

Multi-parameter analysis of the obstacle scattering problem

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Abstract: We consider the acoustic field scattered by a bounded impenetrable obstacle and we study its dependence upon a certain set of parameters. As usual, the problem is modeled by an exterior Dirichlet problem for the Helmholtz equation $\Delta u + k^2 u = 0$. We show that the solution u and its far field pattern u_∞ depend real analytically on the shape of the obstacle, the wave number k , and the Dirichlet datum. We also prove a similar result for the corresponding Dirichlet-to-Neumann map.

Keywords: Helmholtz equation; acoustic scattering; associated exterior Dirichlet problem; Dirichlet-to-Neumann operator; shape sensitivity analysis; perturbed domain; integral equations.

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1 Introduction

Understanding how the shape of an object impacts a certain property is a very old problem and has a variety of applications. We may think, for example, to the problem of finding the best design to maximize some sort of efficiency or to the sort of problems related to *non-destructive test methods*, like the problem of finding the shape of an inclusion from a set of measurements taken on the outer boundary of an object, or the inverse scattering problem, where the shape of an obstacle is inferred from measurements of a scattered wave.

In mathematics, the property that one wants to analyze is often associated with the solution of a boundary value problem or to a quantity related to the solution by a certain functional. Then, understanding how the shape impacts a specific property amounts to studying the dependence of the solution of the boundary value problem upon perturbations of the domain of the partial differential equation.

In mathematical jargon the problem of finding an optimal configuration that maximizes a shape functional goes under the name of *shape optimization*. The reader may find some references in the monographs by Henrot and Pierre [16], Novotny and Sokołowski [30], and Sokołowski and Zolésio [34]. The problems of inferring a shape from measurements on the boundary of an outer domain or a scattered wave are known as *inclusion detection* and *inverse scattering problems*, respectively, and are both examples of *inverse problems*. For some references we mention the books of Colton and Kress [4] and Kirsch [20].

A preliminary task that is common to shape optimization and the above mentioned inverse problems is that of understanding the regularity of the map that associates the shape of an object to the solution of the boundary value problem and to the specific quantity under consideration. For most techniques, indeed, it is desirable to have at least some sort of differentiability (as in Kirsch [19], where the differentiability of the far field pattern is used in the numerical analysis of an inverse scattering problem).

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It is not surprising, then, that many papers deal with the differentiability properties of shape functionals and our paper is one of these. More specifically, we examine an acoustic obstacle scattering problem and study the dependence of the solution and of its far field pattern upon perturbations of the wave number, the Dirichlet datum, and the shape of the obstacle. We also consider the pullback of the Dirichlet-to-Neumann operator and its dependence on the wave number and the shape of the obstacle.

Among the works that precede our paper with similar results we mention those of Potthast [31, 32, 33], where the aim is to prove that the layer potentials of the Helmholtz equation are Fréchet differentiable functions of the support of integration. Potthast's results are obtained in the framework of Schauder spaces and the final goal is that of analyzing the domain derivative of the far field pattern. Related problems are studied in the papers of Haddar and Kress [14], Hettlich [17], Kirsch [19], and Kress and Päiväranta [21]. For similar differentiability results, but for the elastic scattering problem, we mention Charalambopoulos [1]. Finally, the case of Lipschitz domains have been studied by Costabel and Le Louër [5, 6, 29] in the framework of Sobolev spaces.

The novelties that we bring in this list are of two kinds. On the one hand, the regularity properties that we prove are stronger than Fréchet differentiability. More specifically, we obtain *real analyticity* results. On the other hand, we do not confine ourselves only to the shape of the obstacle, but we consider the joint regularity upon the wave number, the Dirichlet datum, and the shape. So, for example, we prove that the far field pattern is a real analytic map of the wave number, the Dirichlet datum, and the shape of the obstacle (a triple that we think as a unique variable in a certain product Banach space).

Incidentally, we observe that there are very few results in literature that go beyond the differentiability of shape functionals. A remarkable example are some recent works on the *shape holomorphy* by Jerez-Hanckes, Schwab, and Zech [18], which deals with the electromagnetic wave scattering problem, by Cohen, Schwab, and Zech [2], about the stationary Navier-Stokes equations, and by Henríquez and Schwab [15], on the Calderón projector for the Laplacian in \mathbb{R}^2 .

We now introduce the geometry of the problem. We fix

$$\alpha \in]0, 1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^3 \text{ of class } C^{1,\alpha} \quad (1)$$

such that $\mathbb{R}^3 \setminus \bar{\Omega}$ is connected.

Here we note that, if Ω is a set, the symbol $\bar{\Omega}$ denotes its closure. Also, if $z \in \mathbb{C}$, we denote by \bar{z} the conjugate of the complex number z . For the definition of sets and functions of the Schauder class $C^{j,\alpha}$ ($j \in \mathbb{N}$) we refer, e.g., to Gilbarg and Trudinger [13]. We also note that, if not otherwise specified, all the functions in the paper are complex-valued.

To consider perturbations of the shape of the obstacle, we take the set Ω of (1) as a reference set. Then we introduce a specific class $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ of $C^{1,\alpha}$ -diffeomorphisms from $\partial\Omega$ to \mathbb{R}^3 : $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ is the set of functions of class $C^{1,\alpha}(\partial\Omega, \mathbb{R}^3)$ that are injective and have injective differential at all points of $\partial\Omega$. By Lanza de Cristoforis and Rossi [25, Lem. 2.2, p. 197] and [24, Lem. 2.5, p. 143], we can see that $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ is open in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^3)$. Moreover, for all $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$ the Jordan-Leray separation theorem ensures that $\mathbb{R}^3 \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, e.g., Deimling [11, Thm. 5.2, p. 26] and [9, §A.4]). We denote by $\mathbb{I}[\phi]$ the bounded connected component of $\mathbb{R}^3 \setminus \phi(\partial\Omega)$ and by $\mathbb{E}[\phi]$ the unbounded one. Then, we have

$$\mathbb{E}[\phi] = \mathbb{R}^3 \setminus \bar{\mathbb{I}[\phi]} \quad \text{and} \quad \bar{\mathbb{E}[\phi]} = \mathbb{R}^3 \setminus \mathbb{I}[\phi].$$

The ϕ -dependent set $\mathbb{I}[\phi]$ models the shape of the impenetrable obstacle (i.e. the scattering object), while $\mathbb{E}[\phi] = \mathbb{R}^3 \setminus \bar{\mathbb{I}[\phi]}$ represents the homogeneous isotropic media where the scattered acoustic waves propagate (see Figure 1).

Now we take $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, $k \in \mathbb{C}$ with $\text{Im}k \geq 0$, and $g \in C^{1,\alpha}(\partial\Omega)$. We consider the following direct obstacle scattering problem: we look for a (complex-valued) function

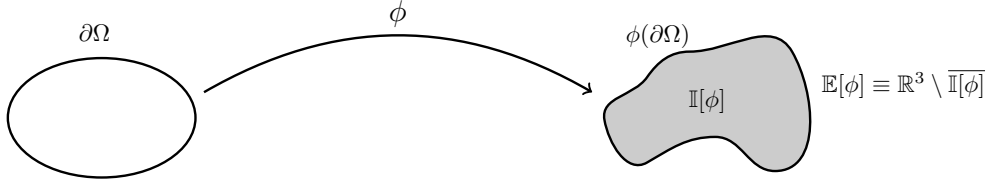


Figure 1: The reference set $\partial\Omega$, a diffeomorphism $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, and the sets $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$.

