our model the effects of these changes on prices and welfare are rather subtle. First we show that a decrease in risk in the first order stochastic dominance (FOSD) sense, i.e., an increase of the mass of risks in the left tail, may be harmful for a positive mass of consumers. The intuition is that the partial screening of types which occurs in equilibrium means that equilibrium prices are affected by changes to the risk distribution within the pool of consumers purchasing each coverage, not in the overall population. When instead we have a decrease in risk according to the stronger ordering given by the monotone likelihood ratio property (MLRP), we show that the relative frequency of riskier types within each pool selecting the same coverage level indeed decreases. As a consequence, a decrease in risk in the MLRP sense leads to a reduction in prices for almost all coverage levels and, as a consequence, almost all consumer types are better off. Since a FOSD reduction in risks may be harmful, our results suggest that policies motivated by their impact on the distribution of risk among consumers buying insurance in the market, such as mandates,³ should be examined carefully in the presence of competitive screening and multi-dimensional heterogeneity.

Finally, we also study changes in the distribution of risk preferences in the population and show that an increase in the share of high-risk-aversion consumers leads to lower equilibrium prices for almost all contracts and higher utility levels for almost all types. Intuitively, when preference heterogeneity is sufficiently small, almost all contracts are purchased by a pair of types: a low-risk-high-risk-aversion type and a high-risk-low-risk-aversion type, with the first one cross-subsidizing the second. Hence, an increase in the relative mass of high-risk-aversion consumers reduces the average risk of such pools and as a result induces a decrease in prices.

Related literature

Our paper is related to the empirical and theoretical literatures on risk classification and market outcomes with multidimensional private information.

HHW study risk classification in health insurance markets. They consider a competitive model where insurance firms provide two exogenously given coverage levels and a mandate is in place, motivated by health care exchanges under the U.S. Affordable Care Act. In this environment a unique equilibrium is shown to exist. Using data from a large employer, HHW estimate the joint distribution of demographic characteristics, preferences and risk levels in the population and perform numerical simulations of their model to study different types of risk classification.

The paper focuses on risk classification based on age and health status. The use of age information in pricing does not fully remove the informational asymmetries in this market; the authors find that all consumers choose the low coverage contract in equilibrium – the same outcome obtained when no individual information is used. Hence, age-based pricing has no effect on coverage levels and only leads to price dispersion, with younger consumers paying lower premia relative to older ones. HHW also study the use of health information in pricing,

³For example, the U.S. code currently states that a health insurance mandate, as proposed by the 2010 Affordable Care Act, “will minimize this adverse selection and broaden the health insurance risk pool to include healthy individuals, which will lower health insurance premiums” (42 U.S.C.A. § 18091(a)(2)(I); see also Parmet (2011)). We do not consider mandates explicitly, so our analysis should be interpreted as simply raising some questions regarding the possible effects of mandates. A comprehensive analysis of mandates in the presence of multi-dimensional heterogeneity and competitive screening is an important topic for future research.

See also Levy and Veiga (2022) for a formal treatment of mandated contracts in a one-dimensional (risk) version of the model developed in this paper. They show that equilibrium generically exhibits partial pooling and increases in the minimal allowed coverage have ambiguous effects on welfare.

potentially removing all information asymmetry between firms and consumers. As a consequence it leads to an increase in coverage, but also generates large variations in prices, for any given level of coverage. This price dispersion decreases ex-ante welfare and HHW find, based on their estimates for the parameters of the model, that this negative effect dominates the positive one due to higher coverage.

Our analysis differ from theirs in three main ways. First, our model allows for a continuum of possible coverage levels. As a consequence, more information about consumers is revealed by their coverage choices, even in the absence of risk classification. Second, we restrict attention to signals that are only partially informative about consumers' risk levels. For example, this includes the use of age, but not health status, as modelled in their paper, in pricing. Finally, we focus on interim welfare, and not ex-ante, making our welfare improvement results stronger. Furthermore, our specification allows us to characterize analytically the effects of risk classification and of changes in the risk-distribution on equilibrium prices and welfare, instead of relying on numerical simulations.

In a related contribution, Finkelstein et al. (2009) study the effects of gender-based pricing in the UK annuities market. Their analysis is based on a two-type model, allows for a large space of contracts, and considers two possible allocations: the equilibrium outcome proposed by Miyazaki (1977); Wilson (1977); Spence (1978), with separation of types potential cross-subsidies across contracts, and complete pooling with actuarially fair prices. The use of gender in pricing has a pure price effect in the presence of full pooling; but affects both prices and coverage in the presence of screening. The effect of a ban on gender-based pricing is quantified numerically through a calibration exercise using annuities data. A ban on gender pricing is shown to hurt women in the population, but this effect is stronger with complete pooling and weaker in the presence of screening. The reason is that the presence of screening leads to endogenous separation of types, and hence less cross-subsidies across genders, even in the absence of gender pricing.

Ericson and Starc (2015) study age-based pricing in the context of the Massachusetts health insurance exchanges. This market features price regulation, represented by a limit on the ratio of prices across groups. A model of imperfect competition where firms only choose prices for each group is proposed to perform counterfactual analysis. Using their estimated costs and demand functions, they show that price regulation in this particular market lead to transfers from the lower-risk group to the higher risk one, and to an overall increase in consumer surplus.

On the empirical side, the presence of multidimensional consumer heterogeneity has been established in the literature on structural estimation of insurance demand. See for instance Cohen and Einav (2007) and Einav et al. (2013). While these papers provide rich estimates of the sources of consumer heterogeneity and drivers of demand, they do not perform model-based counter-factual analysis that fully incorporate supply-side responses to policy interventions.

Garcia and Tsur (2021) study the problem of designing an informative signal in a competitive insurance market with one-dimensional private information about risk and a single exogenously given level of coverage. For any signal realization, they consider the most efficient Akerlof (1970) equilibrium. The optimal signal maximizes the ex-ante expected utility of consumers and is shown to be partially informative: types in both extremes of the risk distribution are "grouped" together by receiving the same signal realization. Levin (2001) investigates in the framework of the Akerlof (1970) model the impact of better information on gains from trade. The paper finds that better information on the uninformed side (analogous to firms in our model) always increases total trade gains. Goldstein and Leitner (2018) study optimal disclosure policy of a regulator that has information about banks. Disclosure can destroy risk-sharing opportunities for banks (the Hirshleifer effect), but some level of disclosure is necessary for risk sharing to occur. Under some conditions the optimal disclosure can take simple forms (full or no disclosure, as well as cutoff disclosure).

A few papers have studied the role of categorical price discrimination in insurance markets focusing on

interim welfare notion similar to ours, taking expectations of payoffs across different categories, for any fixed risk type. Crocker and Snow (1986) study the effect of a risk-informative signal on the incentive-constrained Pareto frontier. Rothschild (2011) considers, in a two-type model, multiple strategic equilibrium concepts proposed in the literature and shows that firms’ access to an informative signal combined with a government-provided partial insurance policy leads to an interim Pareto improvement. Veiga (2022) studies the regulation of price discrimination based on observables (such as age group) in a competitive insurance model when a single contract is available, hence à la Akerlof (1970). The paper characterizes the optimal price discrimination policy, which may correspond to zero, full or partial price discrimination depending on model parameters.

The notion of market outcome we consider is related to the broad competitive screening literature which followed the works of Rothschild and Stiglitz (1976); Miyazaki (1977); Riley (1979). We use a Walrasian market equilibrium notion following Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006). These papers show that, in the presence of one-dimensional risk heterogeneity, competitive equilibria exist and the equilibrium allocation features perfect separation. Guerrieri et al. (2010) show that the same outcome also obtains in the presence of search frictions, as a result of a competitive search equilibrium.

The analysis of multidimensional private information in competitive environments has been tackled by a few notable papers. AG established the existence of competitive equilibria in general environments that allow for multidimensional consumer heterogeneity, but no characterization is provided beyond one-dimensional environments. Results in this respect can be found in Guerrieri and Shimer (2018) and Chang (2018),⁴ using a directed search approach. A common feature of the models considered in both papers is that the willingness to trade of informed agents depends on a one-dimensional sufficient statistic. This implies that types can be partitioned based on their willingness to trade, and all types in the same element of the partition are pooled together in equilibrium regardless of the type distribution and of their level of trade. This property is not present in our model: the set of types sharing the same marginal willingness to pay in equilibrium is not exogenously given and varies with the type distribution. This is a general property of insurance models, since consumers’ risk aversion has only a small effect on their willingness to pay for additional coverage when their risk exposure is small. Fang and Wu (2018) study the correlation between coverage and risk in a multidimensional insurance model and find that this correlation is positive with perfect competition but may be negative in a monopoly environment.

Another strand of the literature introduces a second dimension of heterogeneity in the RS framework besides risk, focusing on two-by-two models. Smart (2000) and Villeneuve (2003) assume that consumers also differ in their privately known degrees of risk aversion, while Wambach (2000) introduces heterogeneity with respect to privately known initial wealth. These papers focus on strategic equilibrium notions — differently from us — and, as a consequence, existence cannot be guaranteed. These authors highlight that pooling and strictly positive profits may obtain in equilibrium. In a follow-up paper, Snow (2009) shows that equilibria with positive profits do not exist if firms can offer multiple contracts but does not characterize what outcomes arise in equilibrium.

2 Model

A continuum of consumers (potential insurees) face income uncertainty due to the possibility of negative shocks. More specifically, suppose that consumers have income W and can suffer a loss \tilde{l} distributed according to the normal distribution $N(\mu, 1)$. Consumers can purchase insurance contracts which are characterized by a pair $(x, p) \in [0, 1] \times \mathbb{R}$, with $x \in (0, 1)$ denoting the insurance coverage (i.e., the fraction of the loss reimbursed) and $p \geq 0$ the premium paid.⁵ Consumer’s preferences are described by a constant absolute risk aversion (CARA)

⁴See also Williams (2021).

⁵We allow for reimbursement policies that are linear in losses. This specification is quite rich and appears fairly natural in applications. In a principal-agent moral hazard model where average loss is controlled by the consumer, Holmstrom and Milgrom

utility function with parameter $\rho > 0$. Hence, the expected utility of a consumer choosing contract (x, p) is given by

$$\begin{aligned} v(x, p; \mu, \rho) &\equiv \mathbb{E} \left\{ -\exp \left[-\rho \left(W - (1-x)\tilde{l} - p \right) \right] \right\} \\ &= -\exp \left\{ -\rho \left[W - (1-x)\mu - \frac{\rho}{2}(1-x)^2 - p \right] \right\}, \end{aligned} \quad (1)$$

where the second equality is a consequence of the exponential structure of both the CARA utility and the normal distribution, which gives us a quadratic expression for the consumers' certainty equivalent (see Einav et al. (2013) and AG for applications of the normal-CARA specification).

We assume that consumers are privately informed about both their risk level μ and their level of risk aversion ρ .⁶ The set of possible risk levels in the population is an interval⁷ $[\mu_L, \mu_H]$ while, for tractability, the set of possible risk aversion values is given by two values, $\rho_l \equiv \rho_0 - \delta/2$ and $\rho_h \equiv \rho_0 + \delta/2$ for some $\delta \geq 0$ describing the level of heterogeneity of risk preference in the population. Hence, the type space is $\Theta \equiv [\mu_L, \mu_H] \times \{l, h\}$ and a generic type is denoted by $\theta = (\mu, i)$.

Types are distributed over Θ according to a probability distribution described by a pair of strictly positive and twice continuously differentiable functions $\phi = (\phi_l, \phi_h)$ on $[\mu_L, \mu_H]$, i.e., for any measurable set $A \subset [\mu_L, \mu_H]$ and $i \in \{l, h\}$

$$\mathbb{P}[\theta \in A \times \{i\}] = \int_{\mu_L}^{\mu_H} \mathbf{1}_{\{\tilde{\mu} \in A\}} \phi_i(\tilde{\mu}) d\tilde{\mu}.$$

Using expression (1), consumer preferences over contracts (x, p) can be represented by the following quasi-linear utility function:

$$u(x, \theta) - p, \quad (2)$$

where $u(x, \theta) = x\mu - \frac{\rho_i}{2}(1-x)^2$. Given the CARA specification, the wealth level W does not affect consumers' preferences for contracts and can then be omitted for simplicity from the expression of $u(\cdot)$. Our equilibrium characterization uses this linear-quadratic expression of consumers' utility.

The insurees' marginal rate of substitution between coverage and price, which determines their willingness to pay for coverage, is

$$u_x(x, \theta) = \mu + \rho_i(1-x), \quad (3)$$

where u_x stands for the partial derivative notation.

From expression (3), we see that a consumer's high willingness to pay for coverage can be due to a high risk level (high μ) or to a high risk aversion level (high ρ). Importantly, the degree to which risk aversion affects one's willingness to pay depends on the level of coverage. For example, risk aversion has a very small effect on one's marginal willingness to pay for coverage if the starting point is close to full coverage. The set of consumers' types who exhibit the same willingness to pay for a given coverage level x plays an important role in our analysis. We see from (3) that this set depends on x and hence will be determined in equilibrium together with the allocation of contracts.⁸

(1987) show that linear contracts are indeed optimal. It is beyond the scope of this paper to show that the linearity of contracts is without loss in our adverse selection competitive setting.

For an example with more complex contracts, see Crocker and Snow (2011), which allow for firms to offer differential coverage for different sources of risk. This richness is shown to improve firms' screening ability, increasing efficiency and mitigating the nonexistence issue.

⁶The model can also allow for private information in the variance of the loss. If $\tilde{l} \sim N(\mu, \sqrt{\sigma})$, it can be shown that the utility of a consumer with contract (x, p) only depends on variance σ and risk aversion ρ through their product $\sigma\rho$. Hence, our analysis extends to this case, provided that we redefine the parameter ρ as representing the product of risk aversion and risk variance.

⁷Levy and Veiga (2021) study competitive insurance markets with unbounded risk distributions, which is out of the scope of our paper. They point out to that equilibria may not exist in general and derive a necessary and sufficient condition for its existence.

⁸As mentioned in the introduction, this feature distinguishes our model from Guerrieri and Shimer (2018) and Chang (2018),

Firms are risk neutral and their expected profit from selling contract (x, p) to a consumer of type $\theta = (\mu, i)$ is given by the price minus the cost of coverage provision for the risk component of the consumer's type:

$$p - c(x, \theta),$$

where $c(x, \theta) = \mu x$. This means that risk aversion ρ is a private-value component of the private information while risk μ is a common-value component of the private information of an individual as it affects the expected costs of firms providing insurance. The number of firms is irrelevant, as they are price takers and have constant returns to scale in insurance provision.

We consider a Walrasian competitive equilibrium notion, meaning that all consumers and firms take prices for all contracts as given. A price function is denoted as $p : [0, 1] \rightarrow \mathbb{R}$.⁹ An allocation is a measurable function $t : \Theta \rightarrow [0, 1]$. We denote the joint distribution over $[0, 1] \times \Theta$ induced by the prior type distribution and an allocation t as \mathbb{P}_t .

Firms have beliefs regarding the set of types that would purchase each possible coverage level if offered, described by $P(\cdot | x) \in \Delta(\Theta)$, for $x \in [0, 1]$. We denote the expected risk level of consumers choosing coverage level $x \in [0, 1]$, using belief system P , by $\mathbb{E}_P[\tilde{\mu} | x]$.

In equilibrium consumers maximize their utility, given prices $p(\cdot)$, which means $t(\theta)$ must solve

$$U(\theta) \equiv \max_{x \in [0, 1]} u(x, \theta) - p(x), \text{ for all } \theta \in \Theta. \quad (4)$$

Insurance firms have constant returns to scale in the quantity traded of each contract. Given prices and belief system P , firms maximize profits if, and only if, no contract generates strictly positive profits and any traded contract makes zero expected profits, i.e., for all $x \in t(\Theta)$ and all $\hat{x} \in [0, 1]$,

$$0 = p(x) - x\mathbb{E}_P[\tilde{\mu} | x] \geq p(\hat{x}) - x\mathbb{E}_P[\tilde{\mu} | \hat{x}]. \quad (5)$$

A competitive equilibrium is given by a price function, an allocation and a belief system that satisfy consumers' and firms' optimality, belief consistency and additional restrictions for non-traded contracts.

Definition 1. A triple of price, allocation and belief (p, t, P) is an equilibrium if it satisfies:

- (1) firm's optimality: condition (5) holds;
- (2) consumer's optimality: $t(\theta)$ solves (4);
- (3) contracts pricing: for any $x \in [0, 1]$

$$p(x) = \max \{u(x, \theta) - U(\theta) \mid \theta \in \Theta\}, \quad (6)$$

where $U(\theta)$ is defined in (4);

(4) belief consistency: $P(\cdot | t(\tilde{\mu}, \tilde{i})) = \mathbb{P}_t(\cdot | t(\tilde{\mu}, \tilde{i}))$ with probability one and, for any $x \in [0, 1]$, the support of $P(\cdot | x)$ is contained in

$$\theta^+(x) \equiv \{\theta \in \Theta \mid U(\theta) = u(x, \theta) - p(x)\}.$$

Conditions (1) and (2) are standard in competitive settings. Condition (3) states that any contract x — even if not traded in equilibrium — has its price determined by the maximal willingness to pay for it across all consumers. That is, the price $p(x)$ is such that at least one type $\theta \in \Theta$ is indifferent between choosing coverage

where the set of types exhibiting the same willingness to pay does not depend on their level of trade. The reason is that in those papers consumers' willingness to trade depends on a one-dimensional sufficient statistic for their type, while this is not true in our environment.

⁹With some abuse of notation, we use p to denote both the price function and the price of a given contract.

x and his equilibrium coverage $t(\theta)$, while other types strictly prefer another contract.¹⁰ This restriction is analogous to the ones proposed by Dubey and Geanakoplos (2002); Bisin and Gottardi (2006); Guerrieri et al. (2010), AG,¹¹ and is in the spirit of subgame perfection. Condition (4) requires that, for all contracts that are traded in equilibrium, beliefs P are consistent with Bayes' rule. For the non-traded contracts it requires that firms believe that the types that would trade them if offered are the ones most willing to trade them at the equilibrium price.

Conditions (3) and (4) are critical, as they rule out trivial equilibria in which any subset of contracts remains non-traded and their prices are set at an arbitrarily high level. If all types most willing to trade a particular contract have costs strictly below their willingness to pay (per unit), this would generate an opportunity for strictly positive profits by firms.

3 Equilibrium characterization

In this section, we characterize equilibrium allocation and prices, starting from the benchmark case where consumers only differ by their risk level, that is where private information is one-dimensional ($\delta = 0$), and extending it subsequently to the environment where there is also preference heterogeneity ($\delta > 0$).

One-dimensional types

Suppose all consumers have the same risk aversion, i.e., $\delta = 0$. In this case, a unique equilibrium exists, the equilibrium is fully separating, with efficiency at the top, and only depends on the support of the risk distribution (the proof is readily obtained by suitably adapting the arguments in Dubey and Geanakoplos (2002) or AG). Let us refer to the map associating, for any traded coverage level $x \in [0, 1]$, the risk of the unique type purchasing x in equilibrium as the type assignment function $m_0(x)$.

We refer to the derivative of equilibrium objects with respect to coverage x by using the dot notation, as in $\dot{p}(x)$, to describe the marginal price function. Definition 1 requires that contracts make zero profits and that consumers choose their coverage level optimally:

$$\frac{p(x)}{x} = m_0(x) \text{ and } \dot{p}(x) = u_x(x, m_0(x), i) = m_0(x) + \rho_0(1 - x). \quad (7)$$

These two equations define a simple differential system which has a unique solution satisfying $m_0(1) = \mu_H$ (efficiency at the top), given by:

$$m_0(x) = \rho_0(1 - x + \ln x) + \mu_H, \quad (8)$$

for all $x \in [x_L, 1]$, where x_L is the lowest traded coverage defined by $m_0(x_L) = \mu_L$. The equilibrium allocation $t_0(\cdot)$ is the inverse of the map $m_0(\cdot)$ and the price function is given by $p_0(x) = xm_0(x)$. These objects will play a role in our equilibrium analysis of the case $\delta > 0$.

¹⁰Note that the prices for certain contracts, as determined by condition (3), may be negative, or more generally firms may make negative profits for some values of x . In this case firms do not offer them and they are not traded in equilibrium. The price of non-traded contracts could be equivalently set at any value lying between the maximal willingness to pay among consumers for such contracts and the expected cost using belief system P .

¹¹In the case where private information is one dimensional, $\delta = 0$ in our setting, all these specifications yield the same, unique, fully separating equilibrium outcome, as in Riley (1979). In more general environments, with multidimensional private information, this equivalence may no longer hold. In particular, as shown by Proposition 1 in Azevedo and Gottlieb (2017) every competitive equilibrium according to AG is also an equilibrium according to our definition, but the reverse may not be true.

Two-dimensional types

For the remainder of the paper we assume that $\delta > 0$, that is, the type space is two-dimensional. In this case we will show that all equilibria feature partial pooling. Hence, the allocation $t : \Theta \rightarrow [0, 1]$ has a generalized inverse represented by a correspondence that gives us the set of types choosing a particular coverage. It may be either empty (in the case of a non-traded contract) or include multiple types (for a contract with pooling). We call this correspondence the type assignment. It can be written as the composition of two mappings $m_i^t : [0, 1] \rightrightarrows [\mu_L, \mu_H]$, for $i \in \{l, h\}$. For each $x \in [0, 1]$, $m_i^t(x)$ represents the risk levels of the types purchasing x and having risk aversion ρ_i , i.e., it is given by $\{\mu \in [\mu_L, \mu_H] \mid t(\mu, i) = x\}$. Alternatively, the set of types choosing coverage x is $[m_l^t(x) \times \{l\}] \cup [m_h^t(x) \times \{h\}]$. Whenever this set is a singleton, we also refer to it as a type assignment function. When the allocation is well understood, the dependence on t is omitted. We then say that the correspondence $m_i(\cdot)$ is non-decreasing if, for any $x' < x$ with $x', x \in [0, 1]$ and $m_i(x), m_i(x') \neq \emptyset$, we have that $\sup m_i(x') \leq \inf m_i(x)$. Additionally, we say that $m_l(x) \geq m_h(x)$ if $\sup m_l(x) \geq \sup m_h(x)$ and $\inf m_l(x) \geq \inf m_h(x)$. The following lemma provides some important properties of the equilibrium allocation and price functions.

Lemma 1. *If (m_l, m_h) are equilibrium type assignments, then:*

- (i) $m_i(\cdot)$ is non-decreasing;
- (ii) for any x such that $m_l(x), m_h(x) \neq \emptyset$, $m_l(x) \geq m_h(x)$;
- (iii) p is an increasing Lipschitz function.

Properties (i) and (ii) of Lemma 1 hold since preferences satisfy the single-crossing property (SCP) on each dimension of the consumer's type (either μ or i), holding the other dimension fixed. Conditional on the consumer's level of risk aversion (risk level), an increase in the risk level (risk aversion) increases the consumer's willingness to pay for coverage. However, the two-dimensional type space does not satisfy globally the single-crossing property, i.e., one can find types (μ, i) and (μ', i') with indifference curves that cross twice. Lemma 1 (iii) follows directly from the equi-Lipschitz property of $u(\cdot, \mu, i)$.

No gap equilibria We restrict attention to equilibria in which the set of traded contracts is convex and of the form $[x_L, 1]$, and we refer to them as no gap equilibria. In Subsection 3.1 (Proposition 2) we show that such equilibria always exist when preference heterogeneity is small. In the benchmark one-dimensional model, with $\delta = 0$, only no-gap equilibria exist.

Our equilibrium characterization relies on a condition on the type distribution which implies that types with higher willingness to pay for insurance have on average higher risk, a property which characterizes the presence of adverse selection in the market. From equation (3) it follows that there are at most two different risk levels of types with the same marginal willingness to pay q for coverage x , given by $(q - (1 - x)\rho_i, i)$, for $i = l, h$ (whenever both values are in $[\mu_L, \mu_H]$). Let us define the function $e(\cdot)$ as the expected risk level of these two types, evaluated with the relative probability weights given by $\phi_l(q - (1 - x)\rho_l)$ and $\phi_h(q - (1 - x)\rho_h)$:

$$e(q, x) \equiv q - (1 - x) \left\{ \rho_0 + \frac{\delta \phi_h(q - (1 - x)\rho_h) - \phi_l(q - (1 - x)\rho_l)}{2 \phi_h(q - (1 - x)\rho_h) + \phi_l(q - (1 - x)\rho_l)} \right\}, \quad (9)$$

with domain $E \equiv \{(q, x) \in \mathbb{R}_+ \times [0, 1] \mid q \in [\mu_L + (1 - x)\rho_h, \mu_H + (1 - x)\rho_l]\}$. We assume the following.

Assumption 1. The function $e(q, x)$ is strictly increasing in q , for all $x \in [0, 1]$.

