Reasoning about Proportional Lumpability

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Abstract. In this paper we reason about the notion of *proportional lumpability*, that generalizes the original definition of lumpability to cope with the state space explosion problem inherent to the computation of the performance indices of large stochastic models. Lumpability is based on a state aggregation technique and applies to Markov chains exhibiting some structural regularity.
Proportional lumpability formalizes the idea that the transition rates of a Markov chain can be altered by some factors in such a way that the new resulting Markov chain is lumpable. It allows one to derive exact performance indices for the original process.
We prove that the problem of computing the coarsest proportional lumpability which refines a given initial partition is well-defined, i.e., it has always a unique solution. Moreover, we introduce a polynomial time algorithm for solving the problem. This provides us further insights on both the notion of proportional lumpability and on generalizations of partition refinement techniques.

Keywords: Markov chains · Lumpability · Algorithms.

1 Introduction

Markov chains constitute the basic underlying semantics model of a plethora of modelling formalism for reliability analysis and performance evaluation of complex systems, such as Stochastic Petri nets [22], Stochastic Automata Networks [24], queuing networks [3] and Markovian process algebras [10, 11].

Although the use of high-level specification formalisms highly simplifies the design of compositional/hierarchical quantitative models, the stochastic process underlying even a very simple model may have a large number of states that makes its analysis a difficult, sometimes impossible, task. In order to study models with a very large state space without resorting to approximation or simulation techniques we can attempt to reduce the state space of the underlying Markov chain by aggregating states with equivalent behaviours (according to a notion of equivalence that captures our concept of behaviour). An interesting class of these aggregation methods that can be decided by the structural analysis of the original Markov chain is known as *lumping*. In the literature, several notions of lumping have been introduced: strong and weak lumping [15], exact lumping [25], and strict lumping [4]. The lumpability method allows one to efficiently compute the exact values of the performance indices when the model is actually lumpable. However, it is well known that not all Markov chains are lumpable. Indeed, Markov chains arising in real-life applications are, in general, not lumpable. To cope with this problem, in [7] the notion of *quasi-lumpability* has been introduced. The idea is that a quasi-lumpable Markov chain can be altered in such a way that the new resulting Markov chain is lumpable and steady state probability bounding methods [5, 7, 8] can be applied to the new lumpable Markov chain in order to obtain bounds on the performance indices of the original model.

In [19], the notion of *proportional lumpability* has been introduced. It extends the original definition of lumpability but, differently than the general definition of quasi-lumpability, it allows one to derive exact performance indices for the original process. In [20] we extended the work presented in [19] by comparing the notion of proportional lumpability with other definitions of lumping such as weak lumpability [15, 17] and the notion of exact lumpability for ordinary differential equations (ODEs) [16, 18].

The definition of proportional lumpability requires to find a function that assigns a positive coefficient to each state of the system. Being the set of all possible such functions infinite, the existence of an efficient algorithmic technique to either check or compute proportional lumpability is not an immediate consequence of the definition.

In this paper we study the properties of proportional lumpability and present two alternative characterizations of it. The first characterization has been proved in [20] and allows one to efficiently verify whether a partition of the state space of a Markov chain is induced by an equivalence relation which is a proportional lumpability. The second charaterization is a novel contribution and it is exploited to design a polynomial time algorithm to compute the coarsest proportional lumpability of a given Markov chain. Indeed, in the case of the classical notion of strong lumpability, partition refinement algorithms are at the basis of the efficient computation of the coarsest lumpability included in a given initial partition. In the same spirit, we prove that the problem of computing the coarsest proportional lumpability which refines a given initial partition is well-defined, i.e., it has always a unique solution. Moreover, we introduce a polynomial time algorithm for solving the problem. This provides us further insights on both the notion of proportional lumpability and on generalizations of partition refinement techniques.

Structure of the paper. The paper is structured as follows: In Section 2 we review the theoretical background on continuous-time Markov chains and recall the concept of strong lumpability. The notion of proportional lumpability is introduced in Section 3 and one novel characterization of it is proved. In Section 4 an algorithm for proportional lumpability is presented and both its correctness and its complexity are proved. Section 5 concludes the paper.

2 Background

In this section we rapidly review the fundamentals of continuous-time Markov chains and the concept of lumpability.

Continuous-Time Markov Chains. A Continuous-Time Markov Chain (CTMC) is a stochastic process $X(t)$ for $t \in \mathbb{R}^+$ taking values into a discrete state space $\mathcal{S}$ such that the *Markov property* holds, i.e., the conditional (on both past and present states) probability distribution of its future behaviour is independent of its past evolution until the present state:

$$Prob(X(t_{n+1}) = s_{n+1} \mid X(t_1) = s_1, X(t_2) = s_2, \ldots, X(t_n) = s_n) = Prob(X(t_{n+1}) = s_{n+1} \mid X(t_n) = s_n).$$

A stochastic process $X(t)$ is said to be *stationary* if the collection of random variables $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as the collection $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}^+$. A CTMC $X(t)$ is said to be *time-homogeneous* if the conditional probability $Prob(X(t+\tau) = s \mid X(t) = s')$ does not depend upon t, and is *irreducible* if every state in $\mathcal{S}$ can be reached from every other state. A state in a Markov process is called *recurrent* if the probability that the process will eventually return to the same state is one. A recurrent state is called *positive-recurrent* if the expected return time is finite. A CTMC is *ergodic* if it is irreducible and all its states are positive-recurrent. In the case of finite Markov chains, irreducibility is sufficient for ergodicity. Henceforth, we consider ergodic CTMCs.