$u \in C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{E}[\phi]})$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{E}[\phi], \\ u = g \circ \phi^{(-1)} & \text{on } \partial\mathbb{E}[\phi], \\ \lim_{x \rightarrow \infty} |x| \left(Du(x) \cdot \frac{x}{|x|} - iku(x) \right) = 0, \\ \lim_{x \rightarrow \infty} u(x) = 0, \end{cases} \quad (2)$$

where, as usual,

$$\Delta = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}.$$

The third condition in problem (2), i.e.

$$\lim_{x \rightarrow \infty} |x| \left(Du(x) \cdot \frac{x}{|x|} - iku(x) \right) = 0, \quad (3)$$

is known as the *outgoing Sommerfeld (k)-radiation condition*. From the point of view of physics, solutions of the Helmholtz equation that satisfy the outgoing Sommerfeld condition describe waves that scatter from a source situated in a bounded domain. In particular, waves with sources situated at infinity do not satisfy condition (3). For $k \neq 0$, the Sommerfeld condition implies the decay at infinity of $u(x)$, and thus it is stronger than the last condition of problem (2) (cf., e.g., Colton and Kress [3, Chap. 3, Rem. 3.4]). For $k = 0$, this is no more the case, as one can easily verify taking u identically constant. In particular, for $k = 0$ a solution u of (2) is a harmonic function that, by the last condition of the system, is also harmonic at infinity (see Folland [12, Chap. 2]). Then, in this case it is the Sommerfeld condition to follow from the decay at infinity of $u(x)$ (see, e.g., Folland [12, Prop. 2.75]).

Either way, from the Sommerfeld condition if $k \in \mathbb{C} \setminus \{0\}$ and $\text{Im}k \geq 0$, or from the decay of $u(x)$ if $k = 0$, we can see that problem (2) has a unique solution in $C_{\text{loc}}^{1,\alpha}(\overline{\mathbb{E}[\phi]})$ for all choice of $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, $k \in \mathbb{C}$ with $\text{Im}k \geq 0$, and $g \in C^{1,\alpha}(\partial\Omega)$ (cf. Colton and Kress [3, Chap. 3] for the case with $k \neq 0$ and Folland [12, Chap. 3] for $k = 0$). From now on, we denote such a solution by $u[\phi, k, g]$.

We stress that we decided to state problem (2) including both the Sommerfeld condition and the decay at infinity for exactly this reason, that is, to have a unique solution both when $k \neq 0$ and $k = 0$. Doing so we can study the dependence of the solution $u[\phi, k, g]$ upon the wave number $k \in \mathbb{C}$ with $\text{Im}k \geq 0$ in a unified way, without the need of introducing two different problems for $k \neq 0$ and $k = 0$.

We also observe that there exists a function $u_\infty[\phi, k, g]$, defined on the boundary $\partial\mathbb{B}_3(0, 1)$ of the three dimensional unit ball $\mathbb{B}_3(0, 1)$ and with values in \mathbb{C} , for which we have the following asymptotic expansion:

$$u[\phi, k, g](x) = \frac{e^{ik|x|}}{|x|} \left(u_\infty[\phi, k, g] \left(\frac{x}{|x|} \right) + O \left(\frac{1}{|x|} \right) \right) \quad \text{as } |x| \rightarrow +\infty.$$

For $k \neq 0$, $u_\infty[\phi, k, g]$ is known as the *far field pattern* of $u[\phi, k, g]$ (see, e.g., Colton and Kress [3, Chap. 3]) and, for $k = 0$, $u_\infty[\phi, k, g]$ is constant (it is indeed the spherical harmonic of degree zero in the expansion $u[\phi, k, g](x) = \sum_{j=0}^{\infty} |x|^{-1-j} Y_j(x/|x|)$), where

every Y_j is a spherical harmonic of degree j). Both for $k = 0$ and $k \neq 0$, $u_\infty[\phi, k, g]$ can be computed from the solution $u[\phi, k, g]$ by the formula

$$u_\infty[\phi, k, g](x) = \frac{1}{4\pi} \int_{\partial\mathbb{B}_3(0,R)} \left(u[\phi, k, g](y) \frac{\partial}{\partial\nu_{\mathbb{B}_3(0,R)}(y)} e^{-ikx \cdot y} - e^{-ikx \cdot y} \frac{\partial u[\phi, k, g](y)}{\partial\nu_{\mathbb{B}_3(0,R)}} \right) d\sigma_y \quad (4)$$

$$\forall x \in \partial\mathbb{B}_3(0,1),$$

where $R > 0$ has to be taken large enough so that $\overline{\mathbb{I}[\phi]} \subseteq \mathbb{B}_3(0, R)$ and where $\nu_{\mathbb{B}_3(0,R)}$ denotes the outward unit normal to $\partial\mathbb{B}_3(0, R)$. By the divergence theorem, we can also verify that the integral in the right-hand side of (4) does not depend on the specific choice of R . From the point of view of physics, the far field pattern represents the main directional (angular) part of a wave away from a scattering object. In inverse scattering theory, one of the main problems is that of reconstructing the properties of an object starting from the knowledge of the far field pattern.

Moreover, if $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, $k \in \mathbb{C}$ with $\text{Im}k \geq 0$, we introduce the pullback of the Dirichlet-to-Neumann operator $\mathcal{D}_{(\phi,k)}$ from $C^{1,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ as the linear operator that takes the Dirichlet datum g to the normal derivative of the solution $u[\phi, k, g]$, i.e.

$$\mathcal{D}_{(\phi,k)}[g] \equiv \left(\frac{\partial}{\partial\nu_{\mathbb{I}[\phi]}} u[\phi, k, g] \right) \circ \phi.$$

Our aim is to investigate the dependence of the solution $u[\phi, k, g]$ and of its far field pattern $u_\infty[\phi, k, g]$ upon the triple (ϕ, k, g) , and of the Dirichlet-to-Neumann operator $\mathcal{D}_{(\phi,k)}$ upon the pair (ϕ, k) . As mentioned above, the rationale of this paper is to prove regularity properties that go beyond the Fréchet differentiability. More specifically, we do not confine to the dependence on the shape: we study the joint dependence on the triple (ϕ, k, g) and we prove (joint) *real analyticity* results. So, for example, in Theorem 4.5 we show that the map

$$\left(\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega) \right) \ni (\phi, k, g) \mapsto u_\infty[\phi, k, g] \in C^2(\partial\mathbb{B}_3(0,1))$$

is real analytic. In the expression above \mathbb{C}^+ is the set of complex numbers k with $\text{Im}k \geq 0$. In Theorems 4.1 and 4.3, we prove similar results also for the solution $u[\phi, k, g]$ and for its normal derivative. In Corollary 4.4 we deduce by Theorem 4.3 a corresponding result for the pullback of the Dirichlet-to-Neumann map.

We stress here that for us the word “analytic” always means “real analytic.” For the definition and properties of real analytic operators, we refer to Deimling [11, §15].