Such condition is always satisfied in the one-dimensional environment where agents only differ in terms of their risk. With two-dimensional heterogeneity it is a joint restriction on the distribution and other model parameters. If risk and risk aversion are independently distributed (i.e., $\phi_i(m) = \mathbb{P}(\tilde{i} = i) \phi(m)$), a sufficient

condition for Assumption 1 is that the risk density be log-convex,¹² with the uniform distribution being a special case. In Appendix A, we show that this assumption holds for $\delta > 0$ small enough, regardless of the form of the distribution functions (ϕ_l, ϕ_h) .¹³

Lemma 2. *There exists $\bar{\delta} > 0$ such that Assumption 1 holds if $\delta \in (0, \bar{\delta})$.*

The next result provides a characterization of equilibria under small preference heterogeneity. The set of traded coverages can be divided into three open sub-intervals and two coverage levels in the transition between them. The highest (lowest) levels of coverage traded in equilibrium feature separation, i.e., each of these contracts is purchased by a single type with high (low) risk-aversion and levels of risk on the right (left) tail of the risk distribution. Coverage levels in the middle open interval feature instead partial pooling as they are purchased by two types: one with low risk aversion and higher risk level, and another with high risk aversion and lower risk. We refer to this type of pooling as discrete pooling. Finally, the two intermediate coverage levels that determine the boundary between these intervals are purchased by a continuum of types with positive measure, which we refer to as continuous pooling.

Proposition 1. *Suppose that Assumption 1 is satisfied. In any (no gap) equilibrium there exist values $0 < x_L < x_d \leq x_u < 1$ such that:*

- (i) (separation) there are two separating intervals $[x_L, x_d)$ and $(x_u, 1]$, with $x_d \leq x_u$; each $x \in (x_L, x_d)$ (resp. $x \in (x_u, 1)$) is chosen by only one type with low (resp. high) risk aversion;
- (ii) (continuous pooling) there are non-degenerate intervals of risk types choosing coverage x_d and x_u ;
- (iii) (discrete pooling) for each $x \in (x_d, x_u)$ there are exactly one low and one high risk aversion types choosing coverage x .

Figure 1 provides a graphical representation of an equilibrium. The left panel represents the type assignments $m_l(\cdot)$ (in red) and $m_h(\cdot)$ (in blue), as well as the price per unit (in black), for varying levels of coverage $x \in [0, 1]$. Coverage levels above x_u (or below x_d) feature separation and are purchased by a single type with high (or low) risk aversion. In the separating regions, the unit price function coincides with the risk level of the type determined by the type assignment function. The middle region between x_d and x_u features discrete pooling, where both type assignments $m_i(\cdot)$ and $i \in \{l, h\}$ are singletons. The unit price of each contract is a convex combination of the risk level of the agents purchasing it and hence lies between the values of the two type assignments. The transition points x_d and x_u feature continuous pooling, which is represented by a point where the type assignment mapping is multi-valued.

The right panel of Figure 1 represents the same equilibrium in the $(\rho \times \mu)$ -space and describes in which of the three equilibrium regions each type lies. Once again, the set of types in the extremes lie in the separating region and are represented in this case by the red region. The middle blue region represents the sets of types in the discrete pooling region, with two dashed blue lines connecting the two types choosing, respectively, coverage x_1 and coverage x_2 . Finally, the green region indicates the positive mass of types choosing coverage levels x_d and x_u , that is the continuous pooling region.

To characterize a no gap equilibrium we need to determine the regions in $[0, 1]$ where separation, continuous and discrete pooling occur, as well as the level of coverage chosen by each type in any of these regions. As specified

¹²Consider any $x \in [0, 1]$ and let $k_i \equiv (1-x)\rho_i$. The second term in (9) is non-decreasing in q as long as

$$\frac{\phi_l(q - (1-x)\rho_l)}{\phi_h(q - (1-x)\rho_l - (1-x)\delta)} = \frac{\mathbb{P}(\tilde{i} = l) \phi(q - k_l)}{\mathbb{P}(\tilde{i} = h) \phi(q - k_h)}$$

is non-decreasing in q , which is implied by log-convexity of $\phi(\cdot)$ as $k_l < k_h$.

¹³Guerrieri and Shimer (2018) make an analogous assumption in their setting. Chang (2018)'s model features equilibria with positive-mass pools which follow from violations of this adverse selection assumption — the average benefit of uninformed traders from trading with informed agents is non-monotonic in the latter's willingness to trade.

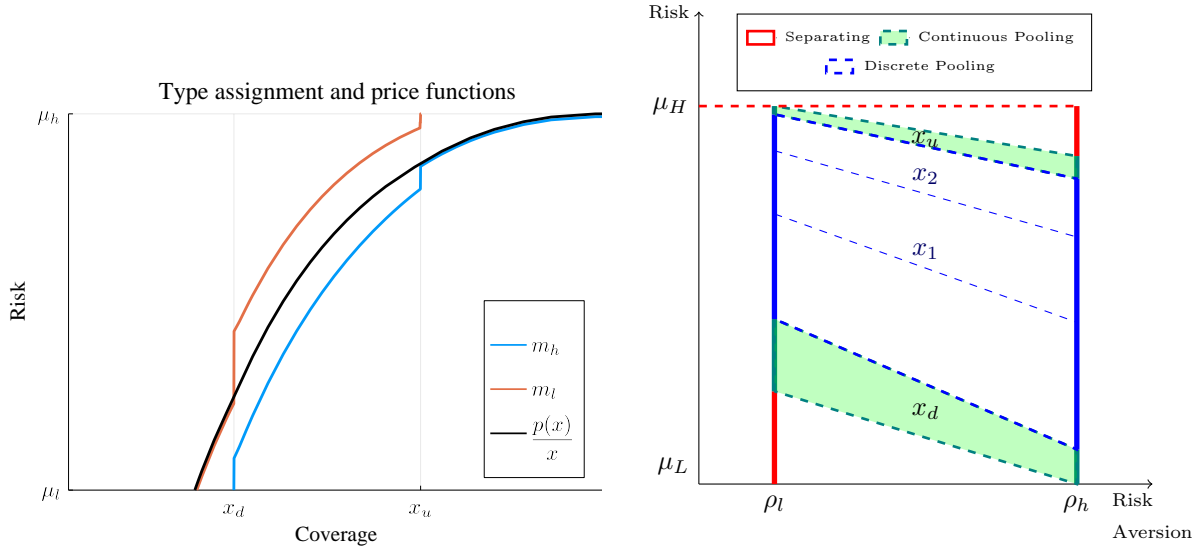


Figure 1: Equilibrium for the case where risk and risk aversion are independently distributed with $\rho_0 = 3$, $\delta = 1$, $\mu \sim U[2, 4]$ and $\mathbb{P}(i = l) = 0.5$.

in Definition 1, firms' beliefs are determined by Bayes' rule for all contracts $x \in [x_L, 1]$. For all contracts $x < x_L$ that are not traded in equilibrium beliefs put all mass on the type most willing to trade x (i.e., type (μ_L, l)) and the price $p(x)$ is such that it makes this type indifferent between any such contract and coverage x_L .¹⁴

Separation

For each contract x in a separating region, there exists a unique type (μ, σ) assigned to it. The characterization of equilibrium prices and trades in this region is similar to the one-dimensional case, since both consumers' and firms' optimality conditions are analogous to (7). Once a boundary condition of the derived differential system is determined, we can then solve for the type assignment and price functions in each of these regions. While the final condition in the top separating region is the same as in the one-dimensional case ($m_h(1) = \mu_H$), the final condition in the bottom separating region depends on the equilibrium type assignment in the other regions.

Continuous pooling

Lemma 1 and Proposition 1 imply that, in this region, coverage x is chosen by all types with risk aversion ρ_i and risk level in an interval $I_i \subset [\mu_L, \mu_H]$, for each $i \in \{l, h\}$. The zero profit condition (5) implies that the unit price of coverage x is determined by the average risk in the pool of consumers choosing x . While Lemma 1 shows that $p(\cdot)$ is continuous, it is not differentiable at x . Indeed, the marginal utility of coverage for all types choosing x must lie in $[\dot{p}(x^-), \dot{p}(x^+)]$,¹⁵ otherwise some types would want to deviate from x . Hence, if I_i has a positive measure, for either $i = l$ or h , the interval $[\dot{p}(x^-), \dot{p}(x^+)]$ also has positive length. Additionally, if the coverage level x is in the interior of the set of traded contracts, the left- and right-hand side limits of the marginal prices must match exactly the minimum and the maximum willingness to pay for coverage among types pooled

¹⁴Equilibria with gaps in the set of traded-contract have a similar structure to the no gap equilibria characterized in Proposition 1, i.e., the same pattern of pooling and separation of types. The only difference is that a set of non-traded coverage levels may exist in the transition between separation and pooling regions.

¹⁵We denote the left and right hand side limits of any function $f(\cdot)$ at x as $f(x^-)$ and $f(x^+)$, respectively, whenever they exist.

at coverage x .¹⁶ Formally, this means:

$$\frac{p(x)}{x} = \mathbb{E} \left[\tilde{\mu} | (\tilde{\mu}, \tilde{i}) \in \bigcup_{i \in \{l, h\}} I_i \times \{i\} \right]$$

$$[\dot{p}(x^-), \dot{p}(x^+)] = \{\mu + \rho_i(1-x) | (\mu, i) \in \bigcup_{i \in \{l, h\}} I_i \times \{i\}\}.$$

Alternatively, continuous pooling is necessary to ensure the continuity of prices, which must hold from Lemma 1, in the transition between discrete pooling regions, involving types with both high and low risk-aversion, to regions of separation, which only involve types with a single risk-aversion level.

Discrete pooling

For every coverage x in this region, the type assignments $m_l(\cdot)$ and $m_h(\cdot)$ are singletons. This means that contract (x, p) is selected by only two types: a low risk aversion type $(m_l(x), l)$ and a high risk aversion type $(m_h(x), h)$. In this case, the posterior probability of low-risk-aversion type $(m_l(x), l)$, conditional on choice x , is given by $w(x) \equiv \mathbb{P}(\tilde{i} = l | t(\tilde{\mu}, \tilde{i}) = x)$. The equilibrium requirements of zero profits (condition (1)) and consumers' optimality (condition (2)) become:

$$\frac{p(x)}{x} = w(x)m_l(x) + (1-w(x))m_h(x) \quad (10)$$

$$\dot{p}(x) = m_i(x) + \rho_i(1-x), \text{ for } i = l, h. \quad (11)$$

The calculation of the posterior beliefs at coverage x in the interior of the discrete pooling region is non-trivial, since the equilibrium distribution of coverages is absolutely continuous in a neighborhood of x (the mass of consumers purchasing contract x is zero). For this reason, the posterior $w(x)$ is determined by two components: the prior distribution and the slopes of the type assignment functions at x . The first element is intuitive: a higher mass of consumers with low, versus high, risk aversion – $\phi_l(\cdot)/\phi_h(\cdot)$ – leads to a higher posterior $w(\cdot)$. However, the slope of functions $m_i(\cdot)$ is also important, since a large slope of $m_l(x)$ means that the mass of consumers with low risk aversion is locally concentrated around x , which is connected with higher posterior $w(x)$. The following lemma formalizes this intuition.

Lemma 3. *In a discrete pooling region, the belief consistency condition 3 of Definition 1 is equivalent to*

$$\frac{w(x)}{1-w(x)} = \frac{\phi_l(m_l(x))\dot{m}_l(x)}{\phi_h(m_h(x))\dot{m}_h(x)}. \quad (12)$$

3.1 Equilibrium existence and multiplicity

In this subsection we show a competitive equilibrium always exists. If preference heterogeneity is small, in fact a continuum of no gap equilibria exist. Proposition 1 implies that all such equilibria have a similar structure and only differ in the continuous pooling coverage levels which determine the transitions between pooling and separating regions.

Proposition 2. *For sufficiently small $\delta > 0$, a continuum of no gap equilibria exist.*

Proposition 2 is proven by construction. The key building blocks of the equilibrium construction are the differential equations that govern prices and allocations within the discrete pooling and separating regions, which

¹⁶Notice that, for some $i \in \{l, h\}$, there must be a sequence of types $(\mu_n, i)_n$ and coverage levels $(x_n)_n$ such that x_n is optimal for type (μ_n, i) , $x_n \searrow x$ and $\mu \equiv \lim \mu_n \in I_i$. Hence, $[p(x_n) - p(x)](x_n - x)^{-1} \leq [u(x_n, \mu_n, i) - u(x, \mu_n, i)](x_n - x)^{-1}$. Taking the limit $n \rightarrow \infty$ we have that $u_x(x, \mu, i) \geq \dot{p}(x^+)$, this inequality must hold as an equality. An analogous argument holds for coverages below x .

have the structure described in Proposition 1. The boundary condition at $x = 1$ is determined by separation and efficiency at the top. Proceeding from the top to the bottom, each transition point corresponds to a continuous pooling point, and determines the boundary condition for the subsequent region of traded contracts. As mentioned before, to ensure the continuity of prices (a necessary condition, from Lemma 1) in the transition between a separating region, where a single type chooses each coverage, and a discrete pooling region, where types with different risk aversion levels are pooled together, the presence of continuous pooling is required. As a consequence, the equilibrium price function has a kink at these transition points. The multiplicity arises from a degree of freedom in the construction, which is given by the transition point between the top separating region and the discrete pooling region, denoted by x_u .

4 Approximation and numerical approach

The main goal of our analysis is to study risk classification, that is, the effects on equilibrium prices and allocations of the availability to firms of a signal on consumers' characteristics. While Proposition 1 provides a characterization of equilibria and shows how they depend on the distribution of consumers' characteristics, to study risk classification on that basis we face two main challenges. First, the presence of multiple equilibria established in Proposition 2 raises the question of whether comparative statics results that are robust to equilibrium selection are possible.¹⁷ Second, the discrete pooling region is governed by a non-linear differential system,¹⁸ which makes an analytical characterization difficult and closed form solutions impossible. In light of these challenges, we follow two complementary approaches to carry out this analysis. In Subsection 4.1 we show that by focusing on the case where preference heterogeneity is sufficiently small, we are able to overcome these two challenges. In Subsection 4.2 we describe an algorithm which is used to investigate the robustness of our results to: (i) non-trivial preference heterogeneity, (ii) more than two levels of risk aversion. The results of our numerical simulations are presented in Subsection 5.5.

4.1 Approximation approach

Our main approach to analyze the effects of the type distribution on the properties of equilibria in the two-dimensional model is to look at the one-dimensional environment as a limit case, and study the behavior of equilibria in its neighborhood. For an arbitrary equilibrium selection, we find a second order Taylor approximation for equilibrium prices and show that it exhibits two key properties: its polynomial coefficients are independent of the equilibrium selection used, and depend on the type distribution in a tractable and explicit way. We then use this tractable approximation to perform comparative statics exercises with respect to the type distribution, which are robust to equilibrium selection. The advantage of such approach is the possibility to use special properties of the one-dimensional case (uniqueness and distribution independence), while still highlighting novel results of the two-dimensional model, especially the fact that the equilibrium depends on the type distribution. In what follows, we make the dependence of equilibrium objects on the parameter δ explicit in our notation, e.g., prices are denoted as $p(x; \delta)$, for $x \in [0, 1]$.

4.1.1 The heuristic approximation

For presentation purposes, we start with a heuristic derivation of our approximation results, using a set of simplifying assumptions on the equilibrium selection and the type distribution, namely, (i) that equilibrium

¹⁷The multiplicity of equilibria is common in situations where we have some pooling and the distribution matters. In AG, for example, the equilibrium defined and numerically studied may not be unique beyond one-dimensional applications.

¹⁸For details, see Lemma 12 in the appendix.

objects are continuously differentiable in δ , (ii) that the set of contracts in the separating and continuous pooling regions vanish as $\delta \rightarrow 0$, (iii) and that risk and risk aversion are independently distributed. On this basis, the approximation of equilibrium prices is obtained simply by taking a total derivative of equilibrium variables with respect to the parameter δ and evaluating them at $\delta = 0$. The technical derivation of the formal approximation result is then presented in the next Subsection 4.1.2 and **does not** use any of the above assumptions.

We start by using the local optimality and zero profits conditions for the discrete pooling region. We can use (11), averaging over both levels of risk aversion, and substituting the average risk for $p(x)/x$ (zero profits) to obtain the following ordinary differential equation on prices:

$$\dot{p}(x; \delta) = \frac{p(x, \delta)}{x} + \bar{\rho}(x; \delta)(1 - x),$$

where the average risk aversion of consumers choosing coverage x is defined as

$$\bar{\rho}(x; \delta) \equiv w(x; \delta)(\rho_0 - \delta/2) + (1 - w(x; \delta))(\rho_0 + \delta/2).$$

The solution of this ODE with the final condition $p(1; \delta) = \mu_H$ is¹⁹

$$\frac{p(x; \delta)}{x} = \mu_H - \int_x^1 \left(\frac{1}{z} - 1 \right) \bar{\rho}(z; \delta) dz. \quad (13)$$

Notice that the equilibrium price for coverage x depends on a cumulative weighted average of the values of risk aversion of the consumers selecting all coverage levels above x . We now use local properties of equilibria, when δ is small, to pin down how the type distribution affects $\bar{\rho}$.

We denote the first- and second-order derivatives of prices with respect to δ as $p_\delta(x; \delta)$ and $p_{\delta\delta}(x; \delta)$, respectively, and use the same convention for other equilibrium objects. Direct differentiation then implies

$$\frac{p_\delta(x; \delta)}{x} = \int_x^1 \left(\frac{1}{z} - 1 \right) \left[w(z; \delta) - \frac{1}{2} + w_\delta(z; \delta)\delta \right] dz.$$

Since we focus (in this subsection) on independently distributed risk and risk aversion we have:

$$\phi_l(\mu) = \omega_0 \phi(\mu) \quad \text{and} \quad \phi_h(\mu) = (1 - \omega_0) \phi(\mu),$$

for a density $\phi : [\mu_L, \mu_H] \mapsto \mathbb{R}_{++}$ and some $\omega_0 \in (0, 1)$. Therefore, taking the limit $\delta \rightarrow 0$ and using (12), we get an expression for the first-order approximation of prices:

$$\frac{p_\delta(x; 0)}{x} = \int_x^1 \left(\frac{1}{z} - 1 \right) \left(\omega_0 - \frac{1}{2} \right) dz,$$

where, under the assumed independence between risk and risk aversion, $w(x, 0) = \omega_0$ is the prior probability of low risk aversion. Hence, the first-order price approximation depends on the relative distribution of risk aversion. A higher relative share of low-risk-aversion consumers leads to higher prices: consumers with high risk aversion demand higher coverage even when having low risk level, which drives down the average cost/price of such contracts.

We are interested in the second order approximation of prices as a function of δ , hence we take the second-order

¹⁹Notice that it corresponds to the one-dimensional equilibrium price (8) when $\delta = 0$, i.e., $\bar{\rho} = \rho_0$.

derivative of prices at $\delta = 0$, which reduces to

$$\frac{p_{\delta\delta}(x; \delta)}{x} = 2 \int_x^1 \left(\frac{1}{z} - 1 \right) w_\delta(z; 0) dz. \quad (14)$$

The term $w_\delta(z, 0)$ is important as it approximates the share of low risk aversion pooled at coverage z when δ is small. Finally, as discussed in Section 3, for a fixed willingness to pay types with lower risk aversion have higher risk. To compute $w_\delta(z, 0)$ we can differentiate equation (12) and evaluate it at $\delta = 0$, which leads to (using the notation $\partial_\delta \equiv \frac{\partial}{\partial \delta}$)

$$\begin{aligned} \frac{w_\delta(x; 0)}{\omega_0(1 - \omega_0)} &= \partial_\delta \left(\log \frac{w(x; \delta)}{1 - w(x; \delta)} \right) \\ &= \frac{\dot{\phi}(m_0(x))}{\phi(m_0(x))} [\partial_\delta (m_l(x; 0) - m_h(x; 0))] + \frac{\partial_\delta (\dot{m}_l(x; 0) - \dot{m}_h(x; 0))}{\dot{m}_0(x)}. \end{aligned}$$

Using then the fact that $m_l(x; \delta) - m_h(x; \delta) = \delta(1 - x)$ (from equation (11)), we get

$$w_\delta(x; 0) = \omega_0(1 - \omega_0) \left[\frac{\dot{\phi}(m_0(x))}{\phi(m_0(x))} (1 - x) - \frac{1}{\dot{m}_0(x)} \right].$$

Finally, since $\dot{m}_0(x) = \rho_0 \frac{1-x}{x}$, equation (14) can be rewritten as

$$\frac{p_{\delta\delta}(x; \delta)}{x} = 2\rho_0^{-1}\omega_0(1 - \omega_0) \left[\int_{m_0(x)}^{\mu_H} \frac{\dot{\phi}(\mu)}{\phi(\mu)} (1 - t_0(\mu)) d\mu - (1 - x) \right].$$

The second-order approximation term allows us to see how the risk distribution affects equilibrium prices. With a positive yet small amount of preference heterogeneity, each coverage level is purchased by types with similar, yet distinct, risk levels. The relative frequency of the higher-risk-lower-aversion types to the lower-risk-higher-aversion ones is represented by the rate of growth of the risk density, $\frac{\dot{\phi}(\cdot)}{\phi(\cdot)}$. A higher rate of growth of the density means that, for any given risk level, there are relatively more types in the population with slightly higher risks. As these two “similar” types are pooled into a single contract, the overall effect on prices is positive. Intuitively, this second-order term is zero in the extreme cases $\omega_0 = 1$ or $\omega_0 = 0$, as the relative share of different risk types is irrelevant in the one-dimensional case.

The terms p_δ and $p_{\delta\delta}$ capture explicitly how the distribution of risks affects prices, at least for small preference heterogeneity. The next Subsection 4.1.2 provides the formal derivation of the price approximation terms, without imposing the conditions of continuous differentiability of equilibrium variables p and w , nor the independence of risk and risk preferences. On that basis we formally establish in the following Subsection 4.1.3 (see Corollary 1) the validity of the approximation results for the independent case discussed here. A reader interested purely in the comparative statics exercises obtained from these formulas can skip the next subsections and go directly to Sections 5 and 6, which show how to use these approximation results to evaluate the welfare impact of signal disclosures and changes in the risk distribution.

4.1.2 Approximation results

In this subsection, we extend the results in Subsection 4.1.1 to the case of non-independent types and arbitrary (potentially non-differentiable) equilibrium selections. Moreover, our approximation results also imply that the separating and continuous pooling regions vanish as $\delta \rightarrow 0$. In order to analyze the behavior of equilibrium prices, it is necessary to study the limiting behavior of the posterior beliefs $w(x)$. When preference heterogeneity

is positive but small, any coverage with discrete pooling features cross-subsidization between types with almost identical level of risk. In the limit $\delta \rightarrow 0$, all consumers selecting the same coverage level x have a risk level equals to $m_0(x)$ and the limiting share of consumers with low risk aversion,²⁰ among the ones with risk level $m_0(x)$, is:

$$\omega_0(x) \equiv \frac{\phi_l(m_0(x))}{\phi_l(m_0(x)) + \phi_h(m_0(x))}, \quad (15)$$

for any $x \in [x_L, 1]$.

Proposition 2 guarantees the existence of equilibria for all $\delta > 0$ sufficiently small. We now consider an arbitrary equilibrium selection and define the following limits, if they exist,

$$p_\delta(x) \equiv \lim_{\delta \rightarrow 0} \frac{p(x; \delta) - p(x; 0)}{\delta}$$

and

$$p_{\delta\delta}(x) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{p(x; \delta) - p(x; 0)}{\delta} - p_\delta(x)}{\delta}.$$

Proposition 3. *Consider an arbitrary equilibrium selection and $\delta > 0$ sufficiently small. For all $x \in (x_L, 1)$, the price and posterior beliefs exhibit the following limiting behavior:*

(a) *Convergence: the function $p(x; \cdot)$ is continuous at zero,*

$$\lim_{\delta \rightarrow 0} \frac{p(x; \delta)}{x} = m_0(x) \quad \text{and} \quad \lim_{\delta \rightarrow 0} w(x; \delta) = \omega_0(x);$$

(b) *Differentiability: the limit $p_\delta(x)$ exists and is given by*

$$\frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2} \right) \left(\frac{1}{z} - 1 \right) dz; \quad (16)$$

(c) *Second-order differentiability: the limit $p_{\delta\delta}(x)$ exists and is given by*

$$\frac{p_{\delta\delta}(x)}{x} = 2 \int_x^1 w_\delta(z) \left(\frac{1}{z} - 1 \right) dz, \quad (17)$$

where

$$\begin{aligned} w_\delta(x) = & \frac{\dot{\omega}_0(x)}{x \dot{m}_0(x)} p_\delta(x) - \frac{x \omega_0(x) [1 - \omega_0(x)]}{\rho_0 (1 - x)} \\ & + (1 - x) [1 - \omega_0(x)] \omega_0(x) \left\{ [1 - \omega_0(x)] \frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x) \frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))} \right\}. \end{aligned} \quad (18)$$

Additionally, all convergence results hold uniformly on any compact subset of $(x_L, 1)$.

Proof. The proof is a combination of the results in Lemmas 17, 18, 19, 22 and 24 in Appendix C. \square

Proposition 3 allows us to use the following approximation of equilibrium prices to perform comparative statics analyses, for any $x \in (x_L, 1)$:

$$p(x; \delta) = p(x; 0) + \delta p_\delta(x) + \frac{\delta^2}{2} p_{\delta\delta}(x) + o(\mu; \delta^2), \quad (19)$$

²⁰Although in the limit $\delta = 0$ we have $\rho_l = \rho_h$, the interpretation is that for each risk type there are exactly two risk-preference types which happen to have the same risk aversion.

where $\delta^{-2}|o(\mu; \delta)|$ converges to zero uniformly on M , for any compact set $M \subset (\mu_L, \mu_H)$. For a fixed $x \in (x_L, 1)$, we refer to the polynomial coefficients $p(x, 0)$, $p_\delta(x)$ and $p_{\delta\delta}(x)$ simply as *approximation coefficients*.

