Advertising policies for a museum temporary exhibition

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Abstract In this paper we use a nondifferentiable optimal control model to analyze the advertising expenditure for a museum institution, which organizes a temporary exhibition and may have higher costs in case of congestion. The laws governing the behaviour of the system through time are defined by two alternative dynamical systems, depending on the visitors attendance rate being higher or lower than a critical level, the congestion threshold. We propose a local search approach in order to determine optimal solutions to the museum visitors flow problem. We focus on special neighbourhood structures in order to develop suitable local search algorithms which take into account some important features of the solutions of the control problem.

Keywords: Marketing; Museum; Optimal control; Local search; Congestion

1. Introduction

Let us consider a cultural organization which wants to promote a special exhibition that takes place in a definite time period. Typically the promoter must compete with a variety of proposals of recreation and entertainment activities for attracting visitors. To this end, the promoter may advertise the

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event in order to enhance the interest for it among the potential audience [17]. There is indeed strong evidence on the increasing importance of marketing strategies also for nonprofit organizations, such as museums, as it is testified for example by [1], [3], [7], [8, pp. 43–45], [11] and [17].

We distinguish between two types of communication channels for transmitting information on museum exhibitions within the social system. The first one is media communication, which is directly controlled by means of the advertising policy of the promoter. The second one is word–of–mouth communication, which is related to museum reputation and is not affected by the advertising policy. Both types of communication are considered in the literature concerning optimal control applications to marketing problems, see for example [12], [4], [5], [6], [10]. As far as word–of–mouth communication is concerned, here we assume in particular that past visitors can spread both favorable and unfavorable information, according to their museum experience being either positive or negative. As Rothenberg [15] notes, museum visits, like many public goods (highways, beaches, parks, tourist attractions of all kinds) are subject to crowding and congestion: the presence of other users adversely affects the level of utility obtained by each consumer. Utility deterioration may be revealed in terms of lenght of queues, psychological tension or aesthetic disfiguration of the exhibition. There exists, in general, a congestion threshold beyond which interference effects become noticeable and the quality of visitor experience decreases. We relate the occurrence of the unsatisfied visitors to the exhibition congestion.

We propose to describe the behaviour of the system through time by means of two alternative dynamical systems. Both of them account for the positive effects on the visitors attendance rate of the advertising expenditure rate and of the cumulative number of satisfied visitors, on the one hand, and account for the negative effects on the visitors attendance rate of the possible congestion of the exhibition and of the cumulative number of unsatisfied visitors, on the other hand. The first dynamical system is associated to a low attendance rate (normal regime), when no actual congestion effects are observed. The second one is associated to a high attendance rate (congested regime), when the visitors attendance rate is also affected negatively by the actual congestion and additional management costs are observed. In a state trajectory we distinguish a sequence of arcs which belong to the normal regime and the congested regime alternatively. Because of the regime switching the optimal control problem is nondifferentiable and the standard maximum principle is unsuitable for analysing optimal policies. Local search techniques are then used in order to formulate a special family of algorithms searching for candidates to optimality.

The paper is organized as follows. In Section 2 we introduce the museum visitors flow problem. In Section 3 we give a suitable definition of neighborhood of an admissible solution and we present the basic local search algorithm. In Section 4 we focus on a family of auxiliary control problems that are needed to specify some steps of the basic local search algorithm. In Section 5 we discuss a cycle based algorithm as a local search algorithm which exploits the different regime characteristics of the admissible solutions.

2. The museum visitors flow (MVF) problem

2.1 Statement of the problem

Let us denote by

T,	the final time, which is the end time of the exhibition, $0 \leq T \leq \overline{T}$;
$\overline{T}$,	the least upper bound of the feasible final times, $\overline{T} > 0$;
$y(t)$,	the visitors attendance rate at time t;
$\overline{y}$,	the congestion threshold, $\overline{y} > 0$;
$x(t)$,	the cumulative number of satisfied visitors at time t;
$z(t)$,	the cumulative number of unsatisfied visitors at time t;
$v(t)$,	the advertising expenditure rate at time t;
$\overline{v}$,	the maximum advertising expenditure rate, $\overline{v} > 0$;
$B(y, v)$,	the museum net benefit rate.