Our analysis relies on the results of [8], where the authors consider layer potentials associated with a family of fundamental solutions of second order differential operators with constant coefficients depending on a parameter. The authors prove the real analytic dependence of the layer potentials upon variations of the diffeomorphism, the density, and the parameter. In the present paper we apply the results of [8] to the k -dependent fundamental solution $-\frac{1}{4\pi|x|} e^{ik|x|}$, $x \in \mathbb{R}^3 \setminus \{0\}$, of the Helmholtz equation $\Delta u + k^2 u = 0$. We also mention the work of Lanza de Cristoforis and Rossi [25] on layer potentials of the Helmholtz equation, where they consider a different family of fundamental solutions (see also the previous work [24] by the same authors, which deals with harmonic layer potentials and [7] for the case of higher order operators). Moreover, analyticity results for integral operators and methods of potential theory have been exploited in the monograph [9] to obtain real analytic continuation properties of the solutions of singularly perturbed boundary value problems. Finally, we point out that an analysis similar to the one of the present paper has been carried out by the authors for other physical quantities arising in fluid mechanics and in material science (see [10, 27, 28] for the longitudinal fluid flow along a periodic array of cylinders and the effective conductivity of a periodic two-phase composite material).

The paper is organized as follows. Section 2 is a section of preliminaries of classical potential theory for the Helmholtz equation. In Section 3 we transform problem (2) into an equivalent integral equation. Finally, in Section 4 we prove our main results on the analyticity of functions related to problem (2).

2 Preliminaries of potential theory

Let $\alpha \in]0, 1[$ and let $\tilde{\Omega}$ be a bounded open connected subset of \mathbb{R}^3 of class $C^{1,\alpha}$. We denote by $\nu_{\tilde{\Omega}}$ the outward unit normal to $\partial\tilde{\Omega}$ and by $d\sigma$ the area element on $\partial\tilde{\Omega}$.

We remark that, in this section, $\tilde{\Omega}$ is a generic open subset of \mathbb{R}^3 that we use as a dummy to define some notation and write some general results. Instead, the set Ω introduced in (1) is a reference domain that we keep fixed for the whole paper.

Our method is based on classical potential theory. In order to construct layer potentials, we introduce for $k \in \mathbb{C}$ the function

$$S(k, x) \equiv -\frac{1}{4\pi|x|} e^{ik|x|} \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

For $k \neq 0$, $S(k, x)$ is a standard fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ that satisfies the outgoing Sommerfeld (k)-radiation condition. For $k = 0$, $S(0, x)$ is a standard fundamental solution of the Laplace equation $\Delta u = 0$, that is

$$S(0, x) = -\frac{1}{4\pi|x|} \quad \forall x \in \mathbb{R}^3 \setminus \{0\},$$

and is harmonic at infinity.

Then, we introduce the layer potentials associated with the fundamental solution $S(k, \cdot)$. We set

$$\begin{aligned} v[\partial\tilde{\Omega}, k, \mu](x) &\equiv \int_{\partial\tilde{\Omega}} S(k, x-y)\mu(y) d\sigma_y \quad \forall x \in \mathbb{R}^3, \\ w[\partial\tilde{\Omega}, k, \mu](x) &\equiv \int_{\partial\tilde{\Omega}} \frac{\partial}{\partial\nu_{\tilde{\Omega}}(y)} S(k, x-y)\mu(y) d\sigma_y \\ &\equiv - \int_{\partial\tilde{\Omega}} \nu_{\tilde{\Omega}}(y) \cdot DS(k, x-y)\mu(y) d\sigma_y \quad \forall x \in \mathbb{R}^3, \end{aligned}$$

for all $\mu \in C^0(\partial\tilde{\Omega})$. Here above, $DS(k, \xi)$ denotes the gradient of $S(k, \cdot)$ computed at the point $\xi \in \mathbb{R}^3 \setminus \{0\}$. We also clarify that, in this paper,

$$\frac{\partial}{\partial\nu_{\tilde{\Omega}}(y)}$$

is the partial derivative (in the normal direction) with respect to the y variable, whereas

$$\frac{\partial}{\partial\nu_{\tilde{\Omega}}(x)}$$

denotes the partial derivative with respect to the x variable. This is why a $-$ (minus) sign appears in front of the last integral. We also set

$$\begin{aligned} V[\partial\tilde{\Omega}, k, \mu](x) &\equiv \int_{\partial\tilde{\Omega}} S(k, x-y)\mu(y) d\sigma_y \quad \forall x \in \partial\tilde{\Omega}, \\ W[\partial\tilde{\Omega}, k, \mu](x) &\equiv \int_{\partial\tilde{\Omega}} \frac{\partial}{\partial\nu_{\tilde{\Omega}}(y)} S(k, x-y)\mu(y) d\sigma_y \quad \forall x \in \partial\tilde{\Omega}, \\ W^*[\partial\tilde{\Omega}, k, \mu](x) &\equiv \int_{\partial\tilde{\Omega}} \frac{\partial}{\partial\nu_{\tilde{\Omega}}(x)} S(k, x-y)\mu(y) d\sigma_y \quad \forall x \in \partial\tilde{\Omega}, \end{aligned}$$

for all $\mu \in C^0(\partial\tilde{\Omega})$. The function $v[\partial\tilde{\Omega}, k, \mu]$ is called ‘‘single layer potential’’ and $w[\partial\tilde{\Omega}, k, \mu]$ is called ‘‘double layer potential.’’ As is well known, if $\mu \in C^0(\partial\tilde{\Omega})$, then $v[\partial\tilde{\Omega}, k, \mu]$ is continuous in \mathbb{R}^3 and we set

$$v^+[\partial\tilde{\Omega}, k, \mu] \equiv v[\partial\tilde{\Omega}, k, \mu]_{|\tilde{\Omega}} \quad \text{and} \quad v^-[\partial\tilde{\Omega}, k, \mu] \equiv v[\partial\tilde{\Omega}, k, \mu]_{|\mathbb{R}^3 \setminus \tilde{\Omega}}.$$

In Theorems 2.1 and 2.2 below, we collect some well-known properties of layer potentials (cf., e.g., [8] with Lanza de Cristoforis, Colton and Kress [3], Lanza de Cristoforis and Rossi [24, 25]).

Theorem 2.1. *Let $\alpha \in]0, 1[$ and let $\tilde{\Omega}$ be a bounded open connected subset of \mathbb{R}^3 of class $C^{1,\alpha}$. Let $k \in \mathbb{C}$ be such that $\text{Im}k \geq 0$. Then the following statements hold.*

(i) *If $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$, then $v^+[\partial\tilde{\Omega}, k, \mu] \in C^{1,\alpha}(\overline{\tilde{\Omega}})$ and $v^-[\partial\tilde{\Omega}, k, \mu] \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3 \setminus \tilde{\Omega})$. Moreover,*

$$\Delta v[\partial\tilde{\Omega}, k, \mu] + k^2 v[\partial\tilde{\Omega}, k, \mu] = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial\tilde{\Omega}$$

and

- *if $k \neq 0$, then $v^-[\partial\tilde{\Omega}, k, \mu]$ satisfies the outgoing Sommerfeld (k)-radiation condition (3),*
- *if $k = 0$, then $v^-[\partial\tilde{\Omega}, k, \mu]$ is harmonic at infinity.*