In broad terms, the insight of Proposition 3 is that, for δ small, equilibrium pools include types with very similar levels of risk. This means that *local* properties of the type distribution can be used to analyze equilibrium prices and, as a consequence, welfare.

The first part of Proposition 3 shows that prices converge pointwise to their one-dimensional counterpart. This occurs as, when the difference of risk-aversion among consumers becomes smaller, the heterogeneity among buyers choosing the same coverage level disappears. Hence, in the limit, the equilibrium allocation and prices fully separate consumers in terms of their risk level. The limit of the endogenous posterior $w(\cdot)$ is then determined using the relative level of risk densities ϕ_l and ϕ_h at any risk level $m_0(x)$.

The second part of Proposition 3 is slightly more subtle and shows that the sign of the first-order approximation of the price does depend on the type distribution through the difference between $\omega_0(x)$ and $\frac{1}{2}$. The reason is that with $\delta > 0$ small, except for very low or very high coverages, all other coverage levels occur in the discrete pooling region where each contract is purchased not by a single risk-type, but by a pair of types: one with low risk and high risk aversion, $(m_h(x), \rho_h)$, and one with high risk and low risk aversion, $(m_l(x), \rho_l)$ (remember that equation (11) implies $m_l(x) > m_h(x)$). Hence, the effect on the equilibrium prices of the heterogeneity in risk levels depends on which type is more prevalent. If there are more low risk-aversion consumers, who exhibit a higher risk than the high risk aversion consumers with whom they pool (i.e., $\omega_0(x) > \frac{1}{2}$), prices are increased; if otherwise (i.e., $\omega_0(x) < \frac{1}{2}$), the effect is reversed.

The third part of Proposition 3 is even more subtle and captures how the probability assigned to the low risk averse consumers exhibiting higher risk ($w(x; \delta)$), conditional on coverage x , differs from its limit $\omega_0(x)$. If $w_\delta(x) > 0$, the introduction of small preference heterogeneity $\delta > 0$ implies that the share of low-risk-aversion-high-risk types purchasing each contract is larger than $\omega_0(x)$, which drives prices up. The way in which the conditional probability vary with the degree of preference heterogeneity δ depends on the type distribution solely through the slopes of the densities of the risk distribution for high and low risk aversion consumers. Building on the discussion in Subsection 4.1.1, the relative rate of growth of the two densities determines the frequency of consumers that have higher risk level within their pool, relative to consumers lower risk within the same pool — who have a slightly lower risk level.

The equilibrium multiplicity issue can be traced to the continuous pooling regions, i.e., for fixed continuous pooling regions a unique corresponding equilibrium exists. An important part of the proof of Proposition 3 is to show that for any $\epsilon > 0$, the mass of types in the discrete pooling region is larger than $1 - \epsilon$ for $\delta > 0$ sufficiently small. As a consequence, as $\delta \rightarrow 0$, the continuous pooling regions become irrelevant, and so does equilibrium multiplicity. Our approximation is then derived using properties of the discrete pooling region. As a result, we are able to perform welfare analysis for a mass of agents whose size approaches one, as laid out in the interim Pareto welfare concept used in Section 5.

4.1.3 Independent risk and risk aversion

Some of our comparative statics results will focus on the case where the distributions of the two dimensions of heterogeneity are independent, so we include an explicit presentation of the approximations coefficients for this case. They coincide with the ones presented and discussed in Subsection 4.1.1.

Corollary 1. *If risk and risk aversion are independently distributed, then*

$$p_\delta(x) = x \left(\omega_0 - \frac{1}{2} \right) (x - 1 + \log x),$$

and

$$\frac{p_{\delta\delta}(x)}{x} = 2\omega_0(1-\omega_0) \left[\int_x^1 (1-z) \left(\frac{1}{z} - 1 \right) \frac{\dot{\phi}(m_0(z))}{\phi(m_0(z))} dz - \frac{(1-x)}{\rho_0} \right].$$

Proof. By direct substitution in Proposition 3. □

4.2 Numerical approach

In Proposition 3 we provided explicit expressions for equilibrium prices which can be used for comparative statics exercises for small, and yet positive, levels of preference heterogeneity. A relevant question is how small the parameter δ has to be for our results to apply. To this end, we also present an algorithm that finds an approximation of a competitive equilibrium in our two-dimensional setting, but allowing both type dimensions to lie on an arbitrary finite grid. We then use this algorithm to investigate the robustness of our results when varying the size of preference heterogeneity, looking at either small but non-trivial levels as well as larger ones. Our algorithm also allows us to investigate the effects of introducing multiple levels of preference heterogeneity.

More precisely, the algorithm uses a finite grid $\hat{\Theta} \subset \mathbb{R}_{++}^2$ for types (μ, ρ) as well as a finite set of coverages $\hat{X} \in [0, 1]$, and relies on a price iteration procedure. For a given price function p_0 , the algorithm calculates the optimal coverage level for each type $t_{p_0} : \hat{\Theta} \mapsto [0, 1]$ and defines a new price function p_1 as follows. For each contract traded under p_0 , the new price p_1 is updated towards the average risk of agents purchasing it. The price for each non-traded contract x' is also updated, but on the basis of the riskiness of the type (μ^+, ρ^+) most willing to pay for it, given the utility level attained by each type under p_0 : if $p_0(x') > \mu^+ x'$, $p_0(x')$ is then lowered. The updating rule for prices of non-traded contracts is crucial and makes sure that, in a fixed point p^* , for all non-traded contracts x' we have $p^*(x') \leq \mu^+ x'$, i.e., the firms' optimality condition (1) in our equilibrium Definition 1 is satisfied.²¹ The algorithm starts with a particular guess for the equilibrium price function and updates prices until the gap between updating stages is approximately zero. Notice that the algorithm does not use the properties of equilibria established in our characterization result (Proposition 1). The results of our simulations are illustrated in Subsection 5.5.²²

5 Signal disclosure

A central policy question concerning the regulation of insurance markets is the extent to which companies should be allowed to discriminate consumers based on observable characteristics. For example, demographic characteristics are useful for firms when pricing insurance contracts as they are correlated with consumers' risk, even if they are not direct determinants of risk.

We analyze this question by considering the effects of the disclosure of an informative signal, or observable characteristic. The realization of the signal for each consumer is public and can then be used by firms in pricing. Their contract offers are based on the distribution of types conditional on the signal realization and, as a consequence, equilibrium prices also depend on the signal realization.

The welfare criterion we use to assess the effects of signal disclosures is interim in the sense that, for any given consumer type, we consider the expected utility in the competitive equilibrium with public disclosure of the signal, taking the expectation across all possible signal realizations. This is the natural measure of consumer's well-being if consumers do not observe the signal realization prior to the intervention. This is the case, for

²¹This iterative procedure is akin to the one used in Azevedo and Gottlieb (2017). Importantly, while their approach relies on "behavioral" types to ensure that all contracts are traded at each iteration, ours prices of non-traded contracts at each iteration are based on conditions (1) and (3) in Definition 1.

²²Section 1 of the Online Appendix provides additional numerical results.

example, if the signal is the result of an imprecise health test or, as in HHW, if one evaluates the situation from the point of view of a young consumer's expected lifetime utility, and considers the effects of the availability of information about individual characteristics that evolve stochastically over time. In many relevant applications, however, such as gender-based pricing, the realization of the signals is known prior to the intervention. In those cases, the expected utility describes the aggregate (utilitarian) welfare of all consumers (in the example, all men and women) sharing the same risk and preference characteristics. We then say that a signal is interim Pareto improving if, for almost all types, the expected utility in the equilibrium with signal disclosure is higher than in the equilibrium with no signal. Any Pareto improvement according to this interim criterion implies also an improvement according to an ex-ante criterion. Proposition 4 is our main result and provides necessary and sufficient conditions for a signal to be interim Pareto improving.

We make two main assumptions regarding signals. First, we focus on pure risk signals, i.e., whose realization is independent of risk preferences, conditional on risk. Signal realizations may still be correlated with risk preferences through their common dependence on risk. Second, it is assumed that the signal is correlated to consumers' unobservable risk level but contains no additional predictive power on their final losses \tilde{l} relative to their risk type μ , which is known by consumers. In other words, the signal is directly informative for firms, but not for consumers and affects them only through its effect on prices. Since the signal has no direct value to consumers, any positive welfare result regarding the disclosure of information (such as Proposition 4) is more surprising.

We denote the (finite) set of possible signal realizations as S . A signal is a function $\pi(\cdot | \cdot) : S \times [\mu_L, \mu_H] \mapsto [0, 1]$, such that $\pi(\cdot | \mu) \in \Delta(S)$ for any $\mu \in [\mu_L, \mu_H]$. That is, for each $\mu \in [\mu_L, \mu_H]$, the mapping $\pi(\cdot | \mu)$ denotes the probability distribution over signal realizations, conditional on risk level μ . We assume that $\pi(\cdot | \cdot)$ is strictly positive, and that, for any $s \in S$, the mapping $\mu \mapsto \pi(s | \mu)$ is continuously differentiable and $\frac{\partial}{\partial \mu} \pi(s | \mu) \neq 0$ for almost all (Lebesgue) $\mu \in [\mu_L, \mu_H]$.²³ The distribution of types conditional on signal s is then given by

$$\phi_i(\mu | s) = \frac{\phi_i(\mu) \pi(s | \mu)}{\Pi(s)}, \quad (20)$$

for $i = l, h$, where $\Pi(s)$ denotes the ex-ante probability of signal realization s :

$$\Pi(s) \equiv \int \pi(s | \mu') [\phi_l(\mu') + \phi_h(\mu')] d\mu'.$$

We use the superscript s to refer to equilibrium variables under the distribution $(\phi_l(\cdot | s), \phi_h(\cdot | s))$ (e.g., $p^s(x; \delta)$), i.e., when public signal s is received, and superscript 0 when referring to equilibrium variables when no signal is received, i.e., under prior distribution (ϕ_l, ϕ_h) (e.g., $p^0(x; \delta)$). The corresponding equilibrium payoffs of a consumer with type $(\mu, i) \in [\mu_L, \mu_H] \times \{l, h\}$ for $s \in S$ with preference heterogeneity $\delta > 0$, is then

$$V_i^s(\mu; \delta) \equiv v(t_i^s(\mu; \delta), p^s(t_i^s(\mu; \delta); \delta); \mu, \rho_i(\delta)), \quad (21)$$

where $v(\cdot)$ is given by (1), and $p^s(\cdot)$ and $t^s(\cdot)$ represent equilibrium price and allocation, respectively. Similarly when the superscript is 0.

Definition 2. A signal is interim Pareto improving if, for any type distribution (ϕ_l, ϕ_h) , any equilibrium selection and for every $\epsilon > 0$, there exists a $\bar{\delta} > 0$ such that

$$\mathbb{P} \left[(\mu, i) \mid \sum_{s \in S} \pi_i(s | \mu) V_i^s(\mu; \delta) > V_i^0(\mu; \delta) \right] > 1 - \epsilon,$$

²³This assumption rules out knife-edge cases where $\pi(s | \cdot)$ is constant in an open interval contained in $[\mu_L, \mu_H]$.

for all $\delta < \bar{\delta}$.

The above welfare criterion is quite strong as it requires a signal to be interim Pareto improving for any prior type distribution with support $[\mu_L, \mu_H]$ and any equilibrium selection. The impact of the availability of a signal on each consumer type (μ, i) is evaluated on the basis of his expected utility gain, or loss, integrating over all signal realizations. Definition 2 is tailored to the use of our approximation approach in two main ways. First, it relies on the price approximation results obtained in Section 4 and hence the analysis refers to the case of arbitrarily small preference heterogeneity $\delta > 0$. Second, given our approximation method, we consider the interim welfare effects for all consumers choosing coverage in the discrete pooling region. The ϵ qualifier relates to the mass of agents outside the discrete pooling region, which becomes arbitrarily small as $\delta \rightarrow 0$.

Using this criterion, our main result provides necessary and sufficient conditions under which a signal is guaranteed to benefit almost all types, regardless of the underlying original type distribution and equilibrium selection. In Subsection 5.1 we show that the welfare impact of signal disclosure follows directly from its effect on expected prices. Subsection 5.2 uses this result to provide necessary and sufficient conditions for a signal to be interim Pareto improving.

5.1 Welfare and price effects

The disclosure of a signal affects consumers by changing the equilibrium price function they face. The main result in this subsection (Lemma 4) shows that the impact of signal disclosure on the expected utility of each type of consumer is determined by the expected price change generated by the signal, averaging over signal realizations and holding the consumer's coverage level constant. This result is surprising since consumers' preferences, represented in (1), feature risk aversion and the disclosure of a signal introduces a new source of uncertainty as equilibrium prices vary with the signal.

Using (19), we can approximate the expected effect of signal disclosure on the price of the coverage chosen by type $(\mu, i) \in [\mu_L, \mu_H] \times \{l, h\}$ prior to the disclosure as

$$\sum_{s \in S} \mu(s | \mu) p^s(t_i^0(\mu; \delta); \delta) - p^0(t_i^0(\mu; \delta); \delta) = \frac{\delta^2}{2} \Delta E[p(t_0(\mu))] + o_i(\delta^2; \mu),$$

with

$$\Delta E[p(x)] \equiv \sum_{s \in S} \pi(s | m_0(x)) [p_{\delta\delta}^s(x) - p_{\delta\delta}^0(x)], \quad (22)$$

where we have used the fact that the constant and the first-order approximation coefficients of prices do not vary with the signal realization, i.e., $p^s(x; 0) - p^0(x; 0) = p_{\delta}^s(x; 0) - p_{\delta}^0(x; 0) = 0$ for all $s \in S$. The independence of the first-order derivative $p_{\delta}^s(\cdot)$ on s follows from the fact that the signal contains no information on risk preferences, given the risk type μ . In other words, the ratio $\omega_0^s(x)$, as defined in (15), does not depend on the signal realization s . The following lemma connects the approximation of the expected price change to the approximation of the expected utility gain of consumers in equilibrium.

Lemma 4. *The welfare effect of signal disclosure on consumer type $(\mu, i) \in (\mu_L, \mu_H) \times \{l, h\}$ is given by*

$$\sum_{s \in S} \pi(s | \mu) V_i^s(\mu; \delta) - V_i^0(\mu; \delta) = \frac{\delta^2}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) \Delta E[p(t_0(\mu))] + o(\delta^2; \mu), \quad (23)$$

where $\delta^{-2} |o(\delta^2; \cdot)|$ converges uniformly to 0 on M , for any compact set $M \subset (\mu_L, \mu_H)$.

Proof. See Appendix C. □

Based on the expression of the equilibrium utility (21), the impact of signal realization s on a particular consumer type can be decomposed in terms of its effect on prices, holding the coverage originally chosen by that type fixed (price effect), and the change in coverage chosen by the consumer as a response to changes in the equilibrium price function (allocation effect). Intuitively, the proof of Lemma 4 relies on two observations: (i) the interim utility change of each type has the same sign and order of magnitude as the price effect, and (ii) the price effect of a signal on payoffs is determined solely by the expected price change generated by the signal, holding the coverage level constant.

While the signal considered may be very informative about risk levels, its impact on equilibrium prices, and also on the allocation, becomes arbitrarily small as $\delta \rightarrow 0$, since the amount of risk heterogeneity within each pool disappears in the limit $\delta = 0$. Hence, observation (i) follows from an envelope-type argument, which implies that the price impact of a shock is of higher order of magnitude than the allocation effect, since in equilibrium consumers optimally choose their coverage amount. It is important to point out that, while the interim expected utility changes are determined purely by price effects at an individual level, allocation effects are still crucial to the analysis. As discussed in the next Subsection 5.2, the price changes induced by signal disclosure are determined by changes in agents' coverage levels.

Observation (ii) is more subtle. The disclosure of the signal introduces a randomization in the agents' consumption level because of the induced randomization in prices. Since equilibrium outcomes in the case of no preference heterogeneity ($\delta = 0$) do not depend on the disclosure of the signal, the impact of signal disclosure on equilibrium prices is *small* for small preference heterogeneity δ , regardless of the informational content of the signal. Since the signal is assumed not to provide additional payoff relevant information to consumers, the realization of the signal is *independent* of their risky losses, conditional on their type. Observation (ii) follows from Arrow-Pratt's approximation result:²⁴ a consumer with a risky consumption level evaluates small lotteries that are independent from the initial consumption according to their expected value.

5.2 Welfare improving signals

The main result of this subsection builds on Proposition 3 to show that a signal is interim Pareto improving if, and only if it satisfies a condition on the information conveyed by the signal referred to as *monotonicity*. The information content of a signal regarding the risk profile of consumers is determined by how the signal distribution varies with the consumers' risk level. Intuitively, monotonicity means that consumers with more distant risk levels must generate more "distinct" distributions over the possible signal realizations. Hence, defining monotonicity requires a measure of how different any two distributions over signal realizations are. For any two distributions $\pi, \tilde{\pi} \in \Delta(S)$, the Kullback–Leibler (KL) divergence of $\tilde{\pi}$ from π is defined as

$$D_{KL}(\pi || \tilde{\pi}) \equiv \sum_s \pi(s) \log \left(\frac{\pi(s)}{\tilde{\pi}(s)} \right). \quad (24)$$

This measure is also referred to as relative entropy and is a non-commutative measure of distribution discrepancy.²⁵ It is always non-negative and equals to zero if and only if $\pi = \tilde{\pi}$. We say that a signal is monotonic if the gap between the signal distributions generated by two types, as measured by the KL divergence measure, varies monotonically with the magnitude of the gap in the risk level of the two types of consumers. Formally:

²⁴For any C^1 Bernoulli $v : \mathbb{R} \mapsto \mathbb{R}$, independent and bounded random variables x and y , and $\varepsilon > 0$, $\mathbb{E}[v(x + \varepsilon y)] = \mathbb{E}[v(x)] + \varepsilon \mathbb{E}[v'(x)] \mathbb{E}[y] + o(\varepsilon)$.

²⁵Information theory (see, for instance, Kraft (1949) and McMillan (1956)) defines this measure as the expected number of extra bits that would be required to code the information if one were to use $\tilde{\pi}$ instead of π . In economics, it is commonly used in the literature on rational inattention as a way to model the cognitive cost of processing information, referred to as mutual information (Sims (2003)).

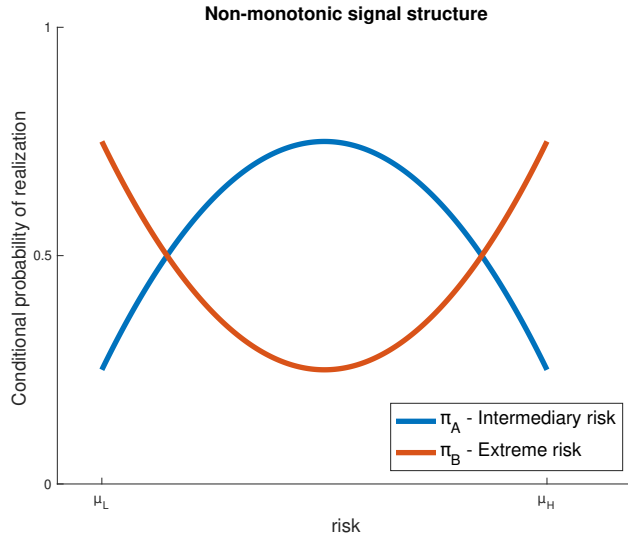


Figure 2: A non-monotonic binary signal. The red curve represents a particular quadratic and convex mapping $\mu \mapsto \pi(B | \mu)$. This means that signal realization B is indicative of extreme risk levels. The blue curve represents the alternative signal realization A with $\pi(A | \cdot) = 1 - \pi(B | \cdot)$.

Definition 3. A signal $(\pi(s|\cdot))_{s \in S}$ is monotonic if, for any $\mu \in [\mu_L, \mu_H]$,

$$D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu}))$$

is strictly increasing in $\tilde{\mu}$ for $\tilde{\mu} > \mu$.

Consider three ordered risk levels $\mu_1, \mu_2, \mu_3 \in [\mu_L, \mu_H]$ with $\mu_1 < \mu_2 < \mu_3$. As the pair of risk levels (μ_1, μ_3) is more distinct relative to pair (μ_1, μ_2) , monotonicity requires that the pair (μ_1, μ_3) should generate signal distributions that “diverge” more from each other, when compared to the pair (μ_1, μ_2) .

Monotonicity is a generalization of common restrictions on signals present in contract theory, such as monotone likelihood ratio ordering. The relationship between the two properties is discussed more in detail in Subsection 5.3.²⁶ Figure 2 illustrates a quadratic non-monotonic binary signal, where one realization of the signal has a higher likelihood for extreme risk levels while the other has higher likelihood for intermediate risk levels. This non-monotonic signal example is further analyzed numerically in Subsection 5.5. We now state our main result, which relates monotonicity to interim Pareto welfare improvements.

Proposition 4. *A signal is interim Pareto improving if and only if it is monotonic.*

Proof. From Lemma 4 we know that the expected utility gain from signal disclosure of a given type depends on the expected price change of the chosen level of coverage x . This change can be obtained from (22), using (17) and satisfies

$$\Delta E[p(x)] = 2 \int_x^1 \sum_{s \in S} \pi(s | m_0(x)) [w_\delta^s(z) - w_\delta^0(z)] \left(\frac{1}{z} - 1\right) dz, \quad (25)$$

where the expressions of $\omega_\delta^s(\cdot)$ and $\omega_\delta^0(\cdot)$ are given by (18).

Expression (25) can then be further simplified using the following properties: (i) signal realizations have no first-order effect on prices, i.e., $p_\delta^s(\cdot) = p_\delta^0(\cdot)$; (ii) signals convey no information on consumers’ risk preference,

²⁶In Section 2 of the Online Appendix we present an alternative and equivalent definition of monotonicity in terms of a statistical property related to the impact of signal realizations on the expected risk of small risk pools.

i.e., the distribution of risk preferences specified in (15) satisfies $\omega^s(\cdot) = \omega^0(\cdot)$; and (iii) Bayes' updating rule (20). As a consequence, we have that

$$\sum_{s \in \mathcal{S}} \pi(s | m_0(x)) [w_\delta^s(z) - w_\delta^0(z)] = (1-z)\omega^0(z) [1 - \omega^0(z)] \sum_{s \in \mathcal{S}} \pi(s | m_0(x)) \frac{\dot{\pi}(s | m_0(z))}{\pi(s | m_0(z))},$$

where $\dot{\pi}(s | \mu) \equiv \frac{\partial}{\partial \mu} \pi(s | \mu)$.

Direct substitution in (25) and a change of variables in the integration (to $\mu' = m_0(z)$) yield then

$$\Delta E[p(t_0(\mu))] = -2 \int_{\mu}^{\mu_H} \frac{(1-z)}{\rho_0} \omega^0(t_0(\mu')) [1 - \omega^0(t_0(\mu'))] \left[\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) \Big|_{\tilde{\mu}=\mu'} \right] d\mu'. \quad (26)$$

This expression establishes a clear relationship between the sign of the expected price change of a given contract and the monotonicity of the Kullback-Leibler divergence measure.²⁷ The rest of the proof is in Appendix D. \square

Proposition 4 implies that the disclosure of a monotonic signal leads to an ex-ante welfare gain (i.e., integrating over all possible types), for $\delta > 0$ sufficiently small. It also states that, for any non-monotonic signal, one can find some type distribution such that the disclosure of this signal leads to an expected utility loss to a positive mass of types. In fact, one can choose the distribution so that the mass of utility-losing types is large enough that signal disclosure leads to an ex-ante utility loss.

Interpretation of the result

In order to provide some intuition for our result and highlight the role of allocation effects in Proposition 4, it is useful to consider first the effects of the disclosure of a monotonic signal in the absence of any change in coverage, and evaluate next the additional price and welfare impact of allocation changes.

Starting from an equilibrium in the absence of signal disclosure, define “no-reallocation” counterfactual prices $\underline{p}^k(\cdot)$, for $k \in \{0\} \cup \mathcal{S}$, as the expected risk level within each pool, conditional on the signal realization, assuming no reallocation effect. That is, such prices are based on type assignment functions m_l and m_h fixed at pre-disclosure levels. These are not equilibrium prices as we ignore the possibility that consumers change their choice of coverage. In the no-reallocation counterfactual, signal disclosure would correspond to a *mean-preserving spread* of prices:

$$\mathbb{E} \left[\underline{p}^{\tilde{s}}(x') - \underline{p}^0(x') \mid t(\tilde{\mu}, \tilde{i}) = x' \right] = 0, \quad (27)$$

where the expectation is taken with respect to the possible signal realizations $\tilde{s} \in \mathcal{S}$.

In contrast, our positive interim Pareto improvement result for disclosure of monotonic signals (Proposition 4) relies on the fact that the expected value of equilibrium prices *decreases* following the signal disclosure (equation 26). Given the definition of counterfactual prices \underline{p}^k , this change can be imputed to the change in the equilibrium allocation.