An ergodic CTMC possesses an *equilibrium* (or *steady-state*) *distribution*, that is the *unique* collection of positive real numbers $\pi(s)$ with $s \in \mathcal{S}$ such that

$$\lim_{t\to\infty} Prob(X(t) = s \mid X(0) = s') = \pi(s)\,.$$

Notice that the above equation for $\pi(s)$ is independent of s'. We denote by $q(s, s')$ the transition rate out of state s to state s', with $s \neq s'$, and by $q(s)$ the sum of all transition rates out of state s to any other state in the chain. A state s for which $q(s) = \infty$ is called an instantaneous state since when entered it is instantaneously left. Whereas such states are theoretically possible, we shall assume throughout that $0 < q(s) < \infty$ for each state s. The infinitesimal generator matrix $\mathbf{Q}$ of a CTMC $X(t)$ with state space $\mathcal{S}$ is the $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose off-diagonal elements are the $q(s, s')$'s and whose diagonal elements are the negative sum of the extra diagonal elements of each row, i.e., $q(s, s) = -\sum_{s' \in \mathcal{S},\ s' \neq s} q(s, s')$. For the sake of simplicity, we use $q(s, s')$ to denote the components of matrix $\mathbf{Q}$. For $s \in \mathcal{S}$ and $S \subseteq \mathcal{S}$ we write $q(s, S)$ to denote $\sum_{s' \in S} q(s, s')$.

Any non-trivial vector of positive real numbers $\boldsymbol{\mu}$ satisfying the system of global balance equations (GBEs) $\boldsymbol{\mu}\mathbf{Q} = \mathbf{0}$ is called *invariant measure* of the CTMC. For an irreducible CTMC $X(t)$, if $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are two invariant measures of $X(t)$, then there exists a constant $k > 0$ such that $\boldsymbol{\mu}_1 = k\boldsymbol{\mu}_2$. If the CTMC is ergodic, then there exists a unique invariant measure $\boldsymbol{\pi}$ whose components sum to unity, i.e., $\sum_{s\in\mathcal{S}} \pi(s) = 1$. In this case $\boldsymbol{\pi}$ is the *equilibrium* or *steady-state distribution* of the CTMC.

Strong Lumpability. In the context of performance and reliability analysis, the notion of *lumpability* provides a model aggregation technique that can be used for generating a Markov chain that is smaller than the original one but allows one to determine exact results for the original process.

The concept of lumpability can be formalized in terms of equivalence relations over the state space of the Markov chain. Any such equivalence induces a *partition* on the state space of the Markov chain and aggregation is achieved by clustering equivalent states into macro-states, thus reducing the overall state space. If the partition can be shown to satisfy the so-called *strong* lumpability condition [15, 2], then the equilibrium solution of the aggregated process may be used to derive an exact solution of the original one.

The notion of strong lumpability has been introduced in [15] and further studied in [1, 4, 21, 26].

Definition 1 (Strong lumpability). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. We say that $X(t)$ is* strongly lumpable *with respect to $\sim$ (resp., $\sim$ is a* strong lumpability *for $X(t)$) if $\sim$ induces a partition on the state space of $X(t)$ such that for any equivalence class $S_i, S_j \in \mathcal{S}/\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$,*

$$q(s, S_j) = q(s', S_j)\,.$$

Thus, an equivalence relation over the state space of a Markov process is a strong lumpability if it induces a partition into equivalence classes such that for any two states within an equivalence class their aggregated transition rates to any other class are the same. Notice that every Markov process is strongly lumpable with respect to the identity relation, and also with respect to the trivial relation having only one equivalence class.

In [15] the authors prove that for an equivalence relation $\sim$ over the state space of a Markov process $X(t)$, the aggregated process is a Markov process for every initial distribution if, and only if, $\sim$ is a strong lumpability for $X(t)$. Moreover, the transition rate between two aggregated states $S_i, S_j \in \mathcal{S}/\sim$ is equal to $q(s, S_j)$ for any $s \in S_i$.

Proposition 1 (Aggregated process for strong lumpability). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Let $\sim$ be a strong lumpability for $X(t)$ and $\widetilde{X}(t)$ be the aggregated process with state space $\mathcal{S}/\sim$ and infinitesimal generator $\widetilde{\mathbf{Q}}$ defined by: for any equivalence class $S_i, S_j \in \mathcal{S}/\sim$,*

$$\widetilde{q}(S_i, S_j) = q(s, S_j)$$

for any $s \in S_i$. Then the equilibrium distribution $\widetilde{\boldsymbol{\pi}}$ of $\widetilde{X}(t)$ is such that for any equivalence class $S \in \mathcal{S}/\sim$,

$$\widetilde{\pi}(S) = \sum_{s \in S} \pi(s)\,.$$

3 Proportional Lumpability

The notion of *proportional lumpability* has been introduced in [19]. As the notion of *quasi-lumpability* [7], also called *near-lumpability* in [4], proportional lumpability extends the original definition of strong lumpability but, differently from the general definition of quasi-lumpability, it allows one to derive an exact solution of the original process.

Definition 2 (Proportional lumpability). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. We say that $X(t)$ is* proportionally lumpable *with respect to $\sim$ (resp., $\sim$ is a* proportional lumpability *for $X(t)$) if there exists a function κ from $\mathcal{S}$ to $\mathbb{R}^+$ such that $\sim$ induces a partition on the state space of $X(t)$ satisfying the property that for any equivalence classes $S_i, S_j \in \mathcal{S}/\!\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$,*

$$\frac{q(s, S_j)}{\kappa(s)} = \frac{q(s', S_j)}{\kappa(s')}.$$

*We say that $X(t)$ is κ-*proportionally lumpable *with respect to $\sim$ (resp., $\sim$ is a κ-*proportional lumpability *for $X(t)$) if $X(t)$ is* proportionally lumpable *with respect to $\sim$ and function κ.*

The following theorem [19] proves that proportional lumpability allows one to compute an exact solution for the original model.