(ii) *The map from $C^{0,\alpha}(\partial\tilde{\Omega})$ to $C^{1,\alpha}(\overline{\tilde{\Omega}})$ that takes μ to $v^+[\partial\tilde{\Omega}, k, \mu]$ is linear and continuous. If $R > 0$ is such that $\overline{\tilde{\Omega}} \subseteq \mathbb{B}_3(0, R)$, then the map from $C^{0,\alpha}(\partial\tilde{\Omega})$ to $C^{1,\alpha}(\overline{\mathbb{B}_3(0, R)} \setminus \tilde{\Omega})$ that takes μ to $v^-[\partial\tilde{\Omega}, k, \mu]_{|\overline{\mathbb{B}_3(0, R)} \setminus \tilde{\Omega}}$ is linear and continuous.*

(iii) *Let $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$. Then*

$$\frac{\partial}{\partial\nu_{\tilde{\Omega}}} v^\pm[\partial\tilde{\Omega}, k, \mu] = \mp \frac{1}{2} \mu + W^*[\partial\tilde{\Omega}, k, \mu] \quad \text{on } \partial\tilde{\Omega}.$$

Moreover, the map that takes $\mu \in C^{0,\alpha}(\partial\tilde{\Omega})$ to $W^[\partial\tilde{\Omega}, k, \mu]$ is a compact operator from $C^{0,\alpha}(\partial\tilde{\Omega})$ to itself.*

Theorem 2.2. *Let $\alpha \in]0, 1[$ and let $\tilde{\Omega}$ be a bounded open connected subset of \mathbb{R}^3 of class $C^{1,\alpha}$. Let $k \in \mathbb{C}$ be such that $\text{Im}k \geq 0$. Then the following statements hold.*

(i) *If $\mu \in C^{1,\alpha}(\partial\tilde{\Omega})$, then $w[\partial\tilde{\Omega}, k, \mu]_{|\partial\tilde{\Omega}}$ can be extended to a continuous function $w^+[\partial\tilde{\Omega}, k, \mu] \in C^{1,\alpha}(\overline{\tilde{\Omega}})$ and $w[\partial\tilde{\Omega}, k, \mu]_{|\mathbb{R}^3 \setminus \overline{\tilde{\Omega}}}$ can be extended to a continuous function $w^-[\partial\tilde{\Omega}, k, \mu] \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3 \setminus \tilde{\Omega})$. Moreover,*

$$\Delta w[\partial\tilde{\Omega}, k, \mu] + k^2 w[\partial\tilde{\Omega}, k, \mu] = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial\tilde{\Omega}$$

and

- *if $k \neq 0$, then $w^-[\partial\tilde{\Omega}, k, \mu]$ satisfies the outgoing Sommerfeld (k)-radiation condition (3),*
- *if $k = 0$, then $w^-[\partial\tilde{\Omega}, k, \mu]$ is harmonic at infinity.*

(ii) *The map from $C^{1,\alpha}(\partial\tilde{\Omega})$ to $C^{1,\alpha}(\overline{\tilde{\Omega}})$ that takes μ to $w^+[\partial\tilde{\Omega}, k, \mu]$ is linear and continuous. If $R > 0$ is such that $\overline{\tilde{\Omega}} \subseteq \mathbb{B}_3(0, R)$, then the map from $C^{1,\alpha}(\partial\tilde{\Omega})$ to $C^{1,\alpha}(\overline{\mathbb{B}_3(0, R)} \setminus \tilde{\Omega})$ that takes μ to $w^-[\partial\tilde{\Omega}, k, \mu]_{|\overline{\mathbb{B}_3(0, R)} \setminus \tilde{\Omega}}$ is linear and continuous.*

(iii) *Let $\mu \in C^{1,\alpha}(\partial\tilde{\Omega})$. Then*

$$w^\pm[\partial\tilde{\Omega}, k, \mu] = \pm \frac{1}{2} \mu + W[\partial\tilde{\Omega}, k, \mu] \quad \text{on } \partial\tilde{\Omega}$$

and

$$\frac{\partial}{\partial\nu_{\tilde{\Omega}}} w^+[\partial\tilde{\Omega}, k, \mu] = \frac{\partial}{\partial\nu_{\tilde{\Omega}}} w^-[\partial\tilde{\Omega}, k, \mu] \quad \text{on } \partial\tilde{\Omega}.$$

Moreover, the map that takes $\mu \in C^{1,\alpha}(\partial\tilde{\Omega})$ to $W[\partial\tilde{\Omega}, k, \mu]$ is a compact operator from $C^{1,\alpha}(\partial\tilde{\Omega})$ to itself.

Our approach is based on integral equations. More precisely, in order to study problem (2), we convert it into an equivalent integral equation. We do so by exploiting a representation formula of the solution $u[\phi, k, g]$ in terms of single and double layer potentials. Therefore, we now show the validity of the following variant of the result of Colton and Kress [3, Thm. 3.33] regarding the solvability of the exterior Dirichlet problem for the Helmholtz equation by means of a combined double and single layer potential.

Theorem 2.3. *Let $\alpha \in]0, 1[$ and let $\tilde{\Omega}$ be a bounded open connected subset of \mathbb{R}^3 of class $C^{1,\alpha}$ such that $\mathbb{R}^3 \setminus \tilde{\Omega}$ is connected. Let $k \in \mathbb{C}$ be such that $\text{Im}k \geq 0$. Then the following statements hold.*

(i) *The integral operator T from $C^{1,\alpha}(\partial\tilde{\Omega})$ to itself defined by*

$$T \equiv -\frac{1}{2}I + W[\partial\tilde{\Omega}, k, \cdot] + (1 - i\text{Re}k)V[\partial\tilde{\Omega}, k, \cdot],$$

where I denotes the identity operator, is a linear homeomorphism.

(ii) *Let $\Gamma \in C^{1,\alpha}(\partial\tilde{\Omega})$. Then problem*

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \tilde{\Omega}, \\ u = \Gamma & \text{on } \partial\tilde{\Omega}, \\ \lim_{x \rightarrow \infty} |x| \left(Du(x) \cdot \frac{x}{|x|} - iku(x) \right) = 0, \\ \lim_{x \rightarrow \infty} u(x) = 0 \end{cases} \quad (5)$$

has a unique solution $u \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3 \setminus \tilde{\Omega})$. Moreover,

$$u = w^-[\partial\tilde{\Omega}, k, \mu] + (1 - i\text{Re}k)v^-[\partial\tilde{\Omega}, k, \mu] \quad \text{in } \mathbb{R}^3 \setminus \tilde{\Omega},$$

where $\mu \in C^{1,\alpha}(\partial\tilde{\Omega})$ is delivered by

$$\mu = \left(-\frac{1}{2}I + W[\partial\tilde{\Omega}, k, \cdot] + (1 - i\text{Re}k)V[\partial\tilde{\Omega}, k, \cdot] \right)^{(-1)}[\Gamma].$$

Proof. We first consider statement (i). We modify the proof of Colton and Kress [3, Thm. 3.33]. We first note that by Theorems 2.1 (ii) and 2.2 (iii), by the continuity of the single layer potential, by the compactness of the embedding of $C^{1,\alpha}(\partial\tilde{\Omega})$ in $C^{0,\alpha}(\partial\tilde{\Omega})$, and by the continuity of the restriction operator from $C^{1,\alpha}(\tilde{\Omega})$ to $C^{1,\alpha}(\partial\tilde{\Omega})$, the operator

$$\psi \mapsto W[\partial\tilde{\Omega}, k, \psi] + (1 - i\text{Re}k)V[\partial\tilde{\Omega}, k, \psi]$$

is compact from $C^{1,\alpha}(\partial\tilde{\Omega})$ to itself. Therefore,

$$T = -\frac{1}{2}I + W[\partial\tilde{\Omega}, k, \cdot] + (1 - i\text{Re}k)V[\partial\tilde{\Omega}, k, \cdot]$$

is a Fredholm operator of index 0. As a consequence, to show that T invertible, it suffices to prove that it is injective. So let $\psi \in C^{1,\alpha}(\partial\tilde{\Omega})$ be such that

$$-\frac{1}{2}\psi + W[\partial\tilde{\Omega}, k, \psi] + (1 - i\text{Re}k)V[\partial\tilde{\Omega}, k, \psi] = 0.$$