To better understand the source of this equilibrium reallocation effect, consider a binary signal with $\mathcal{S} = \{G, B\}$, where G represents a “good” signal (i.e., $\frac{\partial}{\partial \mu} \pi(G | \mu) < 0$), and consider the impact of signal disclosure on an agent, called A (she), with risk level μ_A who chooses coverage x prior to signal disclosure. If we consider the “no-reallocation” prices, A expects the price of coverage $x' > x$ to change, prior to observing signal realization \tilde{s} , by $\mathbb{E}[\underline{p}^{\tilde{s}}(x') - \underline{p}^0(x') \mid \tilde{\mu} = \mu_A]$. If the signal is monotonic, A receives signal $s = G$ with higher probability

²⁷In fact, the proof of Proposition 4 establishes a stronger result: for any signal (potentially non-monotonic), type distribution and risk level μ , an agent with risk level μ has a strict interim utility gain from signal disclosure, when δ is small, as long as expression (26) is strictly negative. In the case of a monotonic signal, this expression is strictly negative for any distribution and any risk level in (μ_L, μ_H) .

than all types choosing coverage $x' > x$. As a consequence, A expects the price of coverage $x' > x$ in the *same segmented market as her*, i.e., including consumers with her signal realization, to go down:

$$\mathbb{E} [\underline{p}^{\bar{s}}(x') - \underline{p}^0(x') \mid \tilde{\mu} = \mu_A] < \mathbb{E} [\underline{p}^{\bar{s}}(x') - \underline{p}^0(x') \mid t(\tilde{\mu}, \tilde{i}) = x'] = 0. \quad (28)$$

Both expectations in (28) are taken with respect to signal realizations, but the left-hand expression uses the distribution of signals generated by A, which puts higher probability on signal G , relative to agents choosing higher coverage x' .²⁸

Once equilibrium effects are fully accounted for, this interim expected price reduction for higher coverages $x' > x$ leads to a lower expected price for coverage x as well, which explains the interim welfare gain for agent A. This top-down equilibrium effect can be seen from our price approximation formulas in Proposition 3: the price of coverage $x \in [0, 1]$ is affected by the risk distribution evaluated at levels $m_0(x')$, with $x' \geq x$. Intuitively, any shift in the risk distribution that increases the risk (and price) of the pools with coverage x' induces agents originally choosing this level to lower their coverage choice, which indirectly impacts the equilibrium price of all contracts with lower coverage.

5.3 Signal-MLRP versus Monotonicity

A common condition on informative signals used in principal-agent models is that signals satisfy the monotone likelihood ratio property (S-MLRP; Mirrlees (1976), Hölmstrom (1979)).

Definition 4. A signal π , with realizations $S = \{s_1, \dots, s_n\}$, satisfies S-MLRP if

$$\frac{\pi(s_{k+1} \mid \mu)}{\pi(s_k \mid \mu)}$$

is strictly increasing in μ , for any $k \in \{1, \dots, n-1\}$.

Proposition 5. Any signal satisfying S-MLRP is monotonic.

Proof. See Appendix D. □

In the case of a binary signal, a distribution over signal realizations is described by a single number and both monotonicity and S-MLRP are equivalent to the requirement that $\pi(s_2 \mid \cdot)$ is strictly increasing. Beyond binary signals, S-MLRP is a stronger requirement in that it imposes restrictions on the relative probability of any pair of signal realizations. Monotonicity, on the other hand, uses the coarser information given by KL divergence measure which integrates over all signal realizations.

Remark 1. An alternative sufficient condition for signal monotonicity is that, for all $s \in S$, $\pi(s \mid \cdot)$ is strictly increasing or decreasing in μ . This condition means that signals can be classified in two categories: the ones that are more likely with high risks and the others that are more likely with low risks. This condition is neither weaker nor stronger than S-MLRP, but both coincide in the case of binary signals.

5.4 Independent risk and risk aversion

In Subsection 5.1, we showed that the welfare effect of signal disclosure is determined by the change in expected prices, and derived an expression for it in (22). If risk and risk aversion are independently distributed (i.e., $\omega_0(\cdot)$

²⁸In fact, Proposition 1 in the Online Appendix shows that a signal is monotonic if, and only if inequality (28) can be guaranteed, for all distributions and traded coverage levels x and x' .

is constant), this expression can be further simplified after integrating by parts to:

$$\Delta E[p(x)] = -2\omega_0(1 - \omega_0) \int_x^1 D_{KL}(\pi(\cdot | m_0(x)) || \pi(\cdot | m_0(z))) dz.$$

We then see that in this case the expected change of the price for coverage x , due to signal disclosure, is determined solely by the integral of the divergence measure $D_{KL}(\cdot)$ for all coverage levels above x , which is non-negative regardless of whether monotonicity holds. Hence, under the assumption of independence, an interim Pareto welfare improvement always obtains, with no condition on the signal. We can then strengthen the welfare result in Proposition 4 as follows.

Corollary 2. *Any signal is interim Pareto improving when the distributions of risk and risk aversion in the population are independent.*

Notice that the definition of an interim Pareto improving signal requires that the signal is interim Pareto improving for all possible distributions of types. Corollary 2 shows that, if we impose some restrictions on the set of allowed type distributions — such as independence — we can find weaker conditions ensuring that a signal is interim Pareto improving. Independence is a special case of a more general property that guarantees that any signal is interim Pareto improving: the property that the mapping $x \mapsto (1 - x)\omega_0(x)[1 - \omega_0(x)]$, for any $x \in (x_L, 1)$, is non-increasing.²⁹

5.5 Larger preference heterogeneity: simulations

In this subsection we use the numerical algorithm described in Subsection 4.2 to present numerical simulations aiming to illustrate the robustness of our results on risk classification with respect to larger levels of preference heterogeneity (δ not too small) and to a richer set of levels of risk aversion. Following Einav et al. (2013), our simulations use a truncated log-normal distribution with positively correlated risk and risk aversion.³⁰ For each distribution, we study binary signal structures with $S = \{A, B\}$ and examine two types of signals: a monotonic signal in which the function $\pi(A | \cdot)$ is affine and strictly increasing, and a non-monotonic signal in which the function $\pi(A | \cdot)$ is quadratic and non-monotonic, as illustrated in Figure 2.

Figure 3 illustrates the price and welfare effects in the case of the monotonic signal for a specification of the model with two levels of risk aversion. The left-hand panel presents the equilibrium prices with and without the signal. We see that signal A, which is indicative of larger risks, leads to higher prices, while signal B, associated with lower risks, leads to lower prices. However, the downward price effect of signal B is considerably larger in magnitude than the upward price effect of signal A. This result is in line with Proposition 4, which states that the disclosure of a monotonic signal leads to an expected price reduction for each contract.³¹

The right-hand side of Figure 3 plots the equivalent variation of income associated with the signal disclosure, i.e., the amount of monetary gain or loss that leads to the same expected utility gain or loss as the by signal disclosure. In this example, the equivalent variation is strictly positive for almost all (99.7%) types, with a small

²⁹For $\delta > 0$ arbitrarily small, the value of the mapping $(1 - x)\omega_0(x)[1 - \omega_0(x)]$ is proportional to the variance of the risk within the pool selecting coverage x : the ratio of risk variance in pools x and x' in $(0, 1)$ converges to the ration of the values of this mappping at x and x' as $\delta \rightarrow 0$.

³⁰The probability mass assigned to each type in the finite support $\hat{\Theta}$ is proportional to the density of a lognormal distribution with covariance matrix

$$\begin{bmatrix} 0.2 & -0.12 \\ -0.12 & 0.25 \end{bmatrix}.$$

This is equivalent to eliminating all but two dimensions of heterogeneity in Einav et al. (2013)'s model: the level of absolute risk aversion and the parameter λ , which represents the expected loss of consumers' income conditional on their information.

³¹The algorithm uses a finite set of possible coverage levels and hence may generate pools at the top instead of separation. This is why the price of full coverage varies with the signal in the example displayed in Figure 3.

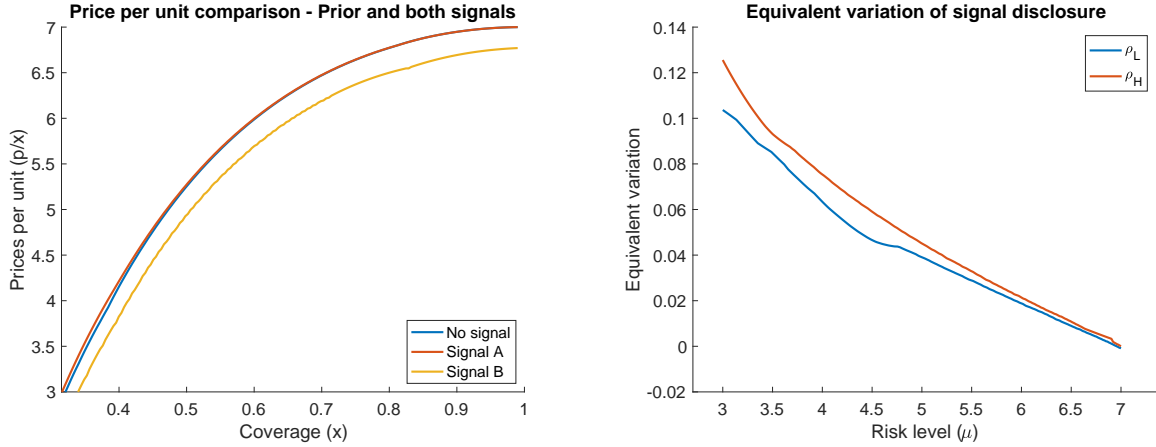


Figure 3: Equilibrium price per unit of coverage for each signal realization and without a signal. The simulation considers a grid of 500 values for risk in the interval $[3, 7]$, and binary grid $\{8, 9.7\}$ for risk aversion levels ($\delta = 1.7$).

mass of types (0.03%) at the very top of the risk distribution having a negative but small welfare loss. We see that the highest interim welfare gain from the signal disclosure is obtained by agents with lower risks.

The analytical results obtained using our approximation approach feature qualitatively similar patterns of interim welfare changes: in the case of δ small, agents with lower risk levels benefit more from the disclosure of a monotonic signal. Looking at points (b) and (c) in Proposition 3 we see that the impact of type the distribution on an agent purchasing coverage $x \in (0, 1)$ is determined by the integration of the distribution effects for all pools with higher coverage. This “top-down” aspect of our approximation indicates that lower-risk agents, who choose lower coverage in equilibrium, benefit the most from disclosure.³²

In order to evaluate the robustness of our findings to different levels of preference heterogeneity, we use the algorithm to study how the welfare effects of signal disclosure vary as we increase the level of preference heterogeneity, as captured by δ , and overall level of risk aversion, represented by ρ_0 . Figure 4 presents the mass of agents whose interim welfare improves with the disclosure of a monotonic signal — represented by colors and values in the heatmap — varying both the amount of preference heterogeneity in relative terms (δ/ρ_0 in the y axis) and the mid-level of risk aversion (ρ_0 in the x axis).

First, consider the left panel of Figure 4, which corresponds to the monotonic signal case. The mass of interim improving agents varies from 1% to 100% depending on model parameters. However, a clear pattern emerges: the mass of improving agents approaches one as either (i) the level of preference heterogeneity is small, or (ii) the level of risk aversion ρ_0 is small. Considering point (i), the simulations suggest that our interim Pareto welfare result (Proposition 4) holds beyond the arbitrarily small δ case. For example, if $\rho_0 = 8.84$ and $\delta \leq 0.35\rho_0$, the mass of interim-improving types is at least 97% (rounded to 1 in the heatmap).

Point (ii) sheds some light on the question of how the requirement of “small δ ” varies with the overall level of risk aversion: we can see that, as the parameter ρ_0 increases, the requirement that δ is small becomes more stringent. To understand this result, observe that the disclosure of an informative signal represents an additional source of dispersion in prices. Even if signal disclosure leads to an interim expected reduction in prices, consumers with sufficiently high risk aversion are still worse off. As a consequence, the set of parameters for which our interim Pareto welfare result applies becomes smaller.

The right-hand side panel displays the results for a similar exercise, but looking at a quadratic non-monotonic signal. For the region of small preference heterogeneity, the monotonic signal leads to an overall higher share of interim-improving agents. In this case, the overall connection between both parameters is less clear, than

³²See our Online Appendix, Subsection 1.1.1, for a numerical example with same parameters and a non-monotonic signal.

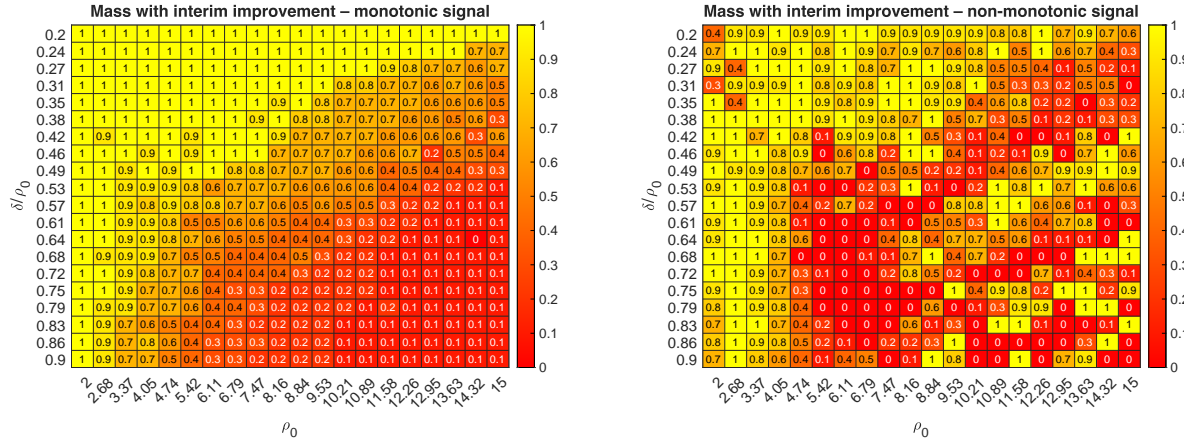


Figure 4: Each point in the grid corresponds to simulation with binary risk aversion set $\{\rho_0 - \delta/2, \rho_0 + \delta/2\}$. The values and colours in the heat-map indicate, for each parameter choice, the mass of types that have an interim welfare improvement from a signal disclosure. The left-hand side plot corresponds to the case of a monotone and linear signal structure, while the right-hand side plot corresponds to the case of a quadratic non-monotone signal. All entries correspond to simulations with a grid of 500 values for risk levels in the interval $[3, 7]$.

the one present in the monotonic case. It is worth clarifying that, while the result obtained in Proposition 4 guarantees that a monotonic signal is interim Pareto improving when δ is sufficiently small, regardless of the type distribution; it only implies that a non-monotonic signal hurts a positive mass of agents for some type distributions.

Finally, Figure 5 reports the welfare effects of the disclosure of the same two types of signals, but allowing for multiple levels of preference heterogeneity so as to approximate the case of continuously distributed risk and risk aversion. We consider a 30×30 grid of the type space for ρ and μ . The qualitative features highlighted in Figure 4 are starker in this case. Comparing the effects for the monotonic signal (presented in the left panel) to those for the non-monotonic signal (presented in the right panel), we see the first one mostly leads to a larger mass of welfare improving agents. Interestingly, the finer risk aversion grid generates a smoother pattern regarding the impact of varying δ and ρ_0 . The smoother impact that model parameters have on simulated outcomes suggests that equilibrium multiplicity may be less of an issue in the case of continuously distributed risk aversion: the “noisier” comparative statics in Figure 4 may be partially driven by implicit equilibrium selection performed by the algorithm. The similarities between Figures 2 and 4 also indicate that the lessons from the binary-risk-aversion model may extend to richer two-dimensional heterogeneity.

6 Effects of changes to the type distribution

In this section we study the effect of changes in the type distribution on prices and consumers’ welfare. This is a relevant question to assess the impact of demographic changes as well as to assess the benefits of policy recommendations motivated by their impact on the risk distribution of consumers present in the insurance market.³³ We show that strong conditions are needed to ensure that an overall “risk reduction” in the risk distribution leads to a price reduction and welfare improvement for almost all consumers in the market. First, we show that a shift in the risk distribution towards lower risks, in the (quite strong) sense given by the monotonic likelihood ratio property (MLRP), leads to lower prices and higher utility for a mass of agents that approaches

³³This issue is also relevant for risk classification, when we consider its effects from an ex-post perspective. Consider a population segmented into a riskier group B and a safer group A . Could a ban on risk classification benefit consumers in group B ? To answer this question, we must compare prices in segmented market B to the ones in the absence of segmentation, with all consumers trading in a single market.

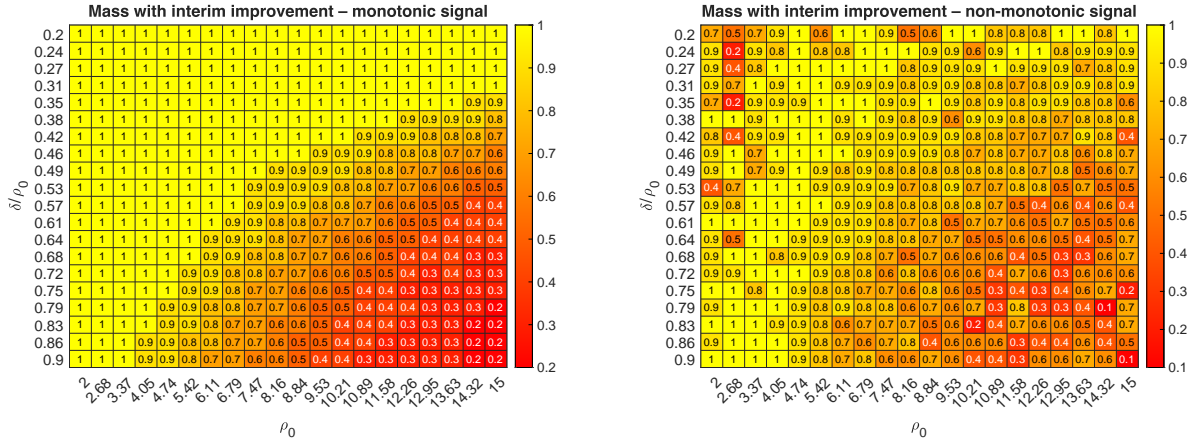


Figure 5: This plot represents numerical simulations with a grid of types containing 30 values for **both** risk μ as well as risk aversion ρ . Each entry corresponds to simulation where the set of risk levels and risk aversion are, respectively, finite grids of $[3, 7]$ and $[\rho_0 - \delta/2, \rho_0 + \delta/2]$. The values and colours in the heatmap indicate, for each parameter choice, the mass of types that have an interim welfare improvement from a signal disclosure. The left-hand side plot corresponds to the case of a monotone and linear signal, while the right-hand side plot corresponds to the case of a quadratic non-monotone signal.

one as δ approaches zero. We then show that, somewhat surprisingly, this strong ordering of distributions cannot be significantly relaxed, by presenting an example where a reduction in risks in the market, in the first order stochastic dominance sense, leads to a reduction in welfare for a positive mass of consumers.

In order to isolate and compare the impact of changes to both the risk and risk aversion distributions, we assume in this section that risk and risk aversion are independently distributed. Hence, the single function ϕ denotes the distribution of risk types in the population, and $\omega \in [0, 1]$ represents the share of low-risk-aversion consumers. We compare prices and consumers' welfare under type distributions (ω^A, ϕ^A) and (ω^B, ϕ^B) , first looking at changes in ϕ and then in ω . We use superscripts to refer to the relevant variables under distributions A and B , e.g., $p^A(\cdot; \delta)$ and \mathbb{P}^A for prices and the probability measure over types under distribution A , respectively.

Definition 5. Type distribution (ω^A, ϕ^A) dominates distribution (ω^B, ϕ^B) if for every $\epsilon > 0$ there exists $\bar{\delta} > 0$ such that

$$\mathbb{P}^k [(\mu, i) \mid V_i^A(\mu; \delta) > V_i^B(\mu; \delta)] > 1 - \epsilon,$$

for any arbitrary equilibrium selection, any $\delta \leq \bar{\delta}$ and $k \in \{A, B\}$.

This definition is in the same vein as Definition 2, but instead of averaging over different signal realizations it simply compares two fixed distributions.

6.1 Risk distribution

We first consider changes in the distribution of risk types in the population in the sense of the monotone likelihood ratio (MLRP). For any two strictly positive densities ϕ^A and ϕ^B on $[\mu_L, \mu_H]$, we say that ϕ^B MLRP-dominates ϕ^A if the ratio

$$\frac{\phi^B(\mu)}{\phi^A(\mu)}$$

is strictly increasing in μ . It is useful to relate this definition to that of S-MLRP in Section 5. Both of them are standard likelihood ratio orderings, but while S-MLRP makes comparisons of distributions over *signal realizations*, MLRP compares two distributions over *risk levels*. However, they are closely connected: if a signal satisfies S-MLRP, then the risk distributions conditional on any two signal realizations $s' \neq s$ are MLRP ordered.

In derivative terms, if ϕ^A and ϕ^B are continuously differentiable, MLRP is equivalent to

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} > \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)}, \quad (29)$$

for almost all (in the Lebesgue sense) risk levels $\mu \in [\mu_L, \mu_H]$. As shown in Proposition 3 (and Corollary 1, in particular), the second order coefficient of our price approximation depends explicitly on the rate of increase of the density of the risk distribution — which represents the relative frequency of pooled agents with very similar, and yet distinct, levels of risk. Building on this, the result which follows establishes that a change in the distribution which reduces risk in the MLRP-sense induces utility gain for almost all consumer types.

Proposition 6. (*Risk distribution effect*) *For any $\omega_0 \in (0, 1)$, if the distribution of risk types ϕ^B MLRP-dominates ϕ^A , then (ω_0, ϕ^A) dominates distribution (ω_0, ϕ^B) .*

Proof. The proof follows a similar argument to the one used in Lemma 4 and Proposition 4 regarding signal disclosure. We show in Appendix C (Lemmas 27 and 26) that the consumers' change in utility is determined by the price change induced by the distribution shift:

$$V_i^B(\mu; \delta) - V_i^A(\mu; \delta) = \frac{\delta^2}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) [p_{\delta\delta}^A(t_0(\mu)) - p_{\delta\delta}^B(t_0(\mu))] + o_i(\mu; \delta^2),$$

where $\delta^{-2}|o_i(\mu; \delta)|$ converges to zero uniformly in any compact subset of (μ_L, μ_H) . We have used in the above expression the fact that $p_{\delta}^A(\cdot) = p_{\delta}^B(\cdot)$, since ω_0 has the same value in both distributions. Using the expressions for the second derivative of prices derived in Corollary 1 we have

$$\frac{p_{\delta\delta}^B(x) - p_{\delta\delta}^A(x)}{x} = 2\omega_0(1 - \omega_0) \int_x^1 \frac{(1-z)^2}{z} \left[\frac{\dot{\phi}^B(m_0(z))}{\phi^B(m_0(z))} - \frac{\dot{\phi}^A(m_0(z))}{\phi^A(m_0(z))} \right] dz,$$

which is strictly positive for all $x \in (x_L, 1)$ and implies the result. \square

The result established in Proposition 6 is fairly intuitive. It shows that if the distribution of risk types changes towards a distribution with more mass on lower risks according to MLRP, then the risk distribution within each pool improves, which is reflected in lower prices and higher utility for consumers.

We should point out that the condition that the decrease in risk in the distribution is according to the MLRP criterion, although quite demanding, cannot be significantly relaxed, as shown by the following example.

Example 1 (A reduction in risk in the FOSD sense may not be welfare improving). Suppose ϕ^A is the uniform distribution on $[\mu_L, \mu_H]$ and ϕ^B is a strictly concave function such that $\dot{\phi}^B(\mu_H) < 0$ and $\phi^B(\mu_H) \geq \phi^A(\mu_H)$.

These properties imply that density ϕ^B crosses ϕ^A only once from below, which ensures that ϕ^B first order stochastically dominates ϕ^A . Figure 6 illustrates an example where ϕ^B is quadratic and ϕ^A is linear. Since $\dot{\phi}^B(\mu_H) < 0$, there exists $\mu_0 \in (\mu_L, \mu_H)$ such that, for any $\mu \in [\mu_0, \mu_H]$,

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} < 0 = \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)}.$$

Using Corollary 1, all types with risk level $\mu \in [\mu_0, \mu_H]$ have strictly higher utility and face strictly lower coverage prices in equilibrium with risk distribution ϕ^B than with risk distribution ϕ^A even though the latter features less risk in the FOSD sense. Hence, this set of consumers benefit from being in a market where there are more high-risk types in the population. The intuition for this result is that each risk pool contains consumers with similar risk levels, for small preference heterogeneity, and hence only the relative frequency of types with

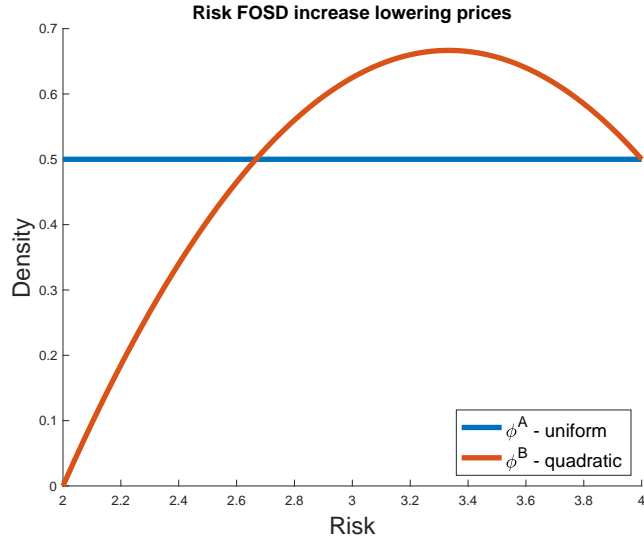


Figure 6: Distribution ϕ^B FOSD dominates ϕ^A and leads to lower prices for high coverage levels.

similar risk levels matters for equilibrium prices. Since ϕ^B is decreasing at the top, the relative frequency of the lower risk types *within the pools* that only include similar risk levels above μ_0 is lower than the one under ϕ^A .