Theorem 1 (Aggregated process for proportional lumpability). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Let κ be a function from $\mathcal{S}$ to $\mathbb{R}^+$, $\sim$ be a κ-proportional lumpability for $X(t)$ and $\widetilde{X}(t)$ be the aggregated process with state space $\mathcal{S}/\!\sim$ and infinitesimal generator $\widetilde{\mathbf{Q}}$ defined by: for any equivalence classes $S_i, S_j \in \mathcal{S}/\!\sim$*

$$\widetilde{q}(S_i, S_j) = \frac{q(s, S_j)}{\kappa(s)}$$

for any $s \in S_i$. Then the invariant measure $\widetilde{\boldsymbol{\mu}}$ of $\widetilde{X}(t)$ is such that for any equivalence class $S \in \mathcal{S}/\!\sim$,

$$\widetilde{\mu}(S) = \sum_{s \in S} \pi(s)\kappa(s)\,. \tag{1}$$

The next Definition 3 introduces a way to perturb a proportionally lumpable CTMC in order to obtain a strongly lumpable one. In contrast with previous perturbation-based approaches, Theorem 2 gives a way to compute the stationary probabilities of a proportionally lumpable chain given those of the perturbed lumpable one. The proof of Theorem 2 is given in [19].

Definition 3 (Perturbed Markov chains). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, and infinitesimal generator $\mathbf{Q}$. Let κ be a function from $\mathcal{S}$ to $\mathbb{R}^+$. We*

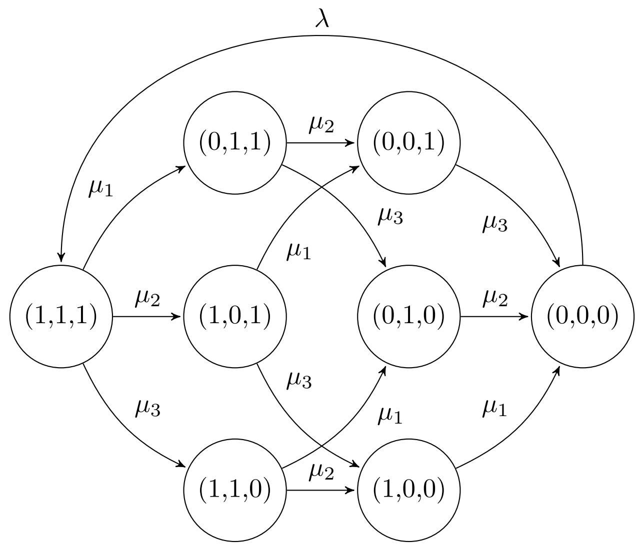

Fig. 1: CTMC representing the reliability of a system with 3 components.

say that a CTMC $X'(t)$ with infinitesimal generator $\mathbf{Q'}$ is a perturbation of $X(t)$ with respect to κ if $X'(t)$ is obtained from $X(t)$ by perturbing its rates such that for all $s, s' \in \mathcal{S}$ with $s \neq s'$,

$$q'(s, s') = \frac{q(s, s')}{\kappa(s)} .$$

Theorem 2 (Equilibrium distribution for proportional lumpability). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, infinitesimal generator $\mathbf{Q}$ and equilibrium distribution $\boldsymbol{\pi}$. Let κ be a function from $\mathcal{S}$ to $\mathbb{R}^+$. Then, for any perturbation $X'(t)$ of the original chain $X(t)$ with respect to κ according to Definition 3 with infinitesimal generator $\mathbf{Q'}$ and equilibrium distribution $\boldsymbol{\pi'}$, the equilibrium distribution $\boldsymbol{\pi}$ of $X(t)$ satisfies the following property: let $K = \sum_{s \in \mathcal{S}} \pi'(s)/\kappa(s)$ then, for all $s \in \mathcal{S}$*

$$\pi(s) = \frac{\pi'(s)}{K \ \kappa(s)} .$$

Example 1. Consider the standard reliability problem for a system consisting of N components. The time to failure of each component $i \in \{1, \ldots, N\}$ is exponentially distributed with rate μ_i and it is independent of the state of the other components. This type of system has been studied in several works, like, e.g., [9, 12–14, 27]. As in [14], we assume that when the system fails it is restored to a new "good" state and the time it takes for this restoration is exponentially

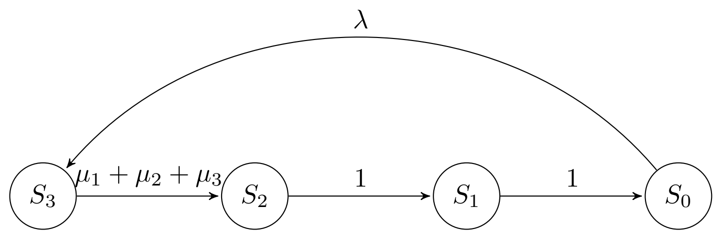

Fig. 2: Aggregated CTMC representing the reliability of the system in Fig.1,

distributed with rate λ. At any point in time, the state of the system can be represented as a boolean vector of size N, $\bar{x} = (x_1, \ldots, x_N)$, where $x_i = 1$ if the i-th component of the system is working, otherwise $x_i = 0$. Hence the set of all possible states is $\mathcal{S} = \{0,1\}^N$. Under these conditions, the time evolution of the state of the system can be described by a continuous time Markov chain. The Markov process corresponding to a system with 3 components, i.e., $N = 3$, is depicted in Figure 1. This system is proportionally lumpable with respect to the partition: $S_n = \{\bar{x} \in \mathcal{S} : \sum x_i = n\}$ with $n \in \{0, 1, 2, 3\}$, i.e.,

$S_0 = \{(0, 0, 0)\}$
$S_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
$S_2 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$
$S_3 = \{(1, 1, 1)\}$
and the function κ such that for each state $s \in S_1 \cup S_2$, $\kappa(s) = q(s)$, while for $s \in S_0 \cup S_3$, $\kappa(s) = 1$.
Thus, we can analyze the aggregated Markov chain represented in Figure 2 and, by Theorems 1 and 2 we can compute the exact solution to the original model.