Then, by the continuity of the single layer potential and by the jump formula for the double layer potential (see Theorem 2.2 (iii)), the function $u \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3 \setminus \tilde{\Omega})$ defined by

$$u = w^-[\partial\tilde{\Omega}, k, \psi] + (1 - i\text{Re}k)v^-[\partial\tilde{\Omega}, k, \psi] \quad \text{in } \mathbb{R}^3 \setminus \tilde{\Omega}$$

solves the homogeneous exterior Dirichlet problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \tilde{\Omega}, \\ u = 0 & \text{on } \partial\tilde{\Omega}, \\ \lim_{x \rightarrow \infty} |x| \left(Du(x) \cdot \frac{x}{|x|} - iku(x) \right) = 0, \\ \lim_{x \rightarrow \infty} u(x) = 0, \end{cases}$$

and thus, by the uniqueness of the solution of problem (5) (cf. Colton and Kress [3, Chap. 3] for the case with $k \neq 0$ and Folland [12, Chap. 3] for $k = 0$), we have

$$u = 0 \quad \text{in } \mathbb{R}^3 \setminus \tilde{\Omega}.$$

Next we set

$$u^\# \equiv w^+[\partial\tilde{\Omega}, k, \psi] + (1 - i\operatorname{Re}k)v^+[\partial\tilde{\Omega}, k, \psi] \quad \text{in } \bar{\Omega}.$$

Clearly, $u^\# \in C^{1,\alpha}(\bar{\Omega})$ and by the jump relations for the layer potentials (see Theorems 2.1 and 2.2), we have

$$u^\#_{|\partial\tilde{\Omega}} = \psi + w^-[\partial\tilde{\Omega}, k, \psi]_{|\partial\tilde{\Omega}} + (1 - i\operatorname{Re}k)v^-[\partial\tilde{\Omega}, k, \psi]_{|\partial\tilde{\Omega}} = \psi$$

and

$$\begin{aligned} \frac{\partial}{\partial\nu_{\tilde{\Omega}}} u^\# &= \frac{\partial}{\partial\nu_{\tilde{\Omega}}} w^-[\partial\tilde{\Omega}, k, \psi] + (1 - i\operatorname{Re}k) \left(-\psi + \frac{\partial}{\partial\nu_{\tilde{\Omega}}} v^-[\partial\tilde{\Omega}, k, \psi] \right) \\ &= (-1 + i\operatorname{Re}k)\psi. \end{aligned}$$

Then, the first Green identity (cf., e.g., Colton and Kress [3, (3.4), p. 68]) implies that

$$(-1 + i\operatorname{Re}k) \int_{\partial\tilde{\Omega}} |\psi|^2 d\sigma = \int_{\partial\tilde{\Omega}} \overline{u^\#} \frac{\partial u^\#}{\partial\nu_{\tilde{\Omega}}} d\sigma = \int_{\tilde{\Omega}} |\nabla u^\#|^2 - k^2 |u^\#|^2 dx. \quad (6)$$

Taking the real part in (6), we obtain

$$- \int_{\partial\tilde{\Omega}} |\psi|^2 d\sigma = \int_{\tilde{\Omega}} |\nabla u^\#|^2 - [(\operatorname{Re}k)^2 - (\operatorname{Im}k)^2] |u^\#|^2 dx \quad (7)$$

and taking the imaginary part in (6), we get

$$\operatorname{Re}k \int_{\partial\tilde{\Omega}} |\psi|^2 d\sigma = -2(\operatorname{Re}k)(\operatorname{Im}k) \int_{\tilde{\Omega}} |u^\#|^2 dx. \quad (8)$$

Now, if $\operatorname{Re}k \neq 0$, then equation (8) implies that $\psi = 0$ (we also remember that $\operatorname{Im}k \geq 0$) and if $\operatorname{Re}k = 0$, then equation (7) implies that $\psi = 0$. Either way, we have $\psi = 0$ and statement (i) follows. The validity of statement (ii) follows from statement (i), from the jump formulas for the double layer potential of Theorems 2.2, and from the continuity of the single layer potential. \square

We now introduce a technical lemma about the real analytic dependence upon the diffeomorphism ϕ of some maps related to the change of variables in integrals and to the outer normal field. For a proof we refer to Lanza de Cristoforis and Rossi [24, p. 166] and to Lanza de Cristoforis [22, Prop. 1].

Lemma 2.4. *Let α, Ω be as in (1). Then the following statements hold.*

(i) *For each $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$, there exists a unique $\tilde{\sigma}[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\phi] > 0$ and*

$$\int_{\phi(\partial\Omega)} w(s) d\sigma_s = \int_{\partial\Omega} w \circ \phi(y) \tilde{\sigma}[\phi](y) d\sigma_y \quad \forall w \in L^1(\phi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ from $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) *The map from $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ to $C^{0,\alpha}(\partial\Omega, \mathbb{R}^3)$ that takes ϕ to $\nu_{\mathbb{I}[\phi]} \circ \phi$ is real analytic.*

By the results of [8] and the definition of $S(k, \cdot)$, we deduce the following lemma on the real analyticity of some maps related to the ϕ -pullback of layer potentials and their derivatives (see also Lanza de Cristoforis and Rossi [25] and Lanza de Cristoforis [26, §3]).

Lemma 2.5. *Let α, Ω be as in (1). Then the following statements hold.*

(i) *The map from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ that takes a triple (ϕ, k, θ) to the function $V[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$ is real analytic.*

(ii) *The map from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ that takes a triple (ϕ, k, θ) to the function $W[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$ is real analytic.*

(iii) The map from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{0,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ that takes a triple (ϕ, k, θ) to the function $W^*[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$ is real analytic.

(iv) The map from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$ that takes a triple (ϕ, k, θ) to the function

$$\left(\frac{\partial}{\partial \nu_{\mathbb{I}[\phi]}} w^-[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \right) \circ \phi$$

is real analytic.

Proof. By a straightforward computation, one verifies that the k -dependent families of fundamental solutions $S(k, \cdot)$ and of differential operators $P[k](u) \equiv \Delta u + k^2 u$ satisfy the assumption in [8, (1.1)]. Then the validity of statements (i)–(iv) follows by [8, Thm. 5.6]. \square

3 Analysis of the integral equation formulation of problem (2)

By Theorem 2.3 we can transform problem (2) into an equivalent integral equation. Then, the dependence of the solution of problem (2) on the shape of the obstacle, the wave number, and the Dirichlet datum, can be analyzed studying the dependence of the solution of the equivalent integral equation on the triple (ϕ, k, g) . We begin with the following Proposition 3.1, which follows from Theorem 2.3 and from a change of variable.