The previous example shows that the ordering of distributions according to the first order stochastic dominance criterion is too weak and does not deliver an unambiguous welfare result as in the case of the MLRP ordering. It shows that a FOSD increase in the risk distribution may have ambiguous welfare effects, where some consumer types benefit from the increase in the mass of risk types in the population. This result illustrates how the effect of changes in the risk distribution in competitive models of insurance markets may be counter-intuitive and hence any policy analysis in this environment requires careful consideration of such effects.

6.2 Preferences distribution

There are two drivers of consumers' demand for coverage in our environment: their risk level, which is the underlying source of adverse selection, and their risk aversion. If consumers have an overall high willingness to pay for coverage that is unrelated to their risk level, the problem of adverse selection is alleviated since the purchase of higher coverage may not be a strong signal of higher risks. More precisely, in our equilibrium analysis the set of consumers choosing the same coverage level in the discrete pooling region is composed by a high-risk-low-risk-aversion and a low-risk-high-risk-aversion type. Hence, an increase in the relative frequency of high-risk-aversion types in the population leads to a lower average risk level in each pool of consumers sharing the same coverage and, as a consequence, to lower prices. The result below shows that this price reduction leads to a utility gain for almost all agents.

Proposition 7. (*Effects of changes in the Risk aversion distribution*) An increase in the share of consumers with high-risk aversion (i.e., $\omega^B < \omega^A$) leads to a uniform utility increase, i.e., type distribution (ω^B, ϕ) dominates (ω^A, ϕ) .

Proof. Once again, using Lemma 27, we can relate changes in consumers' utility to price changes:

$$V_i^B(\mu; \delta) - V_i^A(\mu; \delta) = \delta \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) [p_\delta^A(t_0(\mu)) - p_\delta^B(t_0(\mu))] + o_i(\mu; \delta),$$

where $\delta^{-1}|o_i(\mu; \delta)|$ converges to zero uniformly on M , for any compact set $M \subset (\mu_L, \mu_H)$. Now, suppose that $\omega^B < \omega^A$, we then have that

$$\frac{p_\delta^A(x) - p_\delta^B(x)}{x} = x(\omega^A - \omega^B)(x - 1 + \log x) > 0.$$

□

7 Conclusion

This paper provides a parsimonious competitive equilibrium model of insurance markets where consumers differ both in risk and risk aversion and a rich set of contracts is available. The model illustrates the complex relationship between the distribution of consumer characteristics in the population and the equilibrium pattern of screening, still allowing for tractable analytical results. We also demonstrate how these features of the model provide an important basis for an effective analysis of important policy questions in insurance markets such as risk classification. We characterize the welfare effect of these interventions and provide conditions under which they are beneficial. The monotonicity condition on the public signal available to firms presented here is novel to the literature, has an intuitive interpretation, and is shown to be central to the welfare impact of risk classification. Additionally, we provide conditions under which a reduction in the riskiness in the population benefits all consumer types, and show these conditions are fairly tight.

The results obtained here suggest multiple directions for future work. Our analytical characterization of welfare improving signals points to the empirical question of what set of demographic characteristics constitute monotonic signals. On the theoretical side, the characterization of multidimensional models in which preferences do not satisfy the single-crossing property is a notoriously challenging issue. The tractable approach used here can be applied to analyze other competitive settings with rich heterogeneity, such as labour and credit markets. Our analysis assumes public signals, implying that all insurance firms have access to the same information. If some firms have greater access to these signals, allowing for detailed risk classification may also hinder competition by limiting the ability of less informed firms to operate in the market. Finally, policy makers in practice may be able to design the information to be used by firms. A simple example is the determination of the age brackets that can be used by firms when pricing consumers in health insurance. Garcia and Tsur (2021) analyze this question in the case of a single available contract, but the economic forces generated by a rich set of contracts require a different analysis, which would benefit from the results obtained here.

Appendix A - Equilibrium characterization

Throughout the appendix we use the following definitions, with abuse of notation, for $(\mu_l, \mu_h) \in [\mu_L, \mu_H]^2$:

$$\omega(\mu_l, \mu_h) \equiv \frac{\phi_l(\mu_l)}{\phi_l(\mu_l) + \phi_h(\mu_h)}$$

and

$$R(\mu, x, \delta) \equiv \omega(\mu, \mu - (1 - x)\delta). \tag{30}$$

Proof of Lemma 1

Properties (i) and (ii) are equivalent to monotonicity of demand in each dimension, i.e., $t(\cdot, i)$ is non-decreasing for $i \in \{l, h\}$ and $t(\mu, l) \leq t(\mu, h)$. Monotonicity follows from the fact that preferences satisfy the single-crossing property on each dimension, for a given value of the other dimension.

For property (iii): If $p(x), p(\hat{x}) > 0$, using $(m_i(x), i) \in \theta^+(x)$ and $(m_j(\hat{x}), j) \in \theta^+(\hat{x})$, we have

$$u(x, m_i(x), i) - u(\hat{x}, m_i(x), i) \geq p(x) - p(\hat{x}) \geq u(x, m_j(\hat{x}), j) - u(\hat{x}, m_j(\hat{x}), j)$$

which implies that

$$|p(x) - p(\hat{x})| \leq L|x - \hat{x}|,$$

where $L = \sup \{|u_x(x, \theta)|; x \in [0, 1] \text{ and } \theta \in \Theta\}$. If $p(\hat{x}) = 0 < p(x)$, the first inequality above still holds and implies the result. If $p(\hat{x}) = p(x) = 0$, the result is trivial.

Proof of Lemma 2

Direct differentiation implies

$$\frac{\partial e(q, x)}{\partial q} = 1 - \delta(1-x) \frac{\phi'_h(q - (1-x)\rho_h) \phi_l(q - (1-x)\rho_l) - \phi'_l(q - (1-x)\rho_l) \phi_h(q - (1-x)\rho_h)}{[\phi_h(q - (1-x)\rho_h) + \phi_l(q - (1-x)\rho_l)]^2}.$$

Defining

$$D \equiv \sup_{\mu, \mu' \in [\mu_L, \mu_H]} \left| \frac{\phi'_h(\mu) \phi_l(\mu') - \phi'_l(\mu') \phi_h(\mu)}{\{\phi_h(\mu) + \phi_l(\mu')\}^2} \right|,$$

we have that if

$$\delta < \frac{1}{D+1},$$

then

$$\frac{\partial e(q, x)}{\partial q} > 1 - (1-x) \frac{D}{D+1} > 0,$$

for all $(q, x) \in E$.

Proof of Proposition 1

Consider a no-gap equilibrium with type assignments (m_l, m_h) , posterior $w(\cdot)$, and interval of traded coverages $X \subset [0, 1]$. From the monotonicity property of the equilibrium in Lemma 1, X can be partitioned into the following sets:

- X_s^i the separating coverages chosen by consumers with risk aversion $i \in \{l, h\}$;
- X_d the discrete pooled coverages chosen by exactly two risk averse types;
- X_c the continuous pooled coverages chosen by intervals of risk types.

For each $x \in X_s^h \cup X_s^l \cup X_d$, the type assignment is either empty or a singleton, and we represent it as a function with image in $[\mu_l, \mu_h] \cup \{\emptyset\}$. The posterior $w(\cdot)$ is extended to the separating regions by using $w(x) = 1$ in X_s^l and $w(x) = 0$ in X_s^h . For convenience, we define the product $0 \cdot \emptyset$ to be zero.

The following lemma shows the basic properties of these sets:

Lemma 5. (a) X_c is a countable set of X ;

(b) If $x \in X_s^h$ and $y \in X \cap [x, 1]$, then $y \in X_s^h$ (i.e., X_s^h is an interval in the higher end of X);

(c) If $x \in X_s^l$ and $y \in X \cap [0, x]$, then $y \in X_s^l$ (i.e., X_s^l is an interval in the lower end of X).

Proof. (a) This is a trivial consequence of the monotonicity of m_i , for $i \in \{l, h\}$.

(b) Let us first show that the set $(X_s^l \cup X_d)' \cap (x, 1]$ has zero Lebesgue measure³⁴, where A' is the set of accumulation points of A . Suppose that this is not the case and let $\underline{y} = \inf \left\{ (X_s^l \cup X_d)' \cap (x, 1] \right\}$. If the interval $[x, \underline{y}]$ is non-degenerated, it must be the union of isolated points in $X_s^l \cup X_d$, points in X_c and points in X_s^h . Since the first two sets are countable, \underline{y} is also the limit of a sequence of points in X_s^h on the left, which is also trivially true in the case $x = \underline{y}$ (i.e., when $[x, \underline{y}]$ is a degenerated interval). We can then find a sequence $(y_n^-)_n$ in X_s^h with $y_n^- \nearrow \underline{y}$ and $(y_n^+)_n$ in $[x, 1] \cap [X_s^l \cup X_d]$ with $y_n^+ \searrow \underline{y}$. Optimality requires that $m_l(y_n^+) \geq m_h(y_n^-) + \delta(1 - y_n^-)$, and Lemma 6 implies that $\lim_{n \rightarrow \infty} w(y_n^+) > 0$, which together with the zero profit condition implies

$$\lim_n \frac{p(y_n^-)}{y_n^-} = \lim_n m_h(y_n^-) < \lim_n [(1 - w(y_n^+)) m_h(y_n^+) + w(y_n^+) m_l(y_n^+)] = \lim_n \frac{p(y_n^+)}{y_n^+},$$

a contradiction with the continuity of $p(\cdot)$. Therefore, $[x, 1] \cap X_s^h$ has full measure on $[x, 1] \cap X$, since $X_c \cup X_s^l \cup X_d$ has also zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_h(y) \text{ and } w(y) = 0,$$

for almost all $y \in [x, 1] \cap X$. Hence, by the continuity of the function $p(y)/y$ and the monotonicity of m_h , these equalities must hold for all $y \in [x, 1] \cap X$, showing the result.

(c) Let us first show that the set $(X_s^h \cup X_d)' \cap [0, x]$ has zero measure. Suppose that this is not the case and let $\bar{y} = \sup \left\{ (X_s^h \cup X_d)' \cap [0, x] \right\}$. If the interval $[\bar{y}, x]$ is non-degenerated, it must be the union of isolated points in $X_s^h \cup X_d$, points in X_c and points in X_s^l . Since the first two sets are countable, \bar{y} is also the limit of a sequence of points in X_s^l on the right, which is also trivially true in the case $x = \bar{y}$ (i.e., when $[x, \bar{y}]$ is a degenerated interval). We can then find a sequence $(y_n^+)_n$ in X_s^l with $y_n^+ \searrow \bar{y}$, and $(y_n^-)_n$ in $X_s^h \cup X_d$ with $y_n^- \nearrow \bar{y}$. Optimality implies $m_h(y_n^-) \leq m_l(y_n^+) - \delta(1 - y_n^+)$, and Lemma 6 implies $\lim_n w(y_n^-) < 1$, which together with the zero profit condition implies

$$\lim_n \frac{p(y_n^+)}{y_n^+} = \lim_n m_l(y_n^+) > \lim_n [(1 - w(y_n^-)) m_h(y_n^-) + w(y_n^-) m_l(y_n^-)] = \lim_n \frac{p(y_n^-)}{y_n^-},$$

a contradiction with continuity of $p(\cdot)$. Therefore, $[0, x] \cap X_s^l$ has full measure on $[0, x] \cap X$, since $X_c \cup X_s^h \cup X_d$ has zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_l(y) \text{ and } w(y) = 1,$$

for almost all $y \in [0, x] \cap X$. Hence, from the continuity of the function $p(y)/y$ and monotonicity of m_l , these equalities must hold for all $y \in [0, x] \cap X$, showing the result. \square

The next lemma is the auxiliary result used in the proof of Lemma 5:

Lemma 6. (a) If $(X_s^l \cup X_d) \cap (x, 1]$ has positive measure, then

$$w \left(\inf \left\{ y; y \in (X_s^l \cup X_d)' \cap (x, 1] \right\} \right) > 0;$$

³⁴In what follows, when we refer to positive or zero measure sets we mean Lebesgue measure.

(b) If $X_s^h \cup X_d$ has positive measure, then $\sup \{w(y); y \in X_s^h \cup X_d\} < 1$.

Proof. (a) Define $\underline{y} = \inf \left\{ (X_s^l \cup X_d)' \cap (x, 1] \right\}$. Suppose, by absurd, that $w(\underline{y}) = 0$. We claim that, for every $\epsilon > 0$, $(\underline{y}, \underline{y} + \epsilon) \cap X_d'$ has positive measure. Otherwise, $(\underline{y}, \underline{y} + \epsilon) \cap (X_s^l \cup X_s^h)'$ has full measure for some $\epsilon > 0$. But this implies that, for a fixed $i \in \{l, h\}$, $X_s^i \cap (\underline{y}, \underline{y} + \epsilon)$ has full measure. Otherwise there exists $y_0 \in (\underline{y}, \underline{y} + \epsilon)$ and sequences $(y_n^-)_n$ and $(y_n^+)_n$ such that $y_n^- \nearrow y_0$, $y_n^+ \searrow y_0$, $y_n^- \in X_s^i$ and $y_n^+ \in X_s^j$, for $i \neq j$. For each element of the sequence we have that

$$\frac{p(y_n^-)}{y_n^-} = m_i(y_n^-),$$

with an analogous equality holding for y_n^+ and j . Defining $m_i^0 = \lim_n m_i(y_n^-)$ and $m_j^0 = \lim_n m_j(y_n^+)$, continuity of prices implies $m_i^0 = m_j^0$ and $\dot{p}(y_0^-) = m_i^0 + (1 - y_0)\rho_i$, $\dot{p}(y_0^+) = m_j^0 + (1 - y_0)\rho_j$. This means the price has a kink at y_0 , which is inconsistent with equilibrium. If $i = h$, the marginal price jumps down at y_0 and type (m_h^0, h) wood choose coverage above y_0 , a contradiction. If $i = l$, monotonicity implies that type $(m_h(y_n^+), l)$ must choose coverage in the interval (y_0, y_n^+) , for each n . Hence, there exists sequence $\{y'_n\}_n$ such that $y'_n \searrow y_0$ and $m_l(y'_n) \searrow m_l^0$. But this implies $\dot{p}(y_0^+) = m_l^0 + (1 - y_0)\rho_l < m_h^0 + (1 - y_0)\rho_h = \dot{p}(y_0^+)$, a contradiction as well. Finally, since $w(\underline{y}) = 0$, we must have that $(\underline{y}, \underline{y} + \epsilon) \subset X_s^h$, which contradicts the definition of \underline{y} .

Since m_i is non-decreasing, it must be differentiable for almost all points in $X_d \cap (\underline{y}, \underline{y} + \epsilon)$, for all $\epsilon > 0$. Notice that

$$\frac{p(x)}{x} = m_h(x) + w(x)(m_l(x) - m_h(x)).$$

Then (10), (11) and (12) imply that (m_l, m_h, w) is the solution to the following ordinary differential equation (ODE) system:

$$\dot{m}_l(x) = \frac{1 - R(m_l(x), x, \delta)}{R(m_l(x), x, \delta) - w(x)} w(x)\delta, \quad (31)$$

$$\dot{w}(x) = \left(\frac{1}{x} - \frac{1}{1-x} \right) (1 - w(x)) + \frac{\rho_l}{x\delta} - \frac{\dot{m}_l(x)}{(1-x)\delta}, \quad (32)$$

$$m_h(x) = m_l(x) - (1-x)\delta,$$

on $X_d \cap (\underline{y}, \underline{y} + \epsilon)$, where $R(\mu, x, \delta)$ is defined in (30). By the same argument above, there are two possible cases:

(i) For every $\epsilon > 0$, $(\underline{y} - \epsilon, \underline{y}) \cap X_d$ has positive measure. Since $w(\underline{y}) = 0$, we must have that $\dot{w}(\underline{y}^-) \leq 0$ and $\dot{w}(\underline{y}^+) \geq 0$ (otherwise, this would imply that $p(x)/x < m_l(x)$, for some x close to \underline{y}). Using these properties and the discrete pooling ODE described above, we get

$$\underline{y} = \frac{\delta + \rho_l}{2\delta + \rho_l},$$

which implies that $\dot{w}(\underline{y}) = 0$, and, taking the second derivative, we get

$$\ddot{w}(\underline{y}) = - \left(1 + \frac{\rho_l}{\delta} \right) \frac{1}{\underline{y}^2} - \frac{1}{(1-\underline{y})^2} < 0,$$

which means that \underline{y} is a local maximum of w . Since $w(\underline{y}) = 0$, this is a contradiction.

(ii) There exists $\epsilon > 0$ such that $(\underline{y} - \epsilon, \underline{y}) \subset X_s^h$. Then, for every $z \in (\underline{y} - \epsilon, \underline{y})$, we have that

$$\frac{p(z)}{z} = m_h(z),$$

which implies that

$$\left. \frac{d}{dy} \left(\frac{p(y)}{y} \right) \right|_{y=\underline{y}^-} = \dot{m}_h(\underline{y}^-).$$

On the other hand,

$$\dot{p}(z) = m_h(z) + (1-z)\rho_h,$$

for almost all $z \in (\underline{y}, \underline{y} + \epsilon)$. Since $\lim_{z \rightarrow \underline{y}^+} w(z) = 0$, we have that m_h is continuous at \underline{y} , which implies that \dot{p} is also continuous at \underline{y} . Moreover,

$$\dot{p}(\underline{y}^+) = [\dot{m}_h(\underline{y}^+) + \dot{w}(\underline{y}^+)(m_l(\underline{y}) - m_h(\underline{y}))] \underline{y} + \frac{p(\underline{y})}{\underline{y}},$$

which implies that

$$\left. \frac{d}{dy} \left(\frac{p(y)}{y} \right) \right|_{y=\underline{y}^+} \geq \dot{m}_h(\underline{y}^+),$$

where the last inequality is strict if and only if $\dot{w}(\underline{y}^+) > 0$ since $m_l(\underline{y}) - m_h(\underline{y}) = (1-\underline{y})\delta > 0$. By the continuity of \dot{p} at \underline{y} , $\dot{w}(\underline{y}^+) = 0$ and a proof analogous to the case (i) implies $\ddot{w}(\underline{y}) < 0$, a contradiction.

(b) We know that $x \in X_s^h$ implies $w(x) = 0$, and $x \in X_d$ implies

$$w(x) \leq R(m_l(x), x, \delta) \leq \sup_{\mu, \mu' \in [\mu_L, \mu_H]} \omega(\mu, \mu') < 1.$$

□

Lemma 7. *Under Assumption 1, X_d is an interval.*

Proof. If X_d is not an interval, Lemma 5 shows that the only possibility is for a point $x \in X_c$ to exist such that $(x - \epsilon, x + \epsilon) \subset X_c \cup X_d$ for some $\epsilon > 0$. Take sequences $(x_n^-)_n$ and $(x_n^+)_n$ in X_d such that $x_n^- < x < x_n^+$ and $\lim_n x_n^- = x = \lim_n x_n^+$. Define $m_i^0 \equiv \lim_n m_i(x_n^-)$ and $m_i^1 \equiv \lim_n m_i(x_n^+)$, for $i = l, h$. The optimality condition of types in X_d implies that $m_l^0 = m_h^0 + \delta(1-x)$ and $m_l^1 = m_h^1 + \delta(1-x)$. We then must have that $m_h^0 < m_h^1$,

$$m_h(x) = [m_h^0, m_h^1]$$

and

$$m_l(x) = [m_h^0 + \delta(1-x), m_h^1 + \delta(1-x)].$$

The zero profit condition at coverage x implies that

$$\frac{p(x)}{x} = \frac{\int_{m_h^0}^{m_h^1} e(z + (1-x)\rho_h, x) [\phi_l(z + \delta(1-x)) + \phi_h(z)] dz}{\int_{m_h^0}^{m_h^1} [\phi_l(z + \delta(1-x)) + \phi_h(z)] dz}, \quad (33)$$

where $e(\cdot)$ is defined in (9). But, for each n , the zero profit condition in X_d implies that

$$\frac{p(x_n^-)}{x_n^-} = w(x_n^-) m_l(x_n^-) + (1-w(x_n^-)) m_h(x_n^-),$$

which is lower than $e(m_h(x_n^-) + (1-x_n^-)\rho_h, x_n^-)$, since

$$e(m_h(x_n^-) + (1-x_n^-)\rho_h, x_n^-) = \omega(m_l(x_n^+), m_h(x_n^+)) m_l(x_n^-) + [1 - \omega(m_l(x_n^+), m_h(x_n^+))] m_h(x_n^-)$$

and $w(x_n^-) \leq \omega(m_l(x_n^+), m_h(x_n^+))$. Taking limits, we have that

$$\lim_n \frac{p(x_n^-)}{x_n^-} \leq e(m_h^0 + (1-x)\rho_h, x),$$

which is strictly lower than $\frac{p(x)}{x}$, from (33), by Assumption 1, and contradicts continuity of $p(\cdot)$. Therefore, the result follows from Lemma 5. \square

The following lemma completes the proof and is also useful in the remaining proofs.

Lemma 8. *There exist $0 \leq x_0 < x_d \leq x_u < 1$ with $p(x_0) = \mu_L x_0$ such that:*

- (i) $X_s^h = (x_u, 1]$, $m_l(x_u) = [\underline{a}_u, \mu_H]$ and $m_h(x_u) = [\bar{a}_u, \bar{b}_u]$;
- (ii) $X_s^l = [x_0, x_d)$, $m_l(x_d) = [\underline{a}_d, \underline{b}_d]$ and $m_h(x_d) = [\mu_L, \bar{b}_d]$;
- (iii) if $x_d < x_u$, then $X_d = (x_d, x_u)$.

Proof. Lemmas 5 and 7 imply the result desired with the weaker inequalities $0 \leq x_0 \leq x_d \leq x_u \leq 1$. We will show the result in three steps.

a) $x_0 < x_d$. If $x_0 = x_d$, the equilibrium restriction for off-path contracts requires that, for $\varepsilon > 0$ small, $\theta^+(x_0 - \varepsilon) = (\mu_L, l)$ and $p(x_0 - \varepsilon) \leq (x - \varepsilon)\mu_L$. But this implies a discontinuity in prices: if $x_d \in X_c$ then $\frac{p(x_d)}{x_d}$ is the average of risks over a positive measure set, which is above μ_L strictly; if $[x_d, x_d + \varepsilon] \in X_d$ for some $\varepsilon > 0$, we must have $\omega(x_d) = 0$ which is ruled out by Lemma 6.

b) $x_d < 1$. If $x_d = 1$, for any $x \in (x_0, x_d = 1)$ optimality implies

$$u_x(1, m_l(x), l) < \dot{p}(1-),$$

but $u_x(1, m_l(x), l) = u_x(1, m_l(x), h)$ and hence type $(m_l(x), h)$ would deviate by lowering his coverage level.

c) $x_u < 1$. If $x_u = 1$, differential equation (32) is satisfied in (x_d, x_u) and, since $w(\cdot) \in [0, 1]$ we must have $w(x) \rightarrow 1$ when $x \rightarrow 1$. But (31) implies that $\dot{m}_l(x) \rightarrow \delta \frac{1 - \omega(\mu_H, \mu_H)}{\omega(\mu_H, \mu_H)} > 0$, and using (32), there exists $\underline{x} < 1$ such that, for $x \in (\underline{x}, 1)$

$$\dot{w}(x) < -\delta \frac{1 - \omega(\mu_H, \mu_H)}{2\omega(\mu_H, \mu_H)} \frac{1}{1-x},$$

which implies that $w(x) \rightarrow \infty$ as $x \rightarrow 1$, a contradiction. \square

Proof of Lemma 3

Suppose that $I = [x - \epsilon, x + \epsilon]$ is an interval of coverage with discrete pooling of an equilibrium (p, m_l, m_h, w) such that m_l and m_h are differentiable at x , with $\epsilon > 0$ sufficiently small. From the consistency condition of equilibrium definition we must have

$$\Pr[x \in I | \theta(x) \in [m_l(x - \epsilon/2), m_l(x + \epsilon/2)] \times \{\rho_l\}] = \frac{\int_{m_l(x - \epsilon/2)}^{m_l(x + \epsilon/2)} \phi_l(z) dz}{\int_{m_l(x - \epsilon/2)}^{m_l(x + \epsilon/2)} \phi_l(z) dz + \int_{m_h(x - \epsilon/2)}^{m_h(x + \epsilon/2)} \phi_h(z) dz}$$

and, taking the limit $\epsilon \rightarrow 0$, the left hand side must converge to $w(x)$. However, dividing the numerator and denominator of the fraction on the right hand side by ϵ and taking the limit, we get

$$w(x) = \frac{\phi_l(m_l(x)) \dot{m}_l(x)}{\phi_l(m_l(x)) \dot{m}_l(x) + \phi_h(m_l(x)) \dot{m}_h(x)},$$

which gives the result.