3.1 Alternative characterizations of proportional lumpability

We present two alternative characterizations of proportional lumpability. The first characterization has been proved in [20] and allows one to efficiently verify whether a partition of the state space of a Markov chain is induced by a proportional lumpability. The second charaterization is a novel controbution and is exploited in the next section to design a polynomial time algorithm to compute the coarsest proportional lumpability of a given Markov chain.

First, for a given equivalence relation $\sim$ over the state space of a CTMC, we denote by $q_\sim(s)$ the sum of all transition rates from the state s to any state t such that $s \not\sim t$, i.e., for all $s \in \mathcal{S}$,

$$q_\sim(s) = \sum_{t \not\sim s} q(s, t).$$

The following theorem shows that proportional lumpability can be characterized in terms of $q_\sim(s)$ by replacing $\kappa(s)$ with $q_\sim(s)$ in the original definition.

Theorem 3 (Characterization 1 of proportional lumpability [20]). *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. The relation $\sim$ is a proportional lumpability for $X(t)$ if and only if for any equivalence classes $S_i, S_j \in \mathcal{S}/\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$,*

1. *$q_\sim(s) \neq 0$ if and only if $q_\sim(s') \neq 0$*
2. *if $q_\sim(s) \neq 0$ then*
$$\frac{q(s,S_j)}{q_\sim(s)} = \frac{q(s',S_j)}{q_\sim(s')}.$$

While the above characterization can be exploited to efficiently check whether a given relation is a proportional lumpability, it is not immediate to guess how to use it within an algorithm for the computation of the proportional lumpability that refines a given initial relation. As we will see in Section 4, if the relation changes during the computation, $q_\sim$ also changes. So it could be the case that one of the equalities of item 2 which is not true at the current step will become true later. On the other hand, the following characterization of proportional lumpability is easier to use to define a partition refinement algorithm for proportional lumpability.

Theorem 4 (Characterization 2 of proportional lumpability). *Let $X(t)$ be an ergodic CTMC with state space $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. The relation $\sim$ is a proportional lumpability for $X(t)$ if and only if for any equivalence classes $S_i, S_j, S_k \in \mathcal{S}/\sim$ with $S_i \neq S_j$, $S_i \neq S_k$, and $s, s' \in S_i$,*

1. *$q(s,S_k) \neq 0$ if and only if $q(s',S_k) \neq 0$ and*
2. *if $q(s,S_k) \neq 0$, then*
$$\frac{q(s,S_j)}{q(s,S_k)} = \frac{q(s',S_j)}{q(s',S_k)}$$

Proof. $\Rightarrow$) Suppose that $\sim$ is a κ-proportional lumpability for a function $\kappa : \mathcal{S} \to \mathbb{R}^+$, i.e., for any equivalence classes $S_i, S_j \in \mathcal{S}/\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$,

$$\frac{q(s,S_j)}{\kappa(s)} = \frac{q(s',S_j)}{\kappa(s')}\,. \tag{2}$$

Item 1. follows by the definition of proportional lumpability. Moreover, if $q(s,S_k) \neq 0$ we have that also $q(s',S_k) \neq 0$ and

$$\frac{q(s,S_j)}{q(s,S_k)} = \frac{q(s,S_j)}{\kappa(s)}\frac{\kappa(s)}{q(s,S_k)} = \frac{q(s',S_j)}{\kappa(s')}\frac{\kappa(s')}{q(s',S_k)} = \frac{q(s',S_j)}{q(s',S_k)}\,.$$

$\Leftarrow$) Suppose that $\sim$ is an equivalence relation such that for any equivalence classes $S_i, S_j, S_k \in \mathcal{S}/\sim$ with $S_i \neq S_j$, $S_i \neq S_k$, and $s, s' \in S_i$,

1. $q(s,S_k) \neq 0$ if and only if $q(s',S_k) \neq 0$ and
2. if $q(s,S_k) \neq 0$, then
$$\frac{q(s,S_j)}{q(s,S_k)} = \frac{q(s',S_j)}{q(s',S_k)}$$

For each $S \in \mathcal{S}/\sim$ such that there exists $s \in S$ with $q_\sim(s) \neq 0$ we choose a class $B_S \neq S$ of $\mathcal{S}/\sim$ such that $q(s, B_S) \neq 0$. We define $\kappa : \mathcal{S} \longrightarrow \mathbb{R}^+$ as follows:

- if $q_\sim(s) = 0$, then $\kappa(s) = 1$ otherwise
- if $s \in S$, then $\kappa(s) = q(s, B_S)$.

We prove that $\sim$ is a κ-proportional lumpability. Let $S_i, S_j \in \mathcal{S}/\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$

$$\frac{q(s,S_j)}{\kappa(s)} = \frac{q(s,S_j)}{q(s,B_{S_i})} = \frac{q(s',S_j)}{q(s',B_{S_i})} = \frac{q(s',S_j)}{\kappa(s')} .$$

□

3.2 Comparison with lumpability of the embedded Markov chain

We compare proportional lumpability with lumpability of the embedded Markov chain [20]. The following Examples 2 and 3 are novel.

One standard approach for computing the stationary probability distribution of an ergodic continuous-time Markov chain $X(t)$ is by analyzing its embedded Markov chain $X^E(t)$. Strictly speaking, the embedded Markov chain is a regular discrete-time Markov chain (DTMC), sometimes referred to as its jump process. Given $X(t)$ with state space $\mathcal{S}$, each element of the one-step transition probability matrix of the corresponding embedded Markov chain is denoted by $p(s, s')$, and represents the conditional probability of the transition from state s into state s', defined by:

$$p(s,s') = \frac{q(s,s')}{q(s)} \qquad \text{for } s \neq s'$$

while $p(s, s) = 0$. Assuming that $X^E(t)$ is aperiodic, let $\boldsymbol{\pi}^*$ be its steady-state distribution. Then, one may derive the distribution $\boldsymbol{\pi}$ of $X(t)$ as follows: let $W = \sum_{s\in\mathcal{S}} \pi^*(s)/q(s)$, then

$$\pi(s) = \frac{\pi^*(s)}{Wq(s)} .$$

Notice that, in general, our definition of $q_\sim(s)$ is different from that of $q(s)$, hence the fact that $X(t)$ is proportionally lumpable does not imply that the corresponding embedded Markov chain $X^E(t)$ is lumpable.