Proposition 3.1. *Let α, Ω be as in (1). Let $\phi \in \mathcal{A}_{\partial\Omega}^{1,\alpha}$. Let $k \in \mathbb{C}$ be such that $\text{Im}k \geq 0$. Let $g \in C^{1,\alpha}(\partial\Omega)$. Then the unique solution $u[\phi, k, g] \in C_{\text{loc}}^{1,\alpha}(\mathbb{E}[\bar{\phi}])$ of problem (2) is delivered by*

$$u[\phi, k, g] = w^-[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] + (1 - i\text{Re}k)v^-[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}],$$

where $\theta \in C^{1,\alpha}(\partial\Omega)$ is the unique solution of

$$-\frac{1}{2}\theta + W[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi + (1 - i\text{Re}k)V[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi = g. \quad (9)$$

In view of the previous Proposition 3.1, we find convenient to introduce for all $(\phi, k) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}$ the auxiliary operator

$$\Lambda(\phi, k) : C^{1,\alpha}(\partial\Omega) \rightarrow C^{1,\alpha}(\partial\Omega)$$

defined by setting

$$\Lambda(\phi, k)[\theta] \equiv -\frac{1}{2}\theta + W[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi + (1 - i\text{Re}k)V[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$$

for all $\theta \in C^{1,\alpha}(\partial\Omega)$. Then, we can rewrite the integral equation (9) as

$$\Lambda(\phi, k)[\theta] = g. \quad (10)$$

We plan to show that (10) has a unique solution $\theta[\phi, k, g]$ that depends analytically on (ϕ, k, g) . To do so, we will show that the map that takes (ϕ, k) to $\Lambda(\phi, k)$ is real analytic and invertible and then we will exploit the real analyticity of the inversion map and the formula

$$\theta[\phi, k, g] = \Lambda(\phi, k)^{(-1)}[g].$$

We begin by proving that $(\phi, k) \mapsto \Lambda(\phi, k)$ is real analytic from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}$ to the space

$$\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$$

of linear bounded operators from $C^{1,\alpha}(\partial\Omega)$ to itself equipped, as usual, with the operator norm.

Proposition 3.2. *Let α, Ω be as in (1). Then the map that takes $(\phi, k) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}$ to $\Lambda(\phi, k) \in \mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$ is real analytic.*

Proof. By Lemma 2.5 the maps

$$(\phi, k, \theta) \mapsto V[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$$

from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ and

$$(\phi, k, \theta) \mapsto W[\partial\mathbb{I}[\phi], k, \theta \circ \phi^{(-1)}] \circ \phi$$

from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ are real analytic. Moreover, the map

$$k \mapsto (1 - i\operatorname{Re}k),$$

from \mathbb{C} to itself is real analytic. We deduce that the map

$$\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega) \ni (\phi, k, \theta) \mapsto \tilde{\Lambda}(\phi, k, \theta) \equiv \Lambda(\phi, k)[\theta] \in C^{1,\alpha}(\partial\Omega)$$

is real analytic. Since $\tilde{\Lambda}$ is linear and continuous with respect to the variable θ , we have

$$\Lambda(\phi, k) = d_{\theta}\tilde{\Lambda}(\phi, k, \theta) \quad \forall (\phi, k, \theta) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega).$$

Since the right-hand side equals a partial Fréchet differential of an analytic map, the right-hand side is analytic. Hence Λ is analytic on $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C} \times C^{1,\alpha}(\partial\Omega)$ and, since it does not depend on θ , we conclude that it is analytic on $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}$. \square

Now we find convenient to introduce the set

$$\mathbb{C}^+ \equiv \{k \in \mathbb{C} : \operatorname{Im}k \geq 0\}$$

of complex numbers with nonnegative imaginary part. In the following proposition we see that $\Lambda(\phi, k)$ is an isomorphism for all $(\phi, k) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+$.

Proposition 3.3. *Let α, Ω be as in (1). For all $(\phi, k) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+$ the operator $\Lambda(\phi, k)$ is an isomorphism (i.e. a linear homeomorphism) from $C^{1,\alpha}(\partial\Omega)$ to itself.*

Proof. Since $\Lambda(\phi, k)$ is linear and continuous it suffices to show that it is bijective and then, by the open mapping theorem, we deduce that it is an isomorphism. The fact that $\Lambda(\phi, k)$ is a bijection follows by Theorem 2.3 and by noting that the map from $C^{1,\alpha}(\phi(\partial\Omega))$ to $C^{1,\alpha}(\partial\Omega)$ that takes μ to $\theta \equiv \mu \circ \phi$ is a bijection. \square

By Proposition 3.3 it makes sense to define the map

$$\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega) \ni (\phi, k, g) \mapsto \theta[\phi, k, g] \in C^{1,\alpha}(\partial\Omega)$$

that takes a triple (ϕ, k, g) to the unique solution $\theta[\phi, k, g]$ of equation (10). We now prove that the map above is real analytic. Since \mathbb{C}^+ is not open, we clarify that this means that the map has a real analytic continuation on an open neighborhood of every $(\phi_0, k_0, g_0) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$.

Proposition 3.4. *Let α, Ω be as in (1). Then the map from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ that takes (ϕ, k, g) to $\theta[\phi, k, g]$ is real analytic.*

Proof. Since the map that takes an invertible operator to its inverse is real analytic, Propositions 3.2 and 3.3 imply that $(\phi, k) \mapsto \Lambda(\phi, k)^{(-1)}$ is real analytic from the product $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega))$. Then, since the evaluation map from $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{1,\alpha}(\partial\Omega)) \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ is bilinear and continuous we conclude that

$$(\phi, k, g) \mapsto \theta[\phi, k, g] = \Lambda(\phi, k)^{(-1)}[g]$$

is real analytic from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$. \square

4 Analysis of the solution of problem (2) and of associated functionals

We are now ready to exploit the intermediate result of Proposition 3.4 on the solutions of the equivalent integral equation (10) to prove our main theorems. In particular, Proposition 3.1 gives a representation of the solution of problem (2) by means of layer potentials with a density that, by Proposition 3.4, depends analytically upon (ϕ, k, g) . Then we can use Proposition 3.4 to prove a series of results on the analyticity of functions related to problem (2). We start with a result on the analyticity of the solution $u[\phi, k, g]$.

Theorem 4.1. *Let α, Ω be as in (1). Let Ω' be a bounded open subset of \mathbb{R}^3 . Let $\mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha}$ be the open subset of $\mathcal{A}_{\partial\Omega}^{1, \alpha}$ consisting of the functions ϕ such that*

$$\overline{\Omega'} \subseteq \mathbb{E}[\phi].$$

Then there exists a real analytic map $U_{\Omega'}$ from $\mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega)$ to $C^2(\overline{\Omega'})$ such that

$$u[\phi, k, g]_{|\overline{\Omega'}} = U_{\Omega'}[\phi, k, g] \quad \forall (\phi, k, g) \in \mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega).$$