Appendix B - Existence

Proof of Proposition 2

We can re-write the statement of this proposition as follows:

Proposition 8. *For sufficiently small $\delta > 0$ there exists a continuum of equilibria characterized by (p, m_l, m_h) and $0 < x_d < x_u < 1$ satisfying the following properties:*

- (a) *top separation: $p(x) = m_h(x) = \rho_h(1 - x + \log x) + \mu_H$ and $m_l(x) = \emptyset$, for $x > x_u$;*
- (b) *continuous pooling at x_u : $m_l(x_u) = [\underline{a}_u, \mu_H]$ and $m_h(x_u) = [\bar{a}_u, \bar{b}_u]$ such that:*
 - (b.1) *transversality: $\underline{a}_u = m_l(x_u^-)$, $\bar{a}_u = m_h(x_u^-)$ and $\bar{b}_u = m_h(x_u^+)$;*
 - (b.2) *smooth pasting: $\mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\underline{a}_u, \mu_H], \rho = \rho_l] + \mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\bar{a}_u, \bar{b}_u], \rho = \rho_h] = m_h(x_u^+)$;*
- (c) *discrete pooling in the interval (x_d, x_u) : there exists $w : (x_d, x_u) \rightarrow [0, 1]$ such that:*
 - (c.1) *zero profit: $\frac{p(x)}{x} = w(x)m_l(x) + (1 - w(x))m_h(x)$;*
 - (c.2) *optimality: $\dot{p}(x) = m_h(x) + (1 - x)\rho_h = m_l(x) + (1 - x)\rho_l$;*
 - (c.3) *consistency of beliefs: $\frac{w(x)}{1 - w(x)} = \frac{\phi_l(m_l(x))m_l(x)}{\phi_h(m_h(x))m_h(x)}$;*
 - (c.4) *price continuity at x_d and x_u ;*
- (d) *continuous pooling at x_d : $m_l(x_d) = [\underline{a}_d, \underline{b}_d]$ and $m_h(x_d) = [\mu_L, \bar{b}_d]$ such that:*
 - (d.1) *transversality: $\underline{a}_d = m_l(x_d^-)$, $\underline{b}_d = m_l(x_d^+)$ and $\bar{b}_d = m_h(x_d^+)$;*
 - (d.2) *smooth pasting: $\mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\underline{a}_d, \underline{b}_d], \rho = \rho_l] + \mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\mu_L, \bar{b}_d], \rho = \rho_h] = m_l(x_d^-)$;*
- (e) *bottom separation: $p(x) = m_l(x) = \rho_l(x_d - x + \log(\frac{x}{x_d})) + \underline{a}_d$ and $m_h(x) = \emptyset$, for $x < x_d$.*

In order to show the existence of a non-gap equilibrium we divide the proof in three parts corresponding to the construction of the pooling regions: (i) top continuous pooling region; (ii) discrete pooling region; (iii) bottom continuous region. Let us denote

$$\bar{b}(x) := (\rho_0 + \delta/2)[1 - x + \log x] + \mu_H \quad (34)$$

the top separating equilibrium part.

Top continuous pooling region

For each $\delta > 0$, a top continuous region is characterized by a vector

$$(\underline{a}, \bar{a}, \underline{b}, \bar{b}, x, w) \in [\mu_L, \mu_H]^4 \times [0, 1]^2$$

that satisfies

$$\underline{b} = \mu_H \quad (35)$$

$$\text{Top separating region: } \bar{b} = \bar{b}(x) \quad (36)$$

$$u_x(x_u, \underline{a}_u, l) = u_x(x_u, \bar{a}_u, h) : \underline{a} = \bar{a} + (1 - x)\delta \quad (37)$$

$$\text{Price continuity: } w\underline{a} + (1 - w)\bar{a} = \bar{b} \Leftrightarrow \bar{a} = \bar{b} - w(1 - x)\delta \quad (38)$$

$$\text{Price continuity: } \frac{\int_{\underline{a}}^{\underline{b}} z\phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} z\phi_h(z) dz}{\int_{\underline{a}}^{\underline{b}} \phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} \phi_h(z) dz} = \bar{b} \quad (39)$$

$$\text{No deviation for } (\mu_H, \bar{b}) : \bar{b} + (1 - x)\delta \geq \mu_H \quad (40)$$

and

$$\text{Weight feasibility: } w \in [0, R(\underline{a}, x, \delta)], \quad (41)$$

where we are dropping the sub-index u for convenience. Notice that $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$ defines a top continuous pooling region if and only if $x \in (0, 1)$, $w \in [0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ solves the equation

$$G(x, w) := \int_{\bar{b}(x) + (1-w)(1-x)\delta}^{\mu_H} (z - \bar{b}(x))\phi_l(z)dz + \int_{\bar{b}(x) - w(1-x)\delta}^{\bar{b}(x)} (z - \bar{b}(x))\phi_h(z)dz = 0 \quad (42)$$

and $\bar{b}(x) \geq \mu_H - (1-x)\delta$. In this case, $\underline{a} = \bar{b}(x) + (1-w)(1-x)\delta$, $\bar{a} = \bar{b}(x) - w(1-x)\delta$, $\underline{b} = \mu_H$ and $\bar{b} = \bar{b}(x)$. For simplicity, we will refer only to (x, w) in short instead of the whole vector $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$. Define $\underline{x}^\delta \in (0, 1)$ as the interior solution of the equation

$$\bar{b}(x) + (1-x)\delta - \mu_H = 0$$

which represents the coverage level of risk type with high risk aversion in top separating equilibrium part that pools with risk type μ_h with low risk aversion.

Lemma 9. (*Top continuous pooling region*) *The top continuous pooling region is parametrized by a non-degenerated interval $[\underline{x}^\delta, \bar{x}^\delta] \subset [0, 1]$, with $\underline{x}^\delta < \bar{x}^\delta$, satisfying:*

(a) *for each $x \in [\underline{x}_\delta, \bar{x}_\delta]$, there exists $w \in [0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ such that (x, w) defines a top continuous region;*

(b) *if $x = \bar{x}_\delta < 1$, then $w = R(\bar{b}(\bar{x}^\delta) + (1-w)(1-\bar{x}^\delta)\delta, \bar{x}^\delta, \delta)$.*

Proof. We are trying to find a proper solution to equation (42). We start finding a solution for $x = \underline{x}^\delta \in (0, 1)$, and then show that this solution can be extended to the right of \underline{x}^δ . First, for $x \in (0, 1)$ define

$$\underline{w}(x) \equiv \frac{\bar{b}(x) + (1-x)\delta - \mu_H}{(1-x)\delta},$$

$$g(x, w) \equiv \min \{R(\bar{b}(x) + (1-w')(1-x)\delta, x, \delta) - w \mid w' \in [0, w]\}.$$

Notice that $\bar{b}(x) + (1-w)(1-x)\delta \leq \mu_H$ if, and only if, $w \geq \underline{w}(x)$; $\underline{w}(x) > 0$, for $x \in (\underline{x}^\delta, 1)$; and $G(x, \underline{w}(x)) < 0$.

The derivative of G w.r.t. w is

$$G_w(x, w) = (1-w)(1-x)^2\delta^2\phi_l(\bar{b}(x) + (1-w)(1-x)\delta) - w(1-x)^2\delta\phi_h(\bar{b}(x) - w(1-x)\delta),$$

which is positive if and only if

$$w < R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta). \quad (43)$$

Now considering $x = \underline{x}^\delta$, we have that $\underline{w}(x) = G(\underline{x}^\delta, 0) = 0$,

$$G_w(\underline{x}^\delta, 0) > 0$$

and

$$g(\underline{x}^\delta, 0) = R(\mu_H, \underline{x}^\delta, \delta) > 0.$$

Hence we can find a point $w_0 \in (0, 1)$ such that

$$G(\underline{x}^\delta, w_0) > 0,$$

$$g(\underline{x}^\delta, w_0) > 0.$$

By continuity, we can find $\varepsilon > 0$ sufficiently small such that, for any $x \in (\underline{x}^\delta, \underline{x}^\delta + \varepsilon)$

$$G(x, w_0) > 0 > G(x, \underline{w}(x)),$$

$$g(x, w_0) > 0.$$

From the intermediate value theorem, for each $x \in (\underline{x}^\delta, \underline{x}^\delta + \varepsilon)$, we can find a solution $w^*(x)$ to $G(x, w) = 0$ in $(0, w_0)$. Since $g(x, w_0) > 0$, this solution satisfies $w \in (0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ and hence gives us a top continuous pooling region. Define \bar{x}^δ to be the supremum of all points $x \in (0, 1)$ such that the solution $w^*(\cdot)$ can be extended to $[\underline{x}^\delta, x]$. Our argument above shows that $\underline{x}^\delta < \bar{x}^\delta$. Suppose $\bar{x}^\delta < 1$. If $\underline{w}(\bar{x}^\delta) < w^*(\bar{x}^\delta) < R(\mu_H + w(1-x)\delta, x, \delta)$, the implicit function theorem implies that $w^*(\cdot)$ can be extended beyond \bar{x}^δ , a contradiction. If $\underline{w}(\bar{x}^\delta) < w^*(\bar{x}^\delta)$, we have that $G(\bar{x}^\delta, w^*(\bar{x}^\delta)) < 0$, a contradiction with $w^*(\cdot)$. So we must have $w^*(\bar{x}^\delta) = R(\mu_H + w(1-x)\delta, x, \delta)$. □

Discrete pooling region

For convenience, we repeat the ODE that must be satisfied in a discrete pooling region by type assignment functions and posterior:

$$\begin{aligned} \dot{m}(x) &= \delta \frac{(1 - R(m(x), x, \delta))w(x)}{R(m(x), x, \delta) - w(x)} \\ \dot{w}(x) &= \left(\frac{1}{x} - \frac{1}{1-x} \right) (1 - w(x)) + \delta^{-1} \left(\frac{\rho_l}{x} - \frac{\dot{m}(x)}{1-x} \right) \end{aligned} \quad (44)$$

where we drop the subscript on m_l for convenience.

Lemma 10. *Let $\delta > 0$ be sufficiently small. For each $x_0 \in (0, 1)$, $\mu_0 \in [\mu_L, \mu_H]$ and $w_0 \in [0, R(\mu_0, x_0, \delta)]$, there exists a unique solution (m, w) of (44) with initial condition (x_0, μ_0, w_0) defined in an interval $[x_0, x_1]$ that satisfies:*

- (i) $\dot{m}(x) > 0$;
- (ii) $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in (x_0, x_1]$;
- (iii) $m(x_1) = \mu_H$ or $w(x_1) = 0$.

Proof. Let us consider the following cases:

(a) $w_0 \in [0, R(\mu_0, x_0, \delta))$. By standard theorem for existence and uniqueness of a solution for ODE system (see, for instance, Theorem 3.1 (p. 18) of Hale (1969)), there exists a unique local solution (m, w) of the system (44) with initial condition (x_0, μ_0, w_0) . For this local solution, we have that $\dot{m}(x) > 0$ and $w(x) \in [0, R(m(x), x, \delta))$. We claim that we can extend the solution until we reach x_1 such that $m(x_1) = \mu_H$ or $w(x_1) = 0$, for δ sufficiently small. Otherwise, for arbitrary small δ , we must have $w(x_1) = R(m(x_1), x_1, \delta)$. However, rewriting (44) through a change of variable from x to μ , we can write its solution in terms of μ as:

$$\begin{aligned} \dot{x}(\mu) &= \frac{R(\mu, x(\mu), \delta) - w(\mu)}{\delta w(\mu) (1 - R(\mu, x(\mu), \delta))} \\ \dot{w}(\mu) &= \left(\frac{1}{x(\mu)} - \frac{1}{1-x(\mu)} \right) (1 - w(\mu))\dot{x}(\mu) + \delta^{-1} \left(\frac{\rho_l \dot{x}(\mu)}{x(\mu)} - \frac{1}{1-x(\mu)} \right) \end{aligned} \quad (45)$$

with the same initial condition. With some abuse of notation we will consider the same notation for w function

in both ODE systems. We have that $\dot{x}(\mu_1) = 0$ where $\mu_1 = m(x_1)$. Then

$$\dot{w}(\mu_1) = -\frac{\delta^{-1}}{1-x(\mu_1)} \leq -\delta^{-1} < 0.$$

Taking the second derivative of $x(\cdot)$ for $\mu < \mu_1$ and making $\mu \rightarrow \mu_1$ we get

$$\ddot{x}(\mu_1) > 0 \text{ if and only if } \frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta) - \dot{w}(\mu_1) > 0.$$

Since $\frac{\partial R}{\partial \mu}(\cdot, \cdot, \delta)$ uniformly converges to $\frac{\partial R}{\partial \mu}(\cdot, \cdot, 0)$ and $\dot{w}(\mu_1) \rightarrow -\infty$ when $\delta \rightarrow 0$, $\left| \frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta) \right| < -\dot{w}(\mu_1)$ for sufficiently small δ , which implies that the function $x(\cdot)$ is locally convex around μ_1 and with zero derivative at μ_1 . This contradicts the fact that $\dot{x}(\mu) > 0$, for all $\mu \in [\mu_0, \mu_1]$.

(b) $w_0 = R(\mu_0, x_0, \delta)$. Rewriting (44) through a change of variable from x to μ , by standard theorem for existence and uniqueness of a solution of ODE system, there exists a unique local solution (x, w) of the system (45) with initial condition (x_0, μ_0, w_0) . By the same argument as in case (i), this local solution can be extended to a maximal interval $[\mu_0, \mu_1]$ with $\dot{x}(\mu) > 0$ and $w(\mu) \in [0, R(\mu, x(\mu), \delta))$, for all $\mu \in (\mu_0, \mu_1]$, and $\mu_1 = \mu_H$ or $w(\mu_1) = 0$. We can change μ back to x and get the solution of the original system. \square

Lemma 11. *For each $\delta \in (0, 1)$, there exists infinitely many different solutions (w, m) of (44) with final condition $(x_1, \underline{a}_1, w_1)$ such that:*

- (i) $x_1 \in (\underline{x}^\delta, \bar{x}^\delta)$ and $w_1 \in (0, R(\underline{a}_1, x_1, \delta))$ define a top continuous pooling region;
- (ii) the maximal interval is $[x_0, x_1]$ such that $x_0 \leq \delta$ or $m(x_0) = \mu_L + (1 - x_0)\delta$.

Proof. (i) By Lemma 9, for each $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$, there exists $(x_1, \underline{a}_1, \bar{a}_1, \underline{b}_1, \bar{b}_1, w_1)$ which defines a top continuous pooling region. By Lemma 10, the solution (μ, w) of (44) with final condition $(x_1, \underline{a}_1, w_1)$ and maximal interval $[x_0, x_1]$ is such that: (a) $m(x_0) \geq \mu_L$; (b) $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; (c) $0 \leq w(x_0) \leq R(m(x_0), x_0, \delta)$.

(ii) We start by showing that at least one such solution exists. Suppose not, i.e., $x_0 > \delta$ and $m(x_0) > \mu_L + (1 - x_0)\delta$, for all possible $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. There are three possibilities: (a) $w(x_0) = 0$; (b) $w(x_0) = R(m(x_0), x_0, \delta)$; (c) $0 < w(x_0) < R(m(x_0), x_0, \delta)$. Applying Lemma 10, case (c) cannot hold since $[x_0, x_1]$ is the maximal interval. Therefore, let A (resp. B) be the subset of $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$ such that $m(x_0) > \mu_L + (1 - x_0)\delta$; $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; and $w(x_0) = 0$ (resp. $w(x_0) = R(m(x_0), x_0, \delta)$).

If $\bar{x}^\delta < 1$, $\underline{x}^\delta \in A$ and $\bar{x}^\delta \in B$, $A \cap B = \emptyset$ and $A \cup B = [\underline{x}^\delta, \bar{x}^\delta]$. We claim that A and B are closed sets, a contradiction.

To show that A is closed, let (x_1^n) be a sequence in A that converges to $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. By the definition of A and Lemma 10, we know that there exists a solution (m^n, w^n) of (44) with initial condition $(x_0^n, \mu_0^n, 0)$ in maximal interval $[x_0^n, x_1^n]$ which satisfies: $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (44) we have that there exists a solution (m, w) of (44) with initial condition $(x_0, \mu_0, 0)$ in maximal interval $[x_0, x_1]$ which satisfies: $\dot{m}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in A$.

To show that B is closed, let (x_1^n) be a sequence in B that converges to $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. By the definition of B and Lemma 10, we know that there exists a solution (m^n, w^n) of (44) with initial condition (x_0^n, μ_0^n, w_0^n) in maximal interval $[x_0^n, x_1^n]$ which satisfies: $w_0^n = R(\mu_0^n, x_0^n, \delta)$; $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in (x_0^n, x_1^n]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (44) there exists a solution (μ, w) of (44) with initial condition (x_0, μ_0, w_0) in a maximal interval $[x_0, x_1]$ which satisfies: $w_0 = R(\mu_0, x_0, \delta)$; $\dot{\mu}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in (x_0, x_1]$; and $\mu(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in B$. Therefore, we must have $x_1 \in A \cap B$, which leads to a contradiction by the uniqueness of the ODE solution.

If $\bar{x}^\delta = 1$ and $B = \emptyset$, then take a sequence (x_0^n) in A such that the corresponding sequence of (x_1^n) of the upper bound of the maximal interval which converges to 1 and the solution (w^n, μ^n) of (44). Notice that, by Lemma 13 below, $x_0^n \leq x^\delta < 1$. Using the second equation of (44), we have that

$$(1 - \bar{w}) \int_{a_n}^{x_1^n} \left(\frac{1}{z} - \frac{1}{1-z} \right) dz + \delta^{-1} \rho_l \int_{a_n}^{x_1^n} \frac{1}{z} dz \geq w^n(x_1^n) - w^n(a_n)$$

or

$$(1 - \bar{w} + \delta^{-1} \rho_l) [\ln x_1^n - \ln a_n] + (1 - \bar{w}) [\ln(1 - x_1^n) - \ln(1 - a_n)] \geq w^n(x_1^n) - w^n(a_n),$$

where $\bar{w} = \sup_{\mu, x} R(\mu, x, \delta)$ and $a_n = \max\{1/2, x_0^n\}$. Notice that the left hand side of the above inequality converge to $-\infty$, which implies that $w^n(x_1^n) \rightarrow -\infty$, when $n \rightarrow \infty$. However, this contradicts Lemma 10 and the definition of x_1^n . Therefore, if $\bar{x}^\delta = 1$, then $B \neq \emptyset$ and the previous argument again applies.

Finally, since A and B are mutually exclusive, closed and do not cover $[\bar{x}^\delta, \underline{x}^\delta]$, their complement is (relatively) open and hence infinite. \square

Lemma 12. *If $\delta > 0$ is sufficiently small, the solution (m, w) of (44) with maximum interval $[x_0, x_1]$ in Lemma 11 satisfies $m(x_0) = \mu_L + (1 - x_0)\delta$.*

Proof. Using the notation of Lemma 11, suppose that $x_0 \leq \delta$ for $\delta > 0$ is small enough. Integrating the second equation of the ODE system, we have that

$$m(x) = \underline{a}_1 + \rho_l \int_x^{x_1} \left(1 - \frac{1}{z} \right) dz + \delta \int_x^{x_1} \left(2 - \frac{1}{z} \right) (1 - w(z)) dz + \delta \int_x^{x_1} (1 - z) \dot{w}(z) dz,$$

for all $x \in [\delta, x_1]$. Integrating we have

$$\begin{aligned} m(x) = & \underline{a}_1 + \rho_l [x_1 - x + \ln x - \ln x_1] + \delta \int_x^{x_1} \left(2 - \frac{1}{z} \right) (1 - w(z)) dz \\ & + \delta [(1 - x_1)w(x_1) - (1 - x)w(x) - \int_x^{x_1} w(z) dz]. \end{aligned}$$

Since $w(x)$ is uniformly bounded in the interval $[\delta, 1]$, when $\delta \rightarrow 0$ we have that $\mu(x)$ uniformly converges to $\underline{a}_1 + \rho_l [x_1 - x + \ln x - \ln x_1]$ on the compact interval of $[\delta, x_1]$. Hence, there exist $\delta > 0$ and $\tilde{x} \in [\delta, x_1]$ such that $\mu(\tilde{x}) < \mu_L + (1 - x_0)\delta$, which concludes the proof. \square

Lemma 13. *Suppose that (m, w) is a solution of (44) and $x \in (0, 1)$ is such that $w(x) = 0$. Then, $\dot{w}(x) \geq 0$ if and only if $x \leq x^\delta \equiv \frac{\delta + \rho_l}{2\delta + \rho_l}$.*

Proof. We have that $\dot{m}(x) = 0$ and $\dot{w}(x) = \frac{1}{x} - \frac{1}{1-x} + \frac{\rho_l}{\delta x}$. Then, the result follows. \square

Bottom continuous pooling region

Fix the equilibrium (m, w) of Lemma 12 on the maximal interval $[x_0, x_1]$. A bottom continuous region is characterized by a vector

$$(\underline{a}, \bar{a}, \underline{b}, \bar{b}, x, w) \in [\mu_L, \mu_H]^4 \times [0, 1]^2$$

that satisfies

$$\underline{b} = \mu(x) \tag{46}$$

$$\bar{b} = \underline{b} - (1 - x)\delta \tag{47}$$

$$\bar{a} = \mu_L \tag{48}$$

$$w = w(x) \tag{49}$$

$$\underline{a} = w\underline{b} + (1 - w)\bar{b} = \underline{b} - (1 - w)(1 - x)\delta \tag{50}$$

and

$$\frac{\int_{\underline{a}}^{\underline{b}} z\phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} z\phi_h(z) dz}{\int_{\underline{a}}^{\underline{b}} \phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} \phi_h(z) dz} = \underline{a}, \tag{51}$$

where we are dropping the sub-index d for convenience. Notice that $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$ defines a bottom continuous pooling region if and only if $x \in (0, 1)$ solves the equation

$$\int_{m(x) - (1-w(x))(1-x)\delta}^{m(x)} (z - m(x) + (1 - w(x))(1 - x)\delta)\phi_l(z) dz + \int_{\mu_L}^{\mu(x) - (1-x)\delta} (z - m(x) + (1 - w(x))(1 - x)\delta)\phi_h(z) dz = 0. \tag{52}$$

It is easy to see that there exists $\bar{x}_0 \in (x_0, 1)$ such that $m(\bar{x}_0) - (1 - \bar{x}_0)\delta = \mu_L$ and, therefore, the left hand side of (52) is non-negative at $x = \bar{x}_0$. Moreover, if δ is sufficiently small, the left hand side of (52) at $x = x_1$ is negative. By the intermediate value theorem, there exists $\tilde{x}_0 \in [\bar{x}_0, x_1)$ for which the equation (52) holds at $x = \tilde{x}_0$.

Global deviation

The last step of the proof is to show that the necessary local deviation conditions of the consumer's problem are sufficient for the global deviation conditions. The following lemma states this property.

Lemma 14. *No consumer type has a profitable deviation, given the price and type assignments constructed.*

Proof. For each coverage $x \in [0, 1]$, let $(m_i(x), i)$ be a type that chooses coverage x . We have to show that $(m_i(x), i)$ does not deviate to any contract $\hat{x} \in [0, 1]$. Let $(m_j(\hat{x}), j)$ be a type that finds \hat{x} optimal. By continuity, without loss of generality, we can assume that $x, \hat{x} \in [x_L, x_d] \cup (x_d, x_u) \cup (x_u, 1]$, i.e., they do not belong to kinks of the price function and are on-the-path equilibrium allocations. Thus, from the FOC of consumer's problem we have that

$$\dot{p}(x) = u_x(x, m_i(x), i) \text{ and } \dot{p}(\hat{x}) = u_x(\hat{x}, m_j(\hat{x}), j).$$

Let us consider the following cases:

(a) $i = j$. From the FOC of the consumer's problem,

$$u(x, m_i(x), i) - p(x) \geq u(\hat{x}, m_i(x), i) - p(\hat{x})$$

if and only if

$$\int_{\hat{x}}^x \int_{m_i(\hat{x})}^{m_i(x)} u_{x\mu}(s, t, i) dt ds \geq 0.$$

Since m_i is a non-decreasing function and $u_{x\mu} > 0$, this is always true.