On the other hand, if $X^E(t)$ is lumpable then $X(t)$ is proportionally lumpable with respect to function κ from $\mathcal{S}$ to $\mathbb{R}^+$ such that $\kappa(s) = q(s)$ for all $s \in \mathcal{S}$. In conclusion, we can say that if $X(t)$ has a strongly lumpable embedded process, then it is also proportional lumpable but the opposite does not hold.

Example 2. Consider again the problem of reliability for a system consisting of N components. Suppose that we are now interested in the number of components working at any point time. Thus the state space $\mathcal{S} = \{S_i : 0 \leq i \leq N\}$

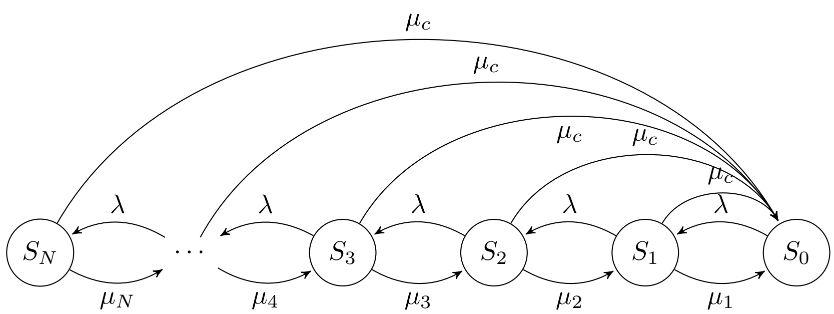

Fig. 3: CTMC for system repair model with common cause failures.

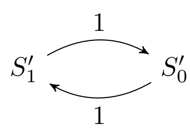

Fig. 4: Aggregated CTMC for system repair model with common cause failures.

where S_i denotes the state of the system where i components are working. We assume that in each state S_i, the time to failure of a component is exponentially distributed with rate μ_i. Each componet can be restored with rate λ. In some cases, the system fails due to the simultaneous failure of components due to common factors. Common cause failures may arise due to the failure of common power supply, environmental conditions (e.g., earthquake, flood, humid-ity, etc.), common maintenance problems, etc. Simultaneous failure due to common cause may occur with failure rate μ_c. The state transition diagram for system repair model is depicted in Figure 3.

This model is proportionally lumpable with respect to the relation $\sim$ over $\mathcal{S}$ given by the reflexive, symmetric and transitive closure of $\{(S_i, S_j) : 1 \leq i, j \leq N\}$, and the function κ such that $\kappa(S_i) = q_\sim(S_i)$ for $i \in \{0, \ldots, N\}$. This relation induces two equivalence classes, $C_0 = \{S_0\}$ and $C_1 = \{S_1, \ldots, S_N\}$, and the model in Figure 3 is proportionally lumpable to the one depicted in Figure 4.

In this case the model in Figure 3 has not a strongly lumpable embedded process due to the fact that $q(S_i) \neq q_\sim(S_i)$ for each $i \in \{0, \ldots, N\}$.

Example 3. Consider the model described in Example 1. We showed that the CTMC depicted in Figure 1 is proportionally lumpable. It is easy to see that this model has also a strongly lumpable embedded process. Indeed, this trivially follows by Theorem 3 and the fact that $q(s) = q_\sim(s)$ for all $s \in \mathcal{S}$ where $\sim$ is the relation inducing the partition $S_n = \{\bar{x} \in \mathcal{S} : \sum x_i = n\}$ with $n \in \{0, 1, 2, 3\}$.

4 Computing Proportional Lumpability

In this section we consider the maximum proportional lumpability problem.

Definition 4 (Maximum Proportional Lumpability Problem). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and let $\mathcal{R}$ be an equivalence relation over $\mathcal{S}$. The* maximum proportional lumpability problem *over $X(t)$ and $\mathcal{R}$ consists in finding the largest equivalence relation $\sim$ such that $\sim \subseteq \mathcal{R}$ and $\sim$ is a proportional lumpability for $X(t)$.*

We have to prove that the maximum proportional lumpability problem is well-defined, i.e., it always admits a unique solution. To this aim, it is convenient to reason in terms of partitions instead of equivalence relations. As a matter of fact, each equivalence relation $\mathcal{R}$ over $\mathcal{S}$ is naturally associated to the partition $\mathcal{S}/\mathcal{R}$ whose blocks correspond to the maximal sets of $\mathcal{R}$-equivalent elements, and vice-versa. This allows us to talk about proportional lumpabilities as both equivalence relations and partitions. In particular, a partition $\mathcal{P}$ is said to be a *proportional partition* when it is associated to an equivalence relation which is a proportional lumpability.

We introduce some notations and terminologies over partitions useful for providing an alternative definition of the maximum proportional lumpability problem.

Given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ over $\mathcal{S}$ we say that $\mathcal{P}_1$ is *finer* than $\mathcal{P}_2$, denoted by $P_1 \sqsubseteq P_2$, if and only if for each block B_1 of $\mathcal{P}_1$ there exists a block B_2 of $\mathcal{P}_2$ such that $B_1 \subseteq B_2$. This is equivalent to say that the blocks of $\mathcal{P}_2$ are unions of blocks of $\mathcal{P}_1$. Equivalently we say that $\mathcal{P}_2$ is coarser than $\mathcal{P}_1$ (also $\mathcal{P}_1$ refines $\mathcal{P}_2$) if $\mathcal{P}_1$ is finer than $\mathcal{P}_2$.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two equivalence relations over $\mathcal{S}$. It holds that $\mathcal{R}_1 \subseteq \mathcal{R}_2$ if and only if the partition $\mathcal{P}_1 \equiv \mathcal{S}/\mathcal{R}_1$ associated to $\mathcal{R}_1$ is finer than the partition $\mathcal{P}_2 \equiv \mathcal{S}/\mathcal{R}_2$ associated to $\mathcal{R}_2$, i.e., $\mathcal{P}_1 \sqsubseteq \mathcal{P}_2$.