Proof. By the definition of $\theta[\phi, k, g]$ and by Lemma 2.4 (i), we have

$$\begin{aligned} u[\phi, k, g](x) &= - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS(k, x - \phi(s)) \theta[\phi, k, g](s) \tilde{\sigma}[\phi](s) d\sigma_s \\ &\quad + (1 - i\operatorname{Re}k) \int_{\partial\Omega} S(k, x - \phi(s)) \theta[\phi, k, g](s) \tilde{\sigma}[\phi](s) d\sigma_s \quad \forall x \in \overline{\Omega'}, \end{aligned}$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega)$. Then we define

$$\begin{aligned} U_{\Omega'}[\phi, k, g](x) &\equiv - \int_{\partial\Omega} \nu_{\mathbb{I}[\phi]} \circ \phi(s) \cdot DS(k, x - \phi(s)) \theta[\phi, k, g](s) \tilde{\sigma}[\phi](s) d\sigma_s \\ &\quad + (1 - i\operatorname{Re}k) \int_{\partial\Omega} S(k, x - \phi(s)) \theta[\phi, k, g](s) \tilde{\sigma}[\phi](s) d\sigma_s \quad \forall x \in \overline{\Omega'}, \end{aligned}$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega)$. By the properties of integral operators with real analytic kernels (cf. Lanza de Cristoforis and Musolino [23, Cor. 3.14]), by Proposition 3.4, and by Lemma 2.4, we verify that $U_{\Omega'}$ is a real analytic map from $\mathcal{A}_{\partial\Omega, \Omega'}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega)$ to $C^2(\overline{\Omega'})$. \square

Remark 4.2. *We note that in Theorem 4.1 we have chosen the target space $C^2(\overline{\Omega'})$ for the sake of simplicity. Indeed, by standard elliptic regularity theory, the solution $u[\phi, k, g]$ is real analytic in the interior of its domain. Therefore, we can easily replace the target space $C^2(\overline{\Omega'})$ with $C^j(\overline{\Omega'})$ for any $j \in \mathbb{N}$ or even with a suitable space of analytic functions.*

Next we consider the normal derivative of the solution.

Theorem 4.3. *Let α, Ω be as in (1). There exists a real analytic map $U^\#$ from $\mathcal{A}_{\partial\Omega}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega)$ to $C^{0, \alpha}(\partial\Omega)$ such that*

$$\left(\frac{\partial}{\partial \nu_{\mathbb{I}[\phi]}} u[\phi, k, g] \right) \circ \phi = U^\#[\phi, k, g] \quad \text{on } \partial\Omega \quad \forall (\phi, k, g) \in \mathcal{A}_{\partial\Omega}^{1, \alpha} \times \mathbb{C}^+ \times C^{1, \alpha}(\partial\Omega).$$

Proof. By Proposition 3.1 and by the jump formulas of the layer potentials, we have

$$\begin{aligned} \left(\frac{\partial}{\partial \nu_{\mathbb{I}[\phi]}} u[\phi, k, g] \right) \circ \phi &= \left(\frac{\partial}{\partial \nu_{\mathbb{I}[\phi]}} w^- [\partial\mathbb{I}[\phi], k, \theta[\phi, k, g] \circ \phi^{(-1)}] \right) \circ \phi \\ &\quad + (1 - i\operatorname{Re}k) \left(\frac{1}{2} \theta[\phi, k, g] + W^* [\partial\mathbb{I}[\phi], k, \theta[\phi, k, g] \circ \phi^{(-1)}] \circ \phi \right), \end{aligned}$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$. Thus, we define

$$U^\#[\phi, k, g] \equiv \left(\frac{\partial}{\partial \nu_{\mathbb{I}[\phi]}} w^- [\partial \mathbb{I}[\phi], k, \theta[\phi, k, g] \circ \phi^{(-1)}] \right) \circ \phi \\ + (1 - i \operatorname{Re} k) \left(\frac{1}{2} \theta[\phi, k, g] + W^* [\partial \mathbb{I}[\phi], k, \theta[\phi, k, g] \circ \phi^{(-1)}] \circ \phi \right),$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$. Then, by Lemma 2.5 and Proposition 3.4, we deduce the analyticity of $U^\#[\phi, k, g]$ and therefore the validity of the theorem. \square

By Theorem 4.3, we deduce the validity of the following corollary on the regularity of the Dirichlet-to-Neumann operator (the proof can be effected by a standard argument of calculus in Banach spaces, see for example the last part of the proof of Proposition 3.2).

Corollary 4.4. *Let α, Ω be as in (1). There exists a real analytic map $\tilde{\mathcal{D}}$ from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega), C^{0,\alpha}(\partial\Omega))$ such that*

$$\mathcal{D}_{(\phi, k)} = \tilde{\mathcal{D}}[\phi, k] \quad \forall (\phi, k) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+.$$

Finally we consider the dependence of the far field pattern $u_\infty[\phi, k, g]$ (cf. (4)) with respect to the perturbation of (ϕ, k, g) .

Theorem 4.5. *Let α, Ω be as in (1). There exists a real analytic map U_∞ from $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ to $C^2(\partial\mathbb{B}_3(0, 1))$ such that*

$$u_\infty[\phi, k, g] = U_\infty[\phi, k, g] \quad \text{on } \partial\mathbb{B}_3(0, 1) \quad \forall (\phi, k, g) \in \mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega). \quad (11)$$

Proof. Let $\rho > 0$ and let $\mathcal{A}_{\partial\Omega, \rho}^{1,\alpha}$ be the open subset of $\mathcal{A}_{\partial\Omega}^{1,\alpha}$ of the functions ϕ such that $\overline{\mathbb{I}[\phi]} \subseteq \mathbb{B}_3(0, \rho)$. By Theorem 4.1 with

$$\Omega' \equiv \mathbb{B}_3(0, 2\rho) \setminus \overline{\mathbb{B}_3(0, \rho)}$$

(notice that $\mathcal{A}_{\partial\Omega, \rho}^{1,\alpha}$ is contained in the set $\mathcal{A}_{\partial\Omega, \Omega'}^{1,\alpha}$ of Theorem 4.1), by the continuity of the trace operator, and by standard calculus in Banach spaces, we deduce that there exist two real analytic maps V_1 and V_2 from $\mathcal{A}_{\partial\Omega, \rho}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ to $C^0(\partial\mathbb{B}_3(0, \rho))$ such that

$$u[\phi, k, g]|_{\partial\mathbb{B}_3(0, \rho)} = V_1[\phi, k, g], \quad \frac{\partial u[\phi, k, g]}{\partial \nu_{\mathbb{B}_3(0, \rho)}} = V_2[\phi, k, g], \quad (12)$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega, \rho}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$. Then, having in mind the expression (4) of the far field pattern, we set

$$U_\rho[\phi, k, g](x) \equiv \frac{1}{4\pi} \int_{\partial\mathbb{B}_3(0, \rho)} \left(V_1[\phi, k, g](y) \frac{\partial}{\partial \nu_{\mathbb{B}_3(0, \rho)}} e^{-ikx \cdot y} - e^{-ikx \cdot y} V_2[\phi, k, g](y) \right) d\sigma_y \\ \forall x \in \partial\mathbb{B}_3(0, 1),$$

for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega, \rho}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$. By the properties of integral operators with real analytic kernels (cf. Lanza de Cristoforis and Musolino [23]), we deduce that U_ρ is a real analytic map from $\mathcal{A}_{\partial\Omega, \rho}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ to $C^2(\partial\mathbb{B}_3(0, 1))$. Moreover, by equalities (12) and by (4) (that does not depend on the specific choice of R), we have

$$u_\infty[\phi, k, g] = U_\rho[\phi, k, g] \quad \text{on } \partial\mathbb{B}_3(0, 1) \quad \forall (\phi, k, g) \in \mathcal{A}_{\partial\Omega, \rho}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega). \quad (13)$$