(b) $i = l$ and $j = h$. Then, $x \in [x_L, x_d] \cup (x_d, x_u)$ and $\hat{x} \in (x_d, x_u) \cup (x_u, 1]$. There are two subcases to consider:

(b.1) $x \leq \hat{x}$. If $\hat{x} \in (x_d, x_u)$, then $u_x(\hat{x}, m_h(\hat{x}), h) = u_x(\hat{x}, m_l(\hat{x}), l)$; if $\hat{x} \in (x_u, 1]$, then $m_l(\hat{x}) = \mu_H$ and

$u_x(\hat{x}, m_h(\hat{x}), h) \geq u_x(\hat{x}, m_l(\hat{x}), l)$.³⁵ In both cases we have

$$u_x(x, m_l(x), l) - u_x(\hat{x}, m_h(\hat{x}), h) \leq u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(\hat{x}), l).$$

Since $u_{x\mu} > 0$ and $m_l(x) \leq m_l(\hat{x})$, then

$$\begin{aligned} u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(\hat{x}), l) &\leq u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(x), l) \\ &= \int_{\hat{x}}^x u_x(s, m_l(x), l) ds. \end{aligned}$$

From the FOC of the consumer's problem and combining the above two inequalities, we have the deviation condition

$$u(\hat{x}, m_l(x), l) - p(\hat{x}) \leq u(x, m_l(x), l) - p(x).$$

(b.2) $x > \hat{x}$. Then, $x, \hat{x} \in (x_d, x_u)$ and we can assume $i = j$ without loss of generality. Therefore, the proof is equivalent to case (a).

(c) $i = h$ and $j = l$. The proof is analogous to case (b). \square

Equilibrium existence and multiplicity

Using the construction in the proof above, we showed that for each solution described in Lemma 11 gives us an equilibrium, and there are infinitely many of these.

Appendix C - Equilibrium approximation

In this section, we obtain approximation results for equilibrium objects for $\delta > 0$ sufficiently small and an arbitrary equilibrium selection. Since, as $\delta \rightarrow 0$, $(x_d, x_u) \rightarrow (x_L, 1)$, almost all traded contracts fall in the discrete pooling region eventually. We start by providing results related to the top continuous pooling region, which are then used to obtain approximation results for the discrete pooling region.

Top continuous pooling region

For $\delta > 0$, a top continuous pooling region is defined by coverage level x_u , set of risk levels pooled with low risk aversion, $[\underline{a}_u, \mu_H]$, high risk aversion, $[\bar{a}_u, \bar{b}_u]$, and the posterior at the top of the discrete pooling region, $w(x_u)$. In this section, for brevity, we denote these objects as $(x_u, \underline{a}_u, \bar{a}_u, \bar{b}_u, w)$. They must satisfy equilibrium equations (35)-(41).

We use the notation $(x_u(\delta), \underline{a}_u(\delta), \bar{a}_u(\delta), \bar{b}_u(\delta), w(\delta))$ in order to make the dependence on parameter δ explicit. We will start by characterizing the convergence of $x_u(\cdot)$ to coverage one as $\delta \rightarrow 0$. We proceed by "sandwiching" $x_u(\delta)$: we find a lower bound $\underline{x}_u(\delta)$ such that $|\underline{x}_u(\delta) - 1|$ is $o(\delta)$. We then study the limiting behavior of all other endogenous variables involved in the top continuous pooling region. Finally, we proceed to study the behavior of certain equilibrium objects in the discrete pooling equilibrium region, which includes all types in the limit as $\delta \rightarrow 0$.

For each $\delta > 0$, the function $x \mapsto \rho_h(\delta)(1 - x + \ln x) + \delta(1 - x)$ is strictly concave, zero and strictly decreasing at $x = 1$, and strictly negative for x sufficiently small, we know that it has a unique zero in $(0, 1)$, denoted by $\underline{x}_u(\cdot)$. From (40), we must have $x_u(\delta) \in [\underline{x}_u(\delta), 1)$ for $\delta > 0$. Finally, define $\underline{x}_u(0) \equiv 1$.

In the next lemma we show that the lower bound on $x_u(\delta)$ converges to 1 at rate δ , which means that any selection $x_u(\delta)$ also converges to 1 at rate δ or faster.

³⁵Here we are using the properties of the top continuous pooling in this proof. In particular, $\bar{b}(x) \geq \mu_H - (1 - x)\delta$, for all $x \in (x_u, 1]$.

Lemma 15. *The lower bound on the pooling point, $\underline{x}_u(\delta)$, is continuously differentiable and satisfies $\underline{x}_u(0) = 1$ and $\dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}$.*

Proof. For $\delta > 0$, continuous differentiability of $\underline{x}_u(\cdot)$ follows from the implicit function theorem, while continuity at $\delta = 0$ follows from the fact that $1 - x + \ln x < 0$, for any $x \in (0, 1)$.

Notice that, for $\delta > 0$,

$$\dot{\underline{x}}_u(\delta) = -\frac{1}{2} \frac{\frac{\ln \underline{x}_u(\delta)}{\delta} + 3 \left(\frac{1 - \underline{x}_u(\delta)}{\delta} \right)}{\frac{\rho_h(\delta)}{\underline{x}_u(\delta)} \frac{1 - \underline{x}_u(\delta)}{\delta} - 2}, \quad (53)$$

which implies that $\dot{\underline{x}}_u(\delta) < 0$. If there exists a sequence $(\delta_n)_n$ such that $\delta_n \rightarrow 0$ and $\dot{\underline{x}}_u(\delta_n)$ converges, then $\underline{x}_u(\cdot)$ is differentiable at zero and, using (53), satisfies

$$1 = -\frac{1}{\rho_0 \dot{\underline{x}}_u(0) + 2} \implies \dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}.$$

Otherwise, we must have $\lim_{\delta \rightarrow 0} \dot{\underline{x}}_u(\delta) = \lim_{\delta \rightarrow 0} \frac{\underline{x}_u(\delta) - 1}{\delta} = -\infty$, and (53) implies that $\lim_{\delta \rightarrow 0} \underline{x}_u(\delta) = -\frac{1}{\rho_0}$, a contradiction. \square

Corollary 3. *For any arbitrary selection $(x_u(\delta))_{\delta > 0}$, we have that*

$$\lim_{\delta \rightarrow 0} x_u(\delta) = 1 \text{ and } \limsup_{\delta \rightarrow 0} \frac{1 - x_u(\delta)}{\delta} \leq -\dot{\underline{x}}_u(0).$$

Proof. The results follow directly from $1 \leq x_u(\delta) \leq \underline{x}_u(\delta)$ and Lemma 15. \square

This means that, potentially passing to a sub-sequence, we can assume that $\frac{1 - x_u(\delta)}{\delta}$ converges. Define $x_u(0) \equiv \lim_{\delta \rightarrow 0} x_u(\delta) = 1$ and $\dot{x}_u(0) \equiv \lim_{\delta \rightarrow 0} \frac{x_u(\delta) - 1}{\delta}$. The same definition will be extended to equilibrium variables $\bar{b}_u(\delta)$, $\bar{a}_u(\delta)$, $\underline{a}_u(\delta)$, $w(x; \delta)$ and $p(x; \delta)$.

We now characterize the changes in the top continuous pooling parameters around $\delta = 0$.

Lemma 16. *For any arbitrary selection we have that:*

- (i) *Convergence:* $\bar{a}_u(0) = \underline{a}_u(0) = \bar{b}_u(0) = \mu_H$
- (ii) δ -*order convergence:* $\dot{\underline{a}}_u(0) = \dot{\bar{a}}_u(0) = \dot{\bar{b}}_u(0) = 0$;
- (iii) δ^2 -*order convergence:* $\ddot{\bar{b}}_u(0) = -\rho_0 (\dot{x}_u(0))^2$.

Proof. (i) follows directly from taking limits on equations (35)-(41). (ii) follows from subtracting both sides of equations (36), (37) and (38) by μ_H , dividing by δ and taking limits $\delta \rightarrow 0$. To obtain (iii) we rewrite equation (34) as follows:

$$\frac{\bar{b}_u - \mu_H}{\delta^2} = \rho_h(\delta) \left(\frac{x_u(\delta) - 1}{\delta} \right)^2 \frac{\left(\frac{\log x_u(\delta)}{x_u(\delta) - 1} - 1 \right)}{x_u(\delta) - 1}.$$

Taking limits we have that $\ddot{\bar{b}}_u(0) = -\rho_0 [\dot{x}_u(0)]^2$. \square

Discrete pooling approximation

For clarity, our results in this section are grouped according to the equilibrium object they refer to. Let the discrete pooling prices and conditional probability of ρ_l be given by $p(x; \delta)$ and $w(x; \delta)$. In the discrete pooling

region of any equilibrium, the price function follows the differential system: from (10) and (11) we have

$$\dot{p}(x; \delta) = \frac{p(x; \delta)}{x} + (1-x) \left[\rho_0 + \delta \left(\frac{1}{2} - w(x; \delta) \right) \right], \quad (54)$$

while (10), (11) and (12) imply

$$\dot{w}(x; \delta) = (1-w(x; \delta)) \frac{1-2x}{x(1-x)} + \frac{\rho_0 - \delta/2}{\delta x} - \frac{w(x; \delta) [1 - \tilde{w}(p(x; \delta), w(x; \delta), x; \delta)]}{[\tilde{w}(p(x; \delta), w(x; \delta), x; \delta) - w(x; \delta)] (1-x)}, \quad (55)$$

where we define the following function

$$\tilde{w}(p, w, x; \delta) \equiv \frac{\phi_l \left(\frac{p}{x} + \delta(1-w)(1-x) \right)}{\phi_l \left(\frac{p}{x} + \delta(1-w)(1-x) \right) + \phi_h \left(\frac{p}{x} - \delta w(1-x) \right)}.$$

Notice that the differential equation (54) has following integral form

$$\frac{p(x; \delta)}{x} = \bar{b}_u(\delta) - \int_x^{x_u(\delta)} \left[\rho_0 + \delta \left(\frac{1}{2} - w(z; \delta) \right) \right] \left(\frac{1}{z} - 1 \right) dz. \quad (56)$$

Hence, characterization of the behavior of the posterior is sufficient to study the price function.

The next lemma characterizes the limits of the functions $p(x; \delta)$ and $w(x; \delta)$, and their derivatives when $\delta \rightarrow 0$.

Price convergence

In this section we will use the fact that the one-dimensional prices and type assignment functions satisfy

$$\left(\frac{p_0(x)}{x} \right)' = m_0'(x) = \rho_0 \left(\frac{1}{x} - 1 \right). \quad (57)$$

Lemma 17. *The price function is continuous in δ at zero: for any $x \in (x_L, 1)$, $\lim_{\delta \rightarrow 0} \frac{p(x; \delta)}{x} = m_0(x)$. This convergence is uniform in any compact set $M \subset (x_L, 1)$.*

Proof. The function $p(x; \delta)$ satisfies (56). From Lemma 16, we know that $\bar{b}_u(\delta) \rightarrow \mu_H$, which, together with (57), imply pointwise convergence. If $z_0 \equiv \inf M$, then for $\delta_0 > 0$ sufficiently small, the family $\left(\frac{p(\cdot; \delta)}{x} \right)_{\delta_0 > \delta > 0}$ is equi-Lipschitz³⁶ in M with constant $L \equiv \left(\frac{1}{z_0} - 1 \right) (\rho_0 + \delta_0)$, which means that the convergence is uniform. \square

Lemma 18. *(First-order price approximation) For any compact $M \subset (x_L, 1)$ and $x \in M$,*

$$\lim_{\delta \rightarrow 0} \frac{\frac{p(x; \delta)}{x} - m_0(x)}{\delta} = \frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2} \right) \left(\frac{1}{z} - 1 \right) dz,$$

with the convergence being uniform in M .

Proof. From equations (56) and (57) we have

$$\frac{\frac{p(x; \delta)}{x} - m_0(x)}{\delta} = \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta} - \int_x^{x_u(\delta)} \left(\frac{1}{2} - w(z; \delta) \right) \left(\frac{1}{z} - 1 \right) dz, \quad (58)$$

³⁶For any family of function $g_\delta : A \mapsto \mathbb{R}$, with index $\delta \in E$ and $A \subset \mathbb{R}$, we say that this family is equi-Lipschitz in $B \subset A$ if there exists a constant L such that

$$|g_\delta(x) - g_\delta(x')| \leq L|x - x'|,$$

for any $x, x' \in B$ and $\delta \in E$.

which, using $m'_0(1) = 0$ as well as Lemmas 16 and 22, implies pointwise convergence of the object of interest. Uniform convergence comes from the fact that the left-hand side of (58) is equi-Lipschitz with constant $L \equiv \left(\frac{1}{z_0} - 1\right)$, where $z_0 \equiv \inf M$. \square

Lemma 19. (Second-order price approximation) *The second-order approximation of equilibrium prices is given by, for any $x \in (x_L, 1)$,*

$$p_{\delta\delta}(x) = 2 \int_0^1 w_\delta(z) \left(\frac{1}{z} - 1\right) dz,$$

with the convergence being uniform for any compact $M \subset (x_L, 1)$.

Proof. Consider any compact $M \subset (x_L, 1)$ and $x \in M$. Using (58) and Lemma 18 we have

$$\begin{aligned} \frac{1}{\delta} \left[\frac{p(x;\delta) - m_0(x)}{\delta} - \frac{p_\delta(x)}{x} \right] &= \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta^2} \\ &+ \int_x^{x_u(\delta)} \frac{(w(z;\delta) - \omega_0(z))}{\delta} \left(\frac{1}{z} - 1\right) dz - \frac{1}{\delta} \int_{x_u(\delta)}^1 \left(\omega_0(z) - \frac{1}{2}\right) \left(\frac{1}{z} - 1\right) dz. \end{aligned} \quad (59)$$

Taking the limit $\delta \rightarrow 0$ and using Lemmas 3, 16 and 24 give us pointwise convergence. Now, consider a compact set $M \subset (x_L, 1)$. Using Lemma 24, we can find $\delta_0 > 0$ sufficiently small such that

$$B_M \equiv \sup_{\delta \leq \delta_0} \sup_{z \in M} \left| \frac{\omega_0(z) - w(z;\delta)}{\delta} \right| < \infty.$$

For $\delta > 0$ smaller than δ_0 , the left-hand side of expression (59) is equi-Lipschitz, for $\delta_0 > \delta > 0$ with constant $L \equiv 2 \left(\frac{1}{z_0} - 1\right) B_M$, given $z_0 \equiv \inf M$, and hence the convergence obtained is uniform on M . \square

Posterior belief convergence

We now obtain some necessary convergence results for the posterior $w(\cdot; \delta)$. For convenience, we refer to the partial derivatives of \tilde{w} as $\tilde{w}_p(p, w, x; \delta) = \frac{\partial}{\partial p} \tilde{w}(p, w, x; \delta)$, with same notation used for derivatives with respect to w , x and δ ; we also denote $\tilde{w}(x; \delta) \equiv \tilde{w}(p(x; \delta), w(x; \delta), x; \delta)$ and the total derivative $\frac{d}{dx} [\tilde{w}(x; \delta)]$ as $d_x \tilde{w}(x; \delta)$. From Lemma 17, it is easy to show that, as $\delta \rightarrow 0$, $\tilde{w}(x; \delta) \rightarrow \omega_0(x)$ and, using direct differentiation,

$$\begin{aligned} (\tilde{w}_p(x; \delta), \tilde{w}_w(x; \delta), \tilde{w}_x(x; \delta)) &\rightarrow \left(\frac{1}{x} \frac{\omega'_0(x)}{m'_0(x)}, 0, -\frac{m_0(x)}{x} \frac{\omega'_0(x)}{m'_0(x)} \right) \\ \tilde{w}_\delta(x; \delta) &\rightarrow (1-x) [1 - \omega_0(x)] \omega_0(x) \left\{ [1 - \omega_0(x)] \frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x) \frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))} \right\}. \end{aligned} \quad (60)$$

Lemma 20. *For $\delta > 0$ sufficiently small and interval $I \subseteq (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$ such that*

$$|\dot{w}(x; \delta)| \leq \frac{1}{D}.$$

Proof. Consider an interval $[a, b]$ with $b - a = D$. Since $w(\cdot; \delta)$ has maximal variation of one in $[a, b]$, the mean value theorem implies the result. \square

Lemma 21. *For $\delta > 0$ sufficiently small and interval $I \subset (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$ and bound $B(\delta, D, x)$ such that*

$$\frac{|\tilde{w}(x; \delta) - w(x; \delta)|}{\delta} \leq B(\delta, D, x),$$

where $B(\cdot)$ is strictly positive, continuous and satisfies $B(0, D, x) < \infty$.

Proof. From Lemma 20 we can find $x \in I$ such that

$$\left| (1 - w(x; \delta)) \frac{1 - 2x}{x(1 - x)} + \frac{1}{\delta} \left[\frac{\rho_0 - \delta/2}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1 - x)} \right] \right| \leq \frac{1}{D}, \quad (61)$$

which implies that, using $\tilde{w}(x; \delta) - w(x; \delta) > 0$ and $\delta > 0$ sufficiently small:

$$\frac{|\tilde{w}(x; \delta) - w(x; \delta)|}{\delta} \leq \frac{xw(x; \delta) [1 - \tilde{w}(x; \delta)]}{(1 - x)(\rho_0 - \delta/2) + \delta [1 - w(x; \delta)] (1 - 2x) - \frac{\delta x(1 - x)}{D}}.$$

Now define $B(\delta, D, x)$ as the right-hand side of this last inequality. \square

Lemma 22. (*Level convergence of w*) For any $x \in (x_L, 1)$, $\limsup_{\delta} \left[\sup_{x \in M} \frac{|w(x; \delta) - \tilde{w}(x; \delta)|}{\delta} \right] < \infty$, which implies that $w(x; \delta)$ converges to $\omega_0(x)$. This convergence is uniform for any compact $M \subset (x_L, 1)$.

Proof. Suppose, by way of contradiction, that there exist sequences $(z_n)_n$ in $M \subset (x_L, 1)$ and $(\delta_n)_n$ such that $\frac{|w(z_n; \delta_n) - \tilde{w}(z_n; \delta_n)|}{\delta_n} \rightarrow \infty$. We first show that one can find another convergent sequence $(x_n)_n$ in M such that

$$\frac{|w(x_n; \delta_n) - \tilde{w}(x_n; \delta_n)|}{\delta_n} \rightarrow \infty \text{ and } \dot{w}(x_n) = d_x \tilde{w}(x_n; \delta_n).$$

We then show that the existence of such sequence leads to a contradiction.

Denote $z_0 \equiv \inf M$ and $z_1 \equiv \sup M$. Consider $D \in (0, \frac{1}{2} \min \{z_0 - x_L, 1 - z_1\})$ and define

$$K \equiv 1 + \sup_{x \in [z_0 - D, z_0] \cup [z_1, z_1 + D]} B(0, D, x).$$

Continuity of B implies that, for n sufficiently large,

$$\sup_{x \in [z_0 - D, z_0] \cup [z_1, z_1 + D]} B(\delta_n, D, x) < K.$$

From Lemma 21, there exist sequences $(x_n^-)_n$ in $[z_0 - D, z_0]$ and $(x_n^+)_n$ in $[z_1, z_1 + D]$ such that

$$\max \left\{ \frac{|\tilde{w}(x_n^-; \delta_n) - w(x_n^-; \delta_n)|}{\delta_n}, \frac{|\tilde{w}(x_n^+; \delta_n) - w(x_n^+; \delta_n)|}{\delta_n} \right\} < K,$$

while defining $x_n \equiv \arg \max_{x \in [x_n^-, x_n^+]} |\tilde{w}(x; \delta_n) - w(x; \delta_n)|$, we have that

$$\lim_n \frac{|\tilde{w}(x_n; \delta_n) - w(x_n; \delta_n)|}{\delta_n} \geq \lim_n \frac{|\tilde{w}(z_n; \delta_n) - w(z_n; \delta_n)|}{\delta_n} = \infty.$$

Hence, $x_n \in (z_0, z_1)$ is an interior optimizer and, hence, satisfies the required properties.

We now show that the construction of sequence $(x_n)_n$ leads to a contradiction. The ordinary differential equation:

$$\dot{w}(x; \delta) = (1 - w(x; \delta)) \frac{1 - 2x}{x(1 - x)} + \frac{1}{\delta} \left[\frac{\rho_0 - \delta}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1 - x)} \right] \quad (62)$$

implies that $\dot{w}(x_n, \delta_n) \rightarrow \infty$. Now notice that

$$\begin{aligned} \dot{w}(x_n; \delta_n) - d_x \tilde{w}(x_n; \delta_n) &= [1 - \tilde{w}_w(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n))] \dot{w}(x_n; \delta_n) \\ &\quad - \tilde{w}_p(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n) \dot{p}(x; \delta_n) - \tilde{w}_x(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n), \end{aligned}$$

which, using (60), diverges as $n \rightarrow \infty$. This is a contradiction with the construction of $(x_n)_n$. \square

Lemma 23. (*Derivative convergence of posterior*) For any compact $M \subset (x_L, 1)$, $\sup_{x \in M} |\dot{w}(x; \delta) - \dot{\omega}_0(x)|$ converges to zero uniformly on M .

Proof. Suppose, by way of contradiction, the desired convergence result fails. Since both $w(x; \delta)$ and $\omega_0(x)$ are twice continuously differentiable and $|w(x; \delta) - \omega_0(x)|$ converges uniformly to zero, we can find constant $\gamma > 0$ and sequence (x^n, δ^n) such that $x^n \in M$, $\delta^n \rightarrow 0$ and

$$\dot{w}(x^n; \delta^n) - \dot{\omega}_0(x^n) = \gamma, \quad (63)$$

$$\ddot{w}(x^n; \delta^n) - \ddot{\omega}_0(x^n) \leq 0.$$

But, using (55) we have that

$$\begin{aligned} \ddot{w}(x^n; \delta^n) &= -\dot{w}(x; \delta) \frac{1-2x}{x(1-x)} - \frac{d}{dx} \left[\frac{w(x; \delta)(1-\tilde{w}(x; \delta))}{(w(x; \delta) - \tilde{w}(x; \delta))(1-x)} \right] \\ &\quad + (1-w(x; \delta)) \frac{d}{dx} \left[\frac{1-2x}{x(1-x)} \right] + \frac{(\rho_0 - \delta/2)}{\delta} \frac{d}{dx} \left[\frac{1}{x} \right] \\ &= \frac{(\dot{w}(x; \delta) - d_x \tilde{w}(x; \delta)) w(x; \delta) (1-w(x; \delta)) + O(1)}{(\tilde{w}(x; \delta) - w(x; \delta))^2}, \end{aligned} \quad (64)$$

where $O(1)$ represents all the terms that are bounded uniformly in $(x, \delta) \in M \times (0, \bar{\delta})$, for $\bar{\delta} > 0$ sufficiently small. Also notice that (63) implies that \dot{w} is bounded and, hence, using (60) we have that

$$\lim_n d_x \tilde{w}(x; \delta) = \dot{\omega}_0(\lim x^n).$$

This, together with (64) imply that $\lim_n \ddot{w}(x^n; \delta^n) = \infty$, contradicting the definition of $(x^n, \delta^n)_n$. \square

Lemma 24. (*First-order approximation of posterior*) The posterior $w(x; \delta)$ is differentiable in δ at zero, for any $x \in (x_L, 1)$, and its derivative is given by

$$\lim_{\delta \rightarrow 0} \frac{w(x; \delta) - \omega_0(x)}{\delta} = w_\delta(x),$$

with $w_\delta(\cdot)$ described in (18). This convergence holds uniformly in any compact $M \subset (x_L, 1)$.

Proof. Consider a compact $M \subset (x_L, 1)$. From (55), using Lemma 23, we can see that on M

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\frac{\rho_0 - \delta/2}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1-x)} \right] = \dot{\omega}_0(x) - (1 - \omega_0(x)) \frac{1-2x}{x(1-x)}, \quad (65)$$

which, together with Lemma 22, imply that on M

$$\lim_{\delta \rightarrow 0} \frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} = \frac{x\omega_0(x) [1 - \omega_0(x)]}{\rho_0(1-x)}. \quad (66)$$

Using the fact that $\omega_0(x) = \tilde{w}(m_0(x), x, w(x; \delta), x; 0)$, continuous differentiability of $\tilde{w}(\cdot)$, Lemma 18 and (60), we have that the following holds uniformly on M

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} &= \lim_{\delta \rightarrow 0} \frac{\tilde{w}(p(x; \delta), w(x; \delta), x; \delta) - \tilde{w}(m_0(x), x, w(x; \delta), x; \delta)}{\delta} \\ &+ \lim_{\delta \rightarrow 0} \frac{\tilde{w}(m_0(x), x, w(x; \delta), x; \delta) - \omega_0(x)}{\delta} + \lim_{\delta \rightarrow 0} \frac{\omega_0(x) - w(x; \delta)}{\delta} \\ &= \frac{\dot{\omega}_0(x)}{x \dot{m}_0(x)} p_\delta(x) + \lim_{\delta \rightarrow 0} \tilde{w}_\delta(p(x; \delta), w(x; \delta), x; \delta) - w_\delta(x), \end{aligned} \quad (67)$$

and hence (60), (66) and (67) give us the result. \square

Welfare approximation

In this section we obtain expressions for the second order approximation of equilibrium utilities, defined in (21). The equilibrium payoffs are related to quasi-linear payoff

$$U_i(\mu; \delta) = u(t_i(\mu; \delta); \mu, \rho_i(\delta)) - p(t_i(\mu; \delta); \delta),$$

by

$$V^i(\mu; \delta) = \gamma_i(U_i(\mu; \delta), \delta), \quad (68)$$

with function $\gamma_i \in C^2$ is defined by

$$\gamma_i(U, \delta) \equiv -\exp\{-\rho_i(\delta)[W - \mu + U]\}.$$

So we proceed by first characterizing the approximation terms of equilibrium allocation $t_i(\cdot; \delta)$, then quasi-linear utility $U_i(\cdot; \delta)$ and finally utility level $V^i(\mu; \delta)$.