Definition 5 (Maximum Proportional Partition Problem). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. The* maximum proportional partition problem *over $X(t)$ and $\mathcal{P}$ consists in finding the coarsest proportional partition $\mathcal{P}_\sim$ refining $\mathcal{P}$.*

Proposition 2 (Equivalence of the two problems). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$. Let $\mathcal{R}$ be an equivalence relation over $\mathcal{S}$ and $\mathcal{S}/\mathcal{R}$ be the partition associated to $\mathcal{R}$. $\sim$ is the solution of the maximum proportional lumpability problem over $X(t)$ and $\mathcal{R}$ if and only if the partition $\mathcal{S}/\sim$ is the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{S}/\mathcal{R}$.*

Proof. This is an immediate consequence of the definitions. □

As a consequence, from now on we will focus on the maximum proportional partition problem.

Notice that the partition $\mathcal{S}/Id$, where Id is the identity relation, is associated to the proportional lumpability Id and it is finer than any other partition $\mathcal{P}$.

In other terms the set of proportional partitions that refine a given partition $\mathcal{P}$ is always not empty. However, it could be the case that for a given partition $\mathcal{P}$ such set contains different elements which are maximal with respect to the partial order $\sqsubseteq$. The following property will allow us to prove that this is never the case, i.e., that the maximum proportional partition problem has always a unique solution. The proofs of the following lemma and theorem are reported in the Appendix.

Lemma 1. *Let $X(t)$ be a CTMC with state space $\mathcal{S}$ and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two proportional partitions over $\mathcal{S}$. Let $\mathcal{P}$ be the smallest partition that is coarser than both $\mathcal{P}_1$ and $\mathcal{P}_2$. $\mathcal{P}$ is a proportional partition.*

Theorem 5 (Uniqueness). *The maximum proportional partition problem has always a unique solution.*

Partition refinement algorithms already defined in the context of bisimulation [23] and lumpabilities [28, 1] are based on the following idea: at every step each existing block B is split into B_1, B_2 using a reference block S, called *splitter*, which witnesses that the elements of B_1 and B_2 are not equivalent, no matter how S will be split during the next steps. In such framework the correctness of the algorithm is proved by proving that:

ST. Step Correctness: at each step the current partition is refined into a new one that is coarser than the solution;
FC. Final Convergence: the final partition is a proportional partition.

In order to be able to proceed along the same lines, we first need to prove that the maximum proportional partition problem has a chance be solved by iteratively applying refinement steps.

Proposition 3 (Iterative Refinements). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. Let $\mathcal{P}_\sim$ be the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$. If $\mathcal{P}'$ is finer than $\mathcal{P}$ and coarser than $\mathcal{P}_\sim$, i.e., $\mathcal{P}_\sim \sqsubseteq \mathcal{P}' \sqsubseteq \mathcal{P}$, then the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}'$ is $\mathcal{P}_\sim$.*

Proof. This is an immediate consequence of the definition of maximum proportional partition problem. □

We now focus on Step Correctness, i.e., we define splitting strategies that approaches the current partition to the result. To this aim we deeply analyse the characterization provided in Theorem 4.

Notice that if $\mathcal{P}$ has a unique class, then $\mathcal{P}$ is a proportional partition, i.e., no refinement is needed. In the case of partitions with only two classes the second condition of Theorem 4 is trivially satisfied, so we get the following characterization for such simple partitions.

Algorithm 1 Fix point computation of BisimSplit

function BisimSplit($X(t)$, $\mathcal{P}$)
  repeat
    $Bool = True$
    for $S, B \in \mathcal{P}$ with $S \neq B$ **do** ▷ S splits B
      $B_1 = \{s \in B \mid q(s, S) \neq 0\}$
      if $B_1 \neq B \wedge B_1 \neq \emptyset$ **then**
        $\mathcal{P} = (\mathcal{P} \setminus \{B\}) \cup \{B_1, B \setminus B_1\}$
        $Bool = False$
  until $Bool$ ▷ Exit when Bool is True
  return $\mathcal{P}$

Lemma 2. *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$ with $|\mathcal{P}| = 2$. $\mathcal{P}$ is a proportional partition if and only if for all $S_i, S_k \in \mathcal{P}$ with $S_i \neq S_k$, and $s, s' \in S_i$ it holds that*

$$q(s, S_k) \neq 0 \qquad \textit{iff} \qquad q(s', S_k) \neq 0$$

Proof. This is an immediate consequence of Theorem 4. □

In the general case only the left to right direction of the above result continues to hold and provides us a first splitting strategy.

Lemma 3. *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. If $\mathcal{P}$ is a proportional partition, then for all $S_i, S_k \in \mathcal{P}$ with $S_i \neq S_k$, and $s, s' \in S_i$ it holds that*

$$q(s, S_k) \neq 0 \qquad \textit{iff} \qquad q(s', S_k) \neq 0$$

Proof. It immediately follows from the definition of proportional lumpability. □

Hence, we can split blocks exploiting the above condition. If s and s' in S_i are such that $q(s, S_k) \neq 0$ while $q(s', S_k) = 0$, no matter how S_k will be split during the computation s and s' will always violate the condition with respect to at least one new class $S'_k \subseteq S_k$. So, we split S_i separating the elements reaching S_k from those that do not reach S_k. We call such splits BisimSplit, since they are exactly the splits performed in classical strong bisimulation algorithms [23]. In Algorithm 1 we describe the function that computes these splits until a fix-point is reached.