Now, let $0 < \rho_1 \leq \rho_2 \leq \dots$ be an increasing sequence of positive real numbers with $\lim_{j \rightarrow +\infty} \rho_j = +\infty$. Then $\mathcal{A}_{\partial\Omega, \rho_1}^{1,\alpha} \subseteq \mathcal{A}_{\partial\Omega, \rho_2}^{1,\alpha} \subseteq \dots$ is an increasing sequence of sets and we have $\cup_{j \geq 1} \mathcal{A}_{\partial\Omega, \rho_j}^{1,\alpha} = \mathcal{A}_{\partial\Omega}^{1,\alpha}$. By equality (13) we see that $U_{\rho_j}[\phi, k, g] = U_{\rho_{j+1}}[\phi, k, g]$ for all $(\phi, k, g) \in \mathcal{A}_{\partial\Omega, \rho_j}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ and all $j \geq 1$. So, we are allowed to “glue together” the maps U_{ρ_j} and define a map U_∞ on the whole of $\mathcal{A}_{\partial\Omega}^{1,\alpha} \times \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$ by taking

$$U_\infty[\phi, k, g] \equiv U_{\rho_j}[\phi, k, g] \quad \text{if } \phi \in \mathcal{A}_{\partial\Omega, \rho_j}^{1,\alpha} \text{ for some } j \geq 1 \quad (14)$$

and $(k, g) \in \mathbb{C}^+ \times C^{1,\alpha}(\partial\Omega)$. By (13) and (14) we see that (11) holds true and, in addition, U_∞ inherits the real analyticity of the maps U_{ρ_j} . \square

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References

- [1] A. Charalambopoulos, On the Fréchet differentiability of boundary integral operators in the inverse elastic scattering problem. *Inverse Problems* 11 (1995), 1137–1161.
- [2] A. Cohen, C. Schwab, and J. Zech, Shape holomorphy of the stationary Navier-Stokes equations. *SIAM J. Math. Anal.* 50 (2018), no. 2, 1720–1752.
- [3] D. Colton and R. Kress, Integral equation methods in scattering theory. Reprint of the 1983 original. *Classics in Applied Mathematics*, 72. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.
- [4] D. Colton and R. Kress, Inverse acoustic and electromagnetic scattering theory. *Applied Mathematical Sciences* 93, Springer, Cham, 2019.
- [5] M. Costabel and F. Le Louër, Shape derivatives of boundary integral operators in electromagnetic scattering. Part I: Shape differentiability of pseudo-homogeneous boundary integral operators. *Integral Equations Oper. Theory* 72 (2012), 509–535.
- [6] M. Costabel and F. Le Louër, Shape derivatives of boundary integral operators in electromagnetic scattering. Part II: Application to scattering by a homogeneous dielectric obstacle. *Integral Equations Oper. Theory* 73 (2012), 17–48.
- [7] M. Dalla Riva, Potential theoretic methods for the analysis of singularly perturbed problems in linearized elasticity, PhD Thesis, University of Padova, 2008.
- [8] M. Dalla Riva and M. Lanza de Cristoforis, A perturbation result for the layer potentials of general second order differential operators with constant coefficients. *J. Appl. Funct. Anal.* 5 (2010), no. 1, 10–30.
- [9] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, Singularly Perturbed Boundary Value Problems: A Functional Analytic Approach. Springer Nature, Cham, 2021.
- [10] M. Dalla Riva, P. Luzzini, P. Musolino, and R. Pukhtaievych, Dependence of effective properties upon regular perturbations. In I. Andrianov, S. Gluzman, V. Mityushev, Editors, *Mechanics and Physics of Structured Media. Asymptotic and Integral Equations Methods of Leonid Fil'shtinsky*. Academic Press, Elsevier, London, 2022, pp. 271–301.
- [11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
- [12] G. B. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.
- [13] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, 2nd Edition, Vol. 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1983.
- [14] H. Haddar and R. Kress, On the Fréchet derivative for obstacle scattering with an impedance boundary condition. *SIAM J. Appl. Math.* 65 (2004), no. 1, 194–208.

- [15] F. Henríquez and C. Schwab, Shape holomorphy of the Calderón projector for the Laplacian in \mathbb{R}^2 . *Integral Equations Operator Theory* 93 (2021), no. 4, Paper No. 43, 40 pp.
- [16] A. Henrot and M. Pierre, *Variation et optimisation de formes*, Vol. 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*, Springer, Berlin, 2005.
- [17] F. Hettlich, Fréchet derivatives in inverse obstacle scattering. *Inverse Problems* 11 (1995), no. 2, 371–382.
- [18] C. Jerez-Hanckes, C. Schwab, and J. Zech, Electromagnetic wave scattering by random surfaces: shape holomorphy. *Math. Models Methods Appl. Sci.* 27 (2017), no. 12, 2229–2259.
- [19] A. Kirsch, The domain derivative and two applications in inverse scattering theory. *Inverse Problems* 9 (1993), no. 1, 81–96.
- [20] A. Kirsch, *An introduction to the mathematical theory of inverse problems*. Applied Mathematical Sciences, 120. Springer, Cham, 2021.
- [21] R. Kress and L. Päivärinta, On the far field in obstacle scattering. *SIAM J. Appl. Math.* 59 (1999), no. 4, 1413–1426.
- [22] M. Lanza de Cristoforis, Perturbation problems in potential theory, a functional analytic approach. *J. Appl. Funct. Anal.* 2 (2007), no. 3, 197–222.
- [23] M. Lanza de Cristoforis and P. Musolino, A real analyticity result for a nonlinear integral operator, *J. Int. Equ. Appl.* 25 (2013), no. 1, 21–46.
- [24] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density. *J. Integral Equations Appl.* 16 (2004), no. 2, 137–174.
- [25] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials for the Helmholtz equation upon perturbation of the support and of the density, in: *Analytic methods of analysis and differential equations: AMADE 2006*, Camb. Sci. Publ., Cambridge, 2008, pp. 193–220.
- [26] M. Lanza de Cristoforis, Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. A functional analytic approach. *Rev. Mat. Complut.* 25 (2012), no. 2, 369–412.
- [27] P. Luzzini and P. Musolino, Perturbation analysis of the effective conductivity of a periodic composite. *Netw. Heterog. Media*, 15 (2020), no. 4, 581–603.
- [28] P. Luzzini, P. Musolino, and R. Pukhtaievych, Shape analysis of the longitudinal flow along a periodic array of cylinders. *J. Math. Anal. Appl.*, 477 (2019), no. 2, 1369–1395.
- [29] F. Le Louër, On the Fréchet derivative in elastic obstacle scattering. *SIAM J. Appl. Math.* 72 (2012), no. 5, 1493–1507.
- [30] A. A. Novotny and J. Sokółowski, *Topological derivatives in shape optimization, Interaction of Mechanics and Mathematics*, Springer, Heidelberg, 2013.
- [31] R. Potthast, Fréchet differentiability of boundary integral operators in inverse acoustic scattering. *Inverse Problems* 10 (1994), no. 2, 431–447.
- [32] R. Potthast, Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain. *J. Inverse Ill-Posed Probl.* 4 (1996), no. 1, 67–84.
- [33] R. Potthast, Domain derivatives in electromagnetic scattering. *Math. Methods Appl. Sci.* 19 (1996), no. 15, 1157–1175.

- [34] J. Sokółowski and J.-P. Zolésio, Introduction to shape optimization. Shape sensitivity analysis, Vol. 16 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1992.