The following equations determine the dynamics of the system:

$$\dot{x}(t) = y(t)(1 - \mathbb{I}(y(t) - \overline{y})), \tag{1.1}$$

$$\dot{y}(t) = -\gamma \max\{0, y(t) - \overline{y}\} + a_x x(t) - a_z z(t) + bv(t), \tag{1.2}$$

$$\dot{z}(t) = y(t)\mathrm{I\!I}(y(t) - \overline{y}), \tag{1.3}$$

$$B(y(t), v(t)) = \alpha y(t) - k \max\{0, y(t) - \overline{y}\} - v(t), \tag{2}$$

where

$$\mathrm{I\!I}(y - \overline{y}) \;=\; \begin{cases} 0, & \text{if} \quad y < \overline{y}, \\ 1, & \text{if} \quad y \geq \overline{y}, \end{cases} \tag{3}$$

and

$$\gamma > 0,\ a_x > 0,\ a_z > 0,\ b > 0,\ \alpha > 0,\ k \geq 0.$$

Equations (1.1) and (1.3) represent the way in which satisfied and unsatisfied visitors appear: all visitors are supposed to be either satisfied or unsatisfied according to the visitors attendance rate being either less or greater than the congestion threshold, respectively. Equation (1.2) represents the growth of museum demand as a function of excess demand, cumulative satisfied visitors, cumulative unsatisfied visitors and advertising. The museum net benefit rate is given by equation (2), which takes into account that each visitor pays a constant admission fee and has a constant exhibition cost, but there is an additional exhibition cost rate at all times in which the visitors rate exceeds the congestion threshold level. In order to make sure that an admissible control $v(t)$ determines a unique state function $(x(t), y(t), z(t))$, we restrict our attention to solutions $(x(t), y(t), z(t), v(t), T)$ such that

i) $v(t)$ is piecewise continuous,

ii) $\dot{y}(t^*-) \neq 0$ at all t^* such that $y(t^*) = \overline{y}$,

iii) if $y(t^*) = \overline{y}$ then there exists $\epsilon > 0$ such that
either $y(t) < \overline{y},\ t \in (t^* - \epsilon, t^*)$ and $\dot{y}(t^*+) \geq \dot{y}(t^*-)$,
or $y(t) > \overline{y},\ t \in (t^* - \epsilon, t^*)$, and $\dot{y}(t^*+) \leq \dot{y}(t^*-)$,

iv) the set $\{t \in [0, T] \mid y(t) = \overline{y}\}$ is finite.

Solutions of this kind are, for instance, those which are determined by the constant control functions.

At time $t = 0$ the state is given by:

$$x(0) = 0, \tag{4.1}$$

$$z(0) = 0, \tag{4.2}$$

$$y(0) = y_0. \tag{4.3}$$

The problem can be stated as follows: find an advertising policy $v(t)$, $t \in [0, T]$, that maximizes the museum total benefit J

$$J = \int_0^T B(y(t), v(t))dt, \tag{5}$$

subject to the differential equations (1.1)–(1.3), with the initial conditions (4.1)–(4.3) and with the further non–negativity constraint

$$y(T) \geq 0, \tag{6}$$

which is also an anticipated stopping condition. The final time T is restricted to vary in the closed interval $[0, \overline{T}]$, whereas feasible controls are constrained by the maximum expenditure rate $\overline{v}$, that is:

$$v(t) \in [0, \overline{v}]. \tag{7}$$

Moreover we assume that the initial visitors attendance rate is less than the congestion threshold level when the exhibition opens to the public:

$$0 < y_0 < \overline{y}. \tag{8}$$

2.2 Normal and congested regimes

We say that the system is in *normal regime* at time t if $y(t) < \overline{y}$, so that its evolution is determined by the motion equations:

$$\dot{x}(t) = y(t), \tag{9.1}$$

$$\dot{y}(t) = a_x x(t) - a_z z(t) + bv(t), \tag{9.2}$$

$$\dot{z}(t) = 0. \tag{9.3}$$

If $y(t) \geq \overline{y}$, we say that the system is in *congested regime* at time t, and the associated motion equations are:

$$\dot{x}(t) = 0, \tag{10.1}$$

$$\dot{y}(t) = -\gamma(y(t) - \overline{y}) + a_x x(t) - a_z z(t) + bv(t), \tag{10.2}$$

$$\dot{z}(t) = y(t). \tag{10.3}$$

Let $(x(t), y(t), z(t), v(t), T)$ be a feasible solution to the museum visitors flow problem, which satisfies the conditions (i)–(iv) of Section 2.1. As condition (8) holds, we can define the following sequence of times:

$$\begin{aligned} t_0 &= 0, \\ t_{2i-1} &= \inf\{t > t_{2i-2} | \quad y(t) \geq \overline{y}\}, \quad i \geq 1, \\ t_{2i} &= \inf\{t > t_{2i-1} | \quad y(t) < \overline{y}\}, \quad i \geq 1, \end{aligned} \tag{11}$$

where $\inf \emptyset = +\infty$. If t_{n-1} is the last real element of the sequence determined recursively by (11), then let

$$t_n = T. \tag{11.1}$$

We call the times $t_0, t_1, ..., t_n$, *transition times* associated with the feasible solution $(x(t), y(t), z(t), v(t), T)$, as we observe a regime change at time t_k, $k \notin \{0, n\}$. In view of the (restrictive) conditions on the feasible solutions of Section 2.1, in particular condition (iv), we have that all feasible solutions have a finite number of transition times. We call ith *epoch* the time interval

$$e_i = [t_{i-1}, t_i], \qquad i = 1, .., n. \tag{12}$$

in which the system is observed staying either in normal regime (i odd) or in congested regime (i even). Finally, if we consider the time interval resulting from the union of two consecutive epochs, we have a *cycle*:

$$c_i = e_i \cup e_{i+1} = [t_{i-1}, t_{i+1}], \qquad i = 1, .., n-1. \tag{13}$$

3. Improvement of admissible policies

We present an algorithm with the goal of searching for candidates to optimality of the MVF problem. The algorithm exploits a local search strategy in a context which is suggested by Bellman's optimality principle.

Local search algorithms, as presented in [2], [9], [13], [14], [18] are iterative algorithms with the aim of finding a (local) optimal solution to a general optimization problem. One first defines the neighbourhood of each feasible solution of the problem as a special subset of feasible solutions "near to" the given solution. Then, starting from an arbitrary initial solution, local search consists in moving from the current solution to another one in its neighbourhood, according to some well–defined rules. When the criterion for selecting the next solution is to choose a solution in the neighbourhood with an improved value of the objective function then the literature refers to it as a *descent algorithm* [9], [14], to indicate that at each step of the iterative process the value of the objective function decreases (in the formulation of the problem as a minimum problem).

3.1 Neighbourhood structure

Let Σ be the set of all admissible solutions to the MVF problem, i.e. the solutions which satisfy the motion equations (1), the assumptions i)–iv) of Section 2.1, the initial conditions (4), the terminal condition (6) and the control constraint (7). Let $\mathcal{I}$ be the set of all closed subintervals of $[0, \overline{T}]$.

Definition Let $F : \Sigma \to \mathcal{P}(\mathcal{I})$ be a mapping from the set of admissible solutions into the power set of the set of closed subintervals of $[0, \overline{T}]$, such that, for all $\xi = (x, y, z, v, T) \in \Sigma$,

i) $F(\xi)$ is finite,

ii) $\bigcup_{I \in F(\xi)} I \ = \ [0, T]$.

Let $\Phi : \bigcup_{\xi \in \Sigma} \{\xi\} \times F(\xi) \to \mathcal{P}(\Sigma)$ be a mapping which maps a couple (ξ, I), where $\xi = (x, y, z, v, T)$ and $I = [t', t''] \in F(\xi)$, into a nonempty subset $\Phi(\xi, I)$ of admissible solutions $\eta \ = \ (x', y', z', v', T') \in \Sigma$ such that $T' \geq t'$ and

$$(x', y', z', v')(t) \ = \ (x, y, z, v)(t), \quad \text{for all } t \in [0, \min\{T, T'\}] \setminus I,$$

where $T' = T$ whenever $t'' < T$.

We define the (F, Φ)–neighbourhood of the solution $\xi \in \Sigma$ as the set

$$N_{F,\Phi}(\xi) \ = \bigcup_{I \in F(\xi)} \Phi(\xi, I).$$

We find it convenient for our purposes not to follow the convention, assumed e.g. in [16], that $\xi \notin N_{F,\Phi}(\xi)$, for all $\xi \in \Sigma$. Nevertheless, we do not exclude choices of Φ such that $\xi \notin \Phi(\xi, I)$ for some (ξ, I) and possibly that