Proposition 4 (BisimSplit Correctness). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. Let $\mathcal{P}_\sim$ be the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$. Let $\mathcal{P}'$ be the partition returned by* BisimSplit*$(X(t), \mathcal{P})$. $\mathcal{P}'$ is finer than $\mathcal{P}$ and coarser than $\mathcal{P}_\sim$.*

Proof. This is a consequence of Lemma 3. □

At this point we focus on the second condition of Theorem 4 and we translate it in a splitting strategy. If s and s' satisfy the first condition of Theorem 4, but not the second one, then one could believe that next splits on S_j and S_k could avoid the problem. In other terms it could be possible for s and s' to remain in the same block thanks to changes in S_j and S_k. The following result proves that this is never the case. The proof is reported in the Appendix.

Lemma 4. *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. Let $\mathcal{P}_\sim$ be the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$. If there exist $S_i, S_j, S_k \in \mathcal{P}$ with $S_i \neq S_j$, $S_i \neq S_k$, and $s, s' \in S_i$ such that $q(s, S_k) \neq 0$, $q(s', S_k) \neq 0$, and*

$$\frac{q(s, S_j)}{q(s, S_k)} \neq \frac{q(s', S_j)}{q(s', S_k)}$$

then s and s' belong to different blocks in $\mathcal{P}_\sim$.

As a consequence we get the splitting strategy described in Algorithm 2.

Algorithm 2 Fix point computation of PropSplit

function PropSplit($X(t), \mathcal{P}$)
  repeat
    $Bool = True$
    for $S, T \in \mathcal{P}$ with $S \neq T$ **do**
      for $B \in \mathcal{P}$ with $B \neq S, B \neq T$ and $\forall s \in B$ it is $q(s,T) \neq 0$ **do**
        $\mathcal{B} = \{B_1, \ldots, B_n\}$ such that $B_f \subseteq B$ and
            for all $s, s' \in B_f$ it is $\frac{q(s,S)}{q(s,T)} = \frac{q(s',S)}{q(s',T)}$
        if $|\mathcal{B}| > 1$ **then**
          $\mathcal{P} = (\mathcal{P} \setminus \{B\}) \cup \mathcal{B}$
          $Bool = False$
  until Bool   ▷ Exit when Bool is True
  return $\mathcal{P}$

Proposition 5 (PropSplit Correctness). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$. Let $\mathcal{P}_\sim$ be the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$. Let $\mathcal{P}'$ be the partition returned by* PropSplit*($X(t), \mathcal{P}$). $\mathcal{P}'$ is finer than $\mathcal{P}$ and coarser than $\mathcal{P}_\sim$.*

Proof. This is a consequence of Lemma 4. □

The algorithm we propose for solving the maximum proportional partition problem alternatively applies the two above described splitting strategies until a fix point is reached. It is described in Algorithm 3.

Algorithm 3 Fix point computation of the Maximum Proportional Partition

function MAXPROP($X(t), \mathcal{P}$)
  repeat
    $\mathcal{P}' = \mathcal{P}$
    $\mathcal{P} =$PROPSPLIT($X(t)$,BISIMSPLIT($X(t), \mathcal{P}$))
  until $\mathcal{P} = \mathcal{P}'$
  return $\mathcal{P}$

Since in Proposition 3 we proved that the problem can be solved through an iterative algorithm and in Propositions 4 and 5 we provided the Step Correctness, it only remains to prove that the final result is a proportional partition and to analyse the complexity of the procedure.

Theorem 6 (Correctness and Complexity). *Let $X(t)$ be a CTMC with state space $\mathcal{S}$, let $\mathcal{P}$ be a partition over $\mathcal{S}$.* MAXPROP*($X(t), \mathcal{P}$) returns the solution of the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$ in time $O(|\mathcal{S}|^4)$.*

Proof. As far as correctness is concerned, in virtue of Propositions 3, 4, and 5 we only have to prove that the output of the algorithm is a proportional partition. The output of the algorithm is a fix-point for the function PROPSPLIT($X(t)$,BISIMSPLIT($X(t), \_$)). We have that BISIMSPLIT implements the first condition of Theorem 4 and PROPSPLIT implement the second condition of Theorem 4. So, since Theorem 4 is a characterization for proportional lumpability, the output of the algorithm is a proportional partition.

During the computation $O(|\mathcal{S}|)$ splits will be performed by either BISIMSPLIT or PROPSPLIT, since in the worst case the final partition has $\Theta(|\mathcal{S}|)$ blocks. Each split performed by BISIMSPLIT can be computed in time $O(|\mathcal{S}|^2)$, e.g., by exploiting [23]. As for the splits performed by PROPSPLIT, from the infinitesimal generator of $X(t)$ and the blocks of the current partition in time $\Theta(|\mathcal{S}|^2)$ we can compute a matrix in which for each state s and each class S we store $q(s, S)$. This matrix has size $O(|\mathcal{S}^2|)$. Each block T of the current partition corresponds to a column t in the matrix. For each column t we compute a new matrix in which for each row s having $q(s, t) \neq 0$ we normalize all the row dividing by $q(s, t)$. This take time $O(|\mathcal{S}|^2)$ and allow us to split each class B with respect to all other classes S, through a single complete scan of the matrix. Hence, for each normalizer T we need time $O(|\mathcal{S}|^2)$. Since T has $O(|\mathcal{S}|)$ possible values, one split of PROPSPLIT requires $O(|\mathcal{S}|^3)$. □

Notice that the above complexity result can be refined by exploiting adjacency lists, hence replacing a factor $|\mathcal{S}|^2$ by the number of non-null elements of the infinitesimal generator of $X(t)$.

5 Conclusion

In this paper we recall the notion of proportional lumpability and present two characterizations of it. These characterizations allow us to develop a computational method for proportional lumpality. More precisely, the first characterization has been proved in [20] and can be exploited to efficiently check whether a given relation is a proportional lumpability, while the second characterization is a novel contribution and allows us to develop an algorithm for the computation of the proportional lumpability that refines a given initial relation.