In order to obtain approximation terms for utility levels, we need to use equilibrium behavior of demand functions $t_i(\mu; \delta)$, which satisfy $t_i(m_i(x; \delta); \delta) = x$. This is obtained through a series of lemmas. Notice that the type assignment functions satisfy, in the pooling region:

$$\begin{aligned} m_l(x; \delta) &= \frac{p(x; \delta)}{x} + \delta(1 - w(x; \delta))(1 - x), \\ m_h(x; \delta) &= \frac{p(x; \delta)}{x} - \delta w(x; \delta)(1 - x). \end{aligned}$$

Our previous approximation results imply that $m_i(x; \delta)$ has a quadratic approximation,

$$m_i(x; \delta) = m_0(x) + \delta m_{i,\delta}(x) + \frac{\delta^2}{2} m_{i,\delta\delta}(x) + o(x; \delta^2),$$

where $\lim_{\delta \rightarrow 0} \sup_{x \in M} \frac{o(x; \delta^2)}{\delta^2}$ for any compact $M \subset (x_L, 1)$.

Moreover, the first-order approximation coefficients are

$$\begin{aligned} m_{l,\delta}(x) &= \frac{p_\delta(x)}{x} + (1 - \omega_0(x))(1 - x), \\ m_{h,\delta}(x) &= \frac{p_\delta(x)}{x} - \omega_0(x)(1 - x), \end{aligned} \quad (69)$$

which implies that $m_{i,\delta}(\cdot)$ only depends on the distribution of types through $\omega_0(\cdot)$.

We can now state our limiting results regarding the demand functions.

Lemma 25. *The demand functions $t_i(\mu; \delta)$, for $i \in \{l, h\}$, satisfy, for $\mu \in (\mu_L, \mu_H)$,*

(a) *continuity:*

$$\lim_{\delta \rightarrow 0} t_i(\mu; \delta) = t_0(\mu);$$

(b) *first-order approximation:*

$$t_{i,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{t_i(\mu; \delta) - t_0(\mu)}{\delta} = -\frac{m_{i,\delta}(t_0(\mu))}{\dot{m}_0(t_0(\mu))};$$

(c) *second-order approximation:*

$$\begin{aligned} t_{i,\delta\delta}(\mu) &\equiv 2 \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{t_i(\mu; \delta) - t_0(\mu)}{\delta} - t_{i,\delta}(\mu) \right\} \\ &= -\frac{1}{\dot{m}_0(t_0(\mu))} \left\{ \ddot{m}_0(t_0(\mu)) [t_{i,\delta}(\mu)]^2 + 2\dot{m}_{i,\delta}(t_0(\mu)) t_{i,\delta}(\mu) + m_{i,\delta\delta}(t_0(\mu)) \right\}, \end{aligned}$$

with the convergence being uniform in any compact $C \subset (\mu_L, \mu_H)$. Additionally, $t_{i,\delta}(\mu)$ only depends on the type distribution through $\omega_0(x)$, for any $x \in (x_L, 1)$.

Proof. Consider a compact $C \subset (\mu_L, \mu_H)$, $i \in \{l, h\}$ and $\delta > 0$ sufficiently small. Let $x_1 \equiv t_0(\inf C - \varepsilon)$ and $x_2 \equiv t_0(\sup C + \varepsilon)$, for $\varepsilon < \min\{1 - \sup C, \inf C - x_L\}$. Continuity of $m_i(\cdot; \delta)$ in δ implies that, as $\delta \rightarrow 0$, $m_i(x_1; \delta) \rightarrow \inf C - \varepsilon$ and $m_i(x_2; \delta) \rightarrow \sup C + \varepsilon$. Hence, monotonicity of m_i and t_i imply that $t_i(m_i(x_1; \delta); \delta) < t_i(\mu; \delta) < t_i(m_i(x_2; \delta); \delta)$ and hence $T_i(\delta) \equiv \{t_i(\mu; \delta) \mid \mu \in C\}$ is a compact set contained in $[x_1, x_2] \subset (x_L, 1)$.

Hence, using uniform convergence of $m_i(\cdot; \delta)$ in $[x_1, x_2]$ we have that

$$m_i(t_i(\mu; \delta); \delta) - m_0(t_i(\mu; \delta)) = \delta m_{i,\delta}(t_i(\mu; \delta)) + \frac{\delta^2}{2} m_{i,\delta\delta}(t_i(\mu; \delta)) + o(t_i(\mu; \delta); \delta), \quad (70)$$

where $\sup_{\mu \in C} \frac{o(t_i(\mu; \delta); \delta)}{\delta} \rightarrow 0$.

Finally, using $m_i(t_i(\mu; \delta); \delta) = m_0(t_0(\mu)) = \mu$ we have

$$m_0(t_i(\mu; \delta)) - m_0(t_0(\mu)) = -[m_i(t_i(\mu; \delta); \delta) - m_0(t_i(\mu; \delta))],$$

and, since the right hand side satisfies approximation equation (70), we have a quadratic approximation of the left hand side, which holds uniformly in $\mu \in C$. Since $m_0(\cdot)$ is twice continuously differentiable and has strictly positive derivative in $[x_1, x_2]$, direct differentiation gives us the result. \square

We are now in position to state the main quasi-linear-payoff approximation result. Define $U^0(\mu) \equiv u(t_0(\mu); \mu, \rho_0) - p_0(\mu)$, for $\mu \in [\mu_L, \mu_H]$ and $\dot{p}_\delta(x) \equiv \frac{d}{dx}[p_\delta(x)]$.

Lemma 26. *For any $\mu \in (\mu_L, \mu_H)$ and $i \in \{l, h\}$, the payoff function has the following limiting behavior, for any compact $C \subset (\mu_L, \mu_H)$:*

(i) *continuity:*

$$\lim_{\delta \rightarrow 0} U_i(\mu; \delta) = U^0(\mu);$$

(ii) *first-order approximation:*

$$U_{h,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{U_h(\mu; \delta) - U^0(\mu)}{\delta} = -p_\delta(t_0(\mu)) - \frac{1}{4}(1 - t_0(\mu))^2,$$

$$U_{l,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{U^l(\mu; \delta) - U^0(\mu)}{\delta} = -p_\delta(t_0(\mu)) + \frac{1}{4}(1 - t_0(\mu))^2;$$

(iii) second-order approximation:

$$U_{h,\delta}(\mu) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{U_h(\mu; \delta) - U^0(\mu)}{\delta} - U_{h,\delta}(\mu)}{\delta} = - \left[\begin{array}{c} p_{\delta\delta}(t_0(\mu)) \\ + [\dot{p}_\delta(t_0(\mu)) - \frac{1}{2}(1 - t_0(\mu))] t_{h,\delta}(\mu) \end{array} \right],$$

$$U_{l,\delta\delta}(\mu) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{U^l(\mu; \delta) - U^0(\mu)}{\delta} - U_{l,\delta}(\mu)}{\delta} = - \left[\begin{array}{c} p_{\delta\delta}(t_0(\mu)) \\ + [\dot{p}_\delta(t_0(\mu)) + \frac{1}{2}(1 - t_0(\mu))] t_{h,\delta}(\mu) \end{array} \right],$$

with the convergence being uniform in C .

Proof. Consider compact $C \subset (\mu_L, \mu_H)$, $\delta > 0$ sufficiently small and $i \in \{l, h\}$ (the argument for $i = l$ is analogous). Uniform continuity in δ and monotonicity in x of $m_i(\cdot; \delta)$ imply that we can find x_1, x_2 such that $t_i(\mu; \delta) \subset [x_1, x_2] \subset (x_L, 1)$ (for details, see the proof of Lemma 25). Notice that the function $H : (x, p, \delta) \mapsto u(x; \mu, \rho_i(\delta)) - p$ is twice continuously differentiable, hence we have that

$$\begin{aligned} U_i(\mu; \delta) - U^0(\mu) &= [H(t_i(\mu; \delta), p(t_i(\mu; \delta); \delta), \delta) - H(t_i(\mu; \delta), p_0(t_i(\mu; \delta)), \delta)] \\ &\quad + [H(t_i(\mu; \delta), p_0(t_i(\mu; \delta)), \delta) - H(t_i(\mu; \delta), p_0(t_0(\mu)), \delta)] \\ &\quad + [H(t_i(\mu; \delta), p_0(t_0(\mu)), \delta) - H(t_0(\mu), p_0(t_0(\mu)), \delta)] \\ &\quad + [H(t_0(\mu), p_0(t_0(\mu)), \delta) - H(t_0(\mu), p_0(t_0(\mu)), 0)], \end{aligned}$$

where twice continuous differentiability of p_0 and Lemmas 18 and 25 imply that each of the four terms in the right hand side have a well-defined quadratic approximation with error term $o(\mu; \delta^2)$ satisfying $\lim_{\delta \rightarrow 0} \sup_{\mu \in C} \frac{o(\mu; \delta^2)}{\delta^2}$, implying that both first- and second-order derivative limits are well defined. The exact formulas obtained come from direct differentiation. \square

Comparing distributions

In Sections 5 and 6, we compare the utility obtained by each type under different type distributions. Consider any two distributions (ϕ_l^k, ϕ_h^k) , for $k \in \{A, B\}$, with the same support. We make the dependence of equilibrium objects by using superscripts, as in $V_i^k(\mu; \delta)$, for $i \in \{l, h\}$. The superscript is omitted whenever the equilibrium object does not depend on k .

Lemma 27. *For any type with risk $(\mu, i) \in (\mu_L, \mu_H) \times \{l, h\}$ and $k \in \{A, B\}$, the following hold:*

$$V^k(\mu, i) \equiv \lim_{\delta \rightarrow 0} V^k(\mu, i; \delta) = \gamma_i(U^0(\mu), 0),$$

$$V_\delta^k(\mu, i) \equiv \lim_{\delta \rightarrow 0} \frac{V^k(\mu, i; \delta) - V^k(\mu, i; 0)}{\delta} = \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial}{\partial \delta} \gamma_i(U^0(\mu), 0),$$

and

$$\begin{aligned} V_{\delta\delta}^k(\mu, i) &\equiv 2 \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\frac{V^k(\mu, i; \delta) - V^k(\mu, i; 0)}{\delta} - \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial}{\partial \delta} \gamma_i(U^0(\mu), 0) \right] = \\ &\quad \left[\begin{array}{c} \frac{\partial^2}{\partial U^2} \gamma_i(U^0(\mu), 0) [U_{i,\delta}^k(\mu)]^2 + \frac{\partial^2}{\partial U \partial \delta} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) \\ + \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta\delta}^k(\mu) \\ + \frac{\partial^2}{\partial \delta \partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial^2}{\partial \delta^2} \gamma_i(U^0(\mu), 0) \end{array} \right], \quad (71) \end{aligned}$$

with convergence of the limits above being guaranteed uniformly on any compact set $C \subset (\mu_L, \mu_H)$.

Proof. The results follow from definition expression (68), twice differentiability of γ_i and the uniform convergence results in Lemma 26. \square

Proof of Lemma 4

We now consider distributions indexed by $k \in \{0\} \cup S$, with index $k = 0$ represents the prior and $k = s$ represents the conditional type distribution with signal realization $s \in S$. Since the signal considered is a pure risk signal, Lemma 26 and Proposition 3 imply that term $U_\delta(\mu, i)$ and, as a consequence, V_δ^k , do not vary with $k \in \{0\} \cup S$. So, we omit notation k in their expressions.

From Lemmas 26 and 27 we have that the signal effect is

$$\begin{aligned} \sum_{s \in S} \pi(s | \mu) V_i^s(\mu; \delta) - V_i^0(\mu; \delta) &= \sum_{s \in S} \pi(s | \mu) [V_i^s(\mu; \delta) - V_i^0(\mu; \delta)] \\ &= -\frac{\delta^2}{2} \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) \sum_{s \in S} [\pi(s | \mu) p_{\delta\delta}^s(t_0(\mu)) - p_{\delta\delta}^0(t_0(\mu))] + o(\mu; \delta^2), \end{aligned}$$

with $\lim_{\delta \rightarrow 0} \sup_{\mu \in C} \frac{o(\mu; \delta^2)}{\delta^2} = 0$. The equality $\frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) = \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0)$ gives us the result.

Appendix D - Signal disclosure and comparative statics

Proof of Proposition 4

(If): Consider an arbitrary $\varepsilon > 0$ and let $M_\varepsilon \subset (\mu_L, \mu_H)$ be compact such that $\int_{M_\varepsilon} \phi_l(\mu') + \phi_h(\mu') d\mu' > 1 - \varepsilon$. From Lemma 4, we can find $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$ and $i = l, h$,

$$\sup_{\mu \in C} \frac{|o_i(\delta^2; \mu)|}{\delta^2} < \inf_{\mu \in M_\varepsilon} \frac{1}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) \Delta E(p(t_0(\mu))),$$

which implies that all types with risk level in M_ε have a strict interim improvement from the signal disclosure.

(Only if): Now suppose that there exist $\mu_1, \mu_2 \in (\mu_L, \mu_H)$ with $\mu_1 < \mu_2$ satisfying $\partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu_2)) < 0$. Now consider a sequence of absolutely continuous full-support distributions on $[\mu_L, \mu_H]$ that weakly converge to the Dirac measure $\delta_{\{\mu_2\}}$, with continuously differentiable densities $\{f^n(\cdot)\}_n$. It is easy to show that there exists C^1 functions $\omega_0^n \circ t_0 : [\mu_L, \mu_H] \mapsto (0, 1)$ such that $\omega_0^n \circ t_0(\cdot) [1 - \omega_0^n \circ t_0(\cdot)] = \frac{1}{4\bar{f}^n} f^n(\cdot)$, where $\bar{f}^n \equiv \sup f^n$.³⁷ Hence, for any density ϕ on $[\mu_L, \mu_H]$, consider type distributions

$$(\phi_l^n(\mu), \phi_h^n(\mu)) = (\omega_0^n \circ t_0(\mu) \phi(\mu), [1 - \omega_0^n \circ t_0(\mu)] \phi(\mu)),$$

for each n . The price effect of disclosing this signal under distribution $(\phi_l^n(\mu), \phi_h^n(\mu))$ on the price of coverage $x = t_0(\mu_1)$ is then given by

$$\Delta E(p^n(x)) = -\frac{1}{2\rho_0 \bar{f}^n} \int_{\mu_1}^{\mu_H} f^n(\mu) (1 - t_0(\mu)) \partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu)) d\mu,$$

³⁷A simple calculation shows that ω_0^n must satisfy $\omega_0^n \circ t_0(\mu) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4f^n(\mu)}$, for all $\mu \in [\mu_L, \mu_H]$. In fact, there are always at least two such solutions.

which converges, as $n \rightarrow \infty$, to

$$\lim_n \Delta E(p^n(x)) \bar{f}^n = -\frac{1}{2\rho_0} (1 - t_0(\mu_2)) \partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu_2)) > 0.$$

This implies that the expected price effect on coverage $t_0(\mu_1)$ is strictly positive.

Proof of Proposition 5

First notice that

$$\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) = \sum_{s \in S} \frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})} [\pi(s | \tilde{\mu}) - \pi(s | \mu)], \quad (72)$$

where we have used the fact that, for any $\mu \in [\mu_L, \mu_H]$, $\sum_{s \in S} \dot{\pi}(s | \mu) = 0$.

From (72),

$$\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) = - \sum_{s \in S} \pi(s | \tilde{\mu}) \frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})} \left[\frac{\pi(s | \mu)}{\pi(s | \tilde{\mu})} - 1 \right] \quad (73)$$

$$= -\text{cov} \tilde{\mu} \left(\frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})}, \frac{\pi(s | \mu)}{\pi(s | \tilde{\mu})} \right), \quad (74)$$

where $\text{cov} \tilde{\mu}$ represents the covariance across different signal realizations with respect to the measure described by $\{\pi(s | \tilde{\mu})\}_{s \in S}$.

Condition MLRP implies that, for any $\ell \in \{1, \dots, n-1\}$,

$$\frac{\pi(s_{\ell+1} | \tilde{\mu})}{\pi(s_{\ell} | \tilde{\mu})} > \frac{\pi(s_{\ell+1} | \mu)}{\pi(s_{\ell} | \mu)} \iff \frac{\pi(s_{\ell} | \mu)}{\pi(s_{\ell} | \tilde{\mu})} > \frac{\pi(s_{\ell+1} | \mu)}{\pi(s_{\ell+1} | \tilde{\mu})},$$

and

$$\frac{\dot{\pi}(s_{\ell+1} | \mu)}{\pi(s_{\ell+1} | \mu)} > \frac{\dot{\pi}(s_{\ell} | \mu)}{\pi(s_{\ell} | \mu)},$$

for almost all $\mu \in [\mu_L, \mu_H]$. Summing up, the first term in the covariance (73), evaluated at s_{ℓ} , is increasing in ℓ while the second term is decreasing in ℓ . Hence, the covariance is negative and the expression in (73) is positive.

The following lemma is an auxiliary result and shows that for the welfare comparison it suffices to consider the first- or second-order terms of the price function.

Lemma 28. *Suppose that, for every compact set $M \subseteq (x_L, 1)$, there exists $\delta > 0$ sufficiently small such that*

$$p_{\delta}^A(x) < p_{\delta}^B(x) \text{ or } p_{\delta}^A(x) = p_{\delta}^B(x) \text{ and } p_{\delta\delta}^A(x) < p_{\delta\delta}^B(x),$$

for all $x \in M$. Then, (ω^A, ϕ^A) welfare-dominates (ω^B, ϕ^B) .

Proof. Consider a compact set $M \subset (x_L, 1)$. Proposition 3 implies that

$$\frac{p^A(x; \delta) - p^B(x; \delta)}{\delta} = \frac{p^A(x; \delta) - m_0(x)x}{\delta} - \frac{p^B(x; \delta) - m_0(x)x}{\delta},$$

which converges, uniformly in M , to $p_{\delta}^A(x) - p_{\delta}^B(x)$ and implies the result if $p_{\delta}^A(x) < p_{\delta}^B(x)$. Alternatively, if

$p_\delta^A(x) = p_\delta^B(x)$, for all $x \in M$, we have that

$$\frac{p^A(x; \delta) - p^B(x; \delta)}{\delta^2} = \frac{\frac{p^A(x; \delta) - m_0(x)x}{\delta} - p_\delta^A(x)}{\delta} - \frac{\frac{p^B(x; \delta) - m_0(x)x}{\delta} - p_\delta^B(x)}{\delta},$$

which converges to $\frac{1}{2} [p_{\delta\delta}^A(x) - p_{\delta\delta}^B(x)]$ uniformly over M and implies again the result. The welfare results follow from Lemma 26. \square

Lemma 28 does guarantee uniform dominance with respect to δ .

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Data Availability Statement

The replication code for the simulations reported in this article are available in Zenodo, at <https://doi.org/10.5281/zenodo.7259228>

References

- AKERLOF, G. (1970): "The market for "lemons": Quality uncertainty and the market mechanism," *The Quarterly Journal of Economics*, 84, 488–500.
- AZEVEDO, E. AND D. GOTTLIEB (2017): "Perfect competition in markets with adverse selection," *Econometrica*, 85, 67–105.
- BISIN, A. AND P. GOTTARDI (2006): "Efficient competitive equilibria with adverse selection," *Journal of Political Economy*, 114, 485–516.
- CHANG, B. (2018): "Adverse selection and liquidity distortion," *The Review of Economic Studies*, 85, 275–306.
- COHEN, A. AND L. EINAV (2007): "Estimating Risk Preferences from Deductible Choice," *American Economic Review*, 97, 745–788.
- CROCKER, K. J. AND A. SNOW (1986): "The efficiency effects of categorical discrimination in the insurance industry," *Journal of Political Economy*, 94, 321–344.

- (2011): “Multidimensional screening in insurance markets with adverse selection,” *Journal of Risk and Insurance*, 78, 287–307.
- DUBEY, P. AND J. GEANAKOPOLOS (2002): “Competitive pooling: Rothschild-Stiglitz reconsidered,” *The Quarterly Journal of Economics*, 117, 1529–1570.
- EINAV, L., A. FINKELSTEIN, S. P. RYAN, P. SCHRIMPF, AND M. R. CULLEN (2013): “Selection on Moral Hazard in Health Insurance,” *American Economic Review*, 103, 178–219.
- ERICSON, K. M. M. AND A. STARC (2015): “Pricing Regulation and Imperfect Competition on the Massachusetts Health Insurance Exchange,” *Review of Economics and Statistics*, 97, 667–682.
- FANG, H. AND Z. WU (2018): “Multidimensional private information, market structure, and insurance markets,” *RAND Journal of Economics*, 43, 751–787.
- FARINHA LUZ, V. (2017): “Characterization and uniqueness of equilibrium in competitive insurance,” *Theoretical Economics*, 12, 1349–1391.
- FINKELSTEIN, A., J. POTERBA, AND C. ROTHSCHILD (2009): “Redistribution by insurance market regulation: Analyzing a ban on gender-based retirement annuities,” *Journal of Financial Economics*, 91, 38–58.
- GALE, D. (1992): “A Walrasian theory of markets with adverse selection,” *The Review of Economic Studies*, 59, 229–255.
- GARCIA, D. AND M. TSUR (2021): “Information design in competitive insurance markets,” *Journal of Economic Theory*, 191, 105160.
- GOLDSTEIN, I. AND Y. LEITNER (2018): “Stress tests and information disclosure,” *Journal of Economic Theory*, 177, 34–69.
- GUERRIERI, V. AND R. SHIMER (2018): “Markets with multidimensional private information,” *American Economic Journal: Microeconomics*, 10, 250–74.
- GUERRIERI, V., R. SHIMER, AND R. WRIGHT (2010): “Adverse selection in competitive search equilibrium,” *Econometrica*, 78, 1823–1862.
- HALE, J. K. (1969): *Ordinary differential equations*, Dover Publications, INC.
- HANDEL, B., I. HENDEL, AND M. D. WHINSTON (2015): “Equilibria in health exchanges: Adverse selection versus reclassification risk,” *Econometrica*, 83, 1261–1313.
- HÖLMSTROM, B. (1979): “Moral hazard and observability,” *The Bell Journal of Economics*, 10, 74–91.
- HOLMSTROM, B. AND P. MILGROM (1987): “Aggregation and Linearity in the Provision of Intertemporal Incentive,” *Econometrica*, 55, 303–328.
- KRAFT, L. G. (1949): “A Device for Quantizing, Grouping, and Coding Amplitude-Modulated Pulses,” *MS Thesis, Electrical Engennering Departmnet, MIT*.
- LEVIN, J. (2001): “Information and the market for lemons,” *RAND Journal of Economics*, 32, 657–666.
- LEVY, Y. J. AND A. VEIGA (2021): “Competitive insurance markets with unbounded cost,” *Journal of Economic Theory*, 192.

- (2022): “Optimal Contract Regulation in Selection Markets,” *Working paper*.
- MCMILLAN, B. (1956): “Two inequalities implied by unique decipherability,” *IEEE Trans. Information Theory*, 2(4), 115–116.
- MIRPLEES, J. A. (1976): “The optimal structure of incentives and authority within an organization,” *The Bell Journal of Economics*, 6, 105–131.
- MIYAZAKI, H. (1977): “The rat race and internal labor markets,” *The Bell Journal of Economics*, 8(2), 394–418.
- PARMET, W. E. (2011): “The individual mandate: implications for public health law,” *The Journal of Law, Medicine & Ethics*, 39, 401–413.
- RILEY, J. G. (1979): “Informational Equilibrium,” *Econometrica*, 47, 331–359.
- ROTHSCHILD, C. (2011): “The efficiency of categorical discrimination in insurance markets,” *Journal of Risk and Insurance*, 78, 267–285.
- ROTHSCHILD, M. AND J. STIGLITZ (1976): “Equilibrium in competitive insurance markets: An essay on the economics of imperfect information,” *The Quarterly Journal of Economics*, 90(4), 629–649.
- SIMS, C. A. (2003): “Implications of rational inattention,” *Journal of Monetary Economics*, 50, 665–690.
- SMART, M. (2000): “Competitive insurance markets with two unobservables,” *International Economic Review*, 41(1), 153–169.
- SNOW, A. (2009): “On the Possibility of Profitable Self-Selection Contracts in Competitive Markets,” *The Journal of Risk and Insurance*, 76, 249–259.
- SPENCE, M. (1978): “Product differentiation and performance in insurance markets,” *Journal of Public Economics*, 10, 427–447.
- VEIGA, A. (2022): “Price Discrimination in Markets for Lemons,” *Working paper*.
- VILLENEUVE, B. (2003): “Concurrence et antisélection multidimensionnelle en assurance,” *Annales d’Économie et de Statistique*, 69, 119–42.
- WAMBACH, A. (2000): “Introducing heterogeneity in the Rothschild-Stiglitz model,” *Journal of Risk and Insurance*, 67(4), 579–591.
- WILLIAMS, B. (2021): “Search, liquidity, and retention: Screening multidimensional private information,” *Journal of Political Economy*, 129, 1487–1507.
- WILSON, C. (1977): “A model of insurance markets with incomplete information,” *Journal of Economic Theory*, 16, 167–207.