The algorithm we presented for proportional lumpability at the moment does not exploit any ad-hoc technique for reducing the computational complexity, such as the process the smallest half policy presented in [23] for bisimulation computation and extended to lumpability in [6, 28]. As future work we plan to investigate along this direction.

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A Appendix

Proof of Lemma 1

First notice that each block $A \in \mathcal{P}$ can be written both as a union of blocks of $\mathcal{P}_1$ and as a union of blocks of $\mathcal{P}_2$, i.e.,

$$A = A_{11} \cup A_{12} \cup \cdots \cup A_{1k_1} = A_{21} \cup A_{22} \cup \cdots \cup A_{2k_2}$$

with $A_{ij} \in \mathcal{P}_i$.

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are proportional partitions, there exist two functions κ_1, κ_2 from $\mathcal{S}$ to $\mathbb{R}^+$ that witness this fact. This implies that if we take two states s and s' which are not in A and are in a block B_i of $\mathcal{P}_i$, it holds that:

$$\frac{q(s, A)}{\kappa_i(s)} = \frac{\sum_{j=1}^{k_i} q(s, A_{ij})}{\kappa_i(s)} = \frac{\sum_{j=1}^{k_i} q(s', A_{ij})}{\kappa_i(s')} = \frac{q(s', A)}{\kappa_i(s')}$$

This last can be rewritten as:

$$q(s, A) = \frac{\kappa_i(s)}{\kappa_i(s')} q(s', A)$$

For each block $B \in \mathcal{P}$ we fix a representative element $b \in B$. For each $b' \in B$ there exists at least one finite sequence $b_0, b_1, \ldots, b_m$ such that $b_0 = b$, $b_m = b'$ and for each $h = 0, \ldots, m-1$ there exists B_h such that $b_h, b_{h+1} \in B_h$ and either $B_h \in \mathcal{P}_1$ or $B_h \in \mathcal{P}_2$. For each $b' \in B$ we fix one of such sequences. For the sake of clarity, let us consider a simple case where $b, b_1 \in B_0 \in \mathcal{P}_1$, $b_1, b_2 \in B_1 \in \mathcal{P}_2$, and $b_2, b' \in B_2 \in \mathcal{P}_1$. Let $A \in \mathcal{P}$ with $A \neq B$. In virtue of the last equation, we have:

$$q(b, A) = \frac{\kappa_1(b_1)}{\kappa_1(b)} q(b_1, A) = \frac{\kappa_1(b)}{\kappa_1(b_1)} \frac{\kappa_2(b_1)}{\kappa_2(b_2)} q(b_2, A) = \frac{\kappa_1(b)}{\kappa_1(b_1)} \frac{\kappa_2(b_1)}{\kappa_2(b_2)} \frac{\kappa_1(b_2)}{\kappa_1(b')} q(b', A)$$

In the general case we obtain:

$$q(b, A) = K(b, b') q(b', A)$$

where $K(b, b')$ is a product of fractions involving values of κ_1 and κ_2 that depends on the sequence that we have fixed from b to b'. Since both b and the sequence have been fixed we can define $\overline{K}(b') = K(b, b')$. As a consequence, if $b', b'' \in B$ we obtain that for each $A \in \mathcal{P}$ with $A \neq B$ it holds

$$\overline{K}(b') q(b', A) = q(b, A) = \overline{K}(b'') q(b'', A)$$

This means that $\mathcal{P}$ is a proportional partition. $\square$

Proof of Lemma 4

Let $S_j = A_1 \cup \cdots \cup A_n$ and $S_k = B_1 \cup \ldots B_m$ with $A_f, B_h \in \mathcal{P}_\sim$. Let κ be a function witnessing that $\mathcal{P}_\sim$ is a proportional lumpability. If by contradiction there exists a block $C \in \mathcal{P}_\sim$ such that $s, s' \in C$, then we would have

$$\frac{q(s, A_f)}{\kappa(s)} = \frac{q(s', A_f)}{\kappa(s')}$$

for each $f = 1, \ldots, n$ and

$$\frac{q(s, B_h)}{\kappa(s)} = \frac{q(s', B_h)}{\kappa(s')}$$

for each $h = 1, \ldots, m$. As a consequence by summing for $f = 1, \ldots, n$ and $h = 1, \ldots, m$ we have

$$\frac{q(s, S_j)}{\kappa(s)} = \frac{q(s', S_j)}{\kappa(s')} \text{ and } \frac{q(s, S_k)}{\kappa(s)} = \frac{q(s', S_k)}{\kappa(s')}$$

Since by hypothesis it holds $q(s, S_k) \neq 0$ and $q(s', S_k) \neq 0$ we get

$$\frac{q(s, S_j)}{q(s, S_k)} = \frac{q(s', S_j)}{q(s', S_k)}$$

which contradicts the hypothesis. □

Proof of Theorem 5

The existence of at least one solution is trivial, since the identity relation is a proportional lumpability.

As far as the uniqueness is concerned, let us consider the maximum proportional partition problem over $X(t)$ and $\mathcal{P}$. Let us assume by contraddiction that the set of proportional partitions that refines $\mathcal{P}$ has at least two different maximal elements. This means that there are two different partitions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that:

a. $\mathcal{Q}_i$ is a proportional partition;
b. $\mathcal{Q}_i$ refines $\mathcal{P}$;
c. each $\mathcal{Q}'$ coarser than $\mathcal{Q}_i$ and refining $\mathcal{P}$ is not a proportional partition.

By Lemma 1 the smallest partition $\mathcal{Q}$ that is coarser than both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is a proportional partition. Moreover, since both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ refine $\mathcal{P}$, it holds that $\mathcal{Q}$ refines $\mathcal{P}$. This contradicts item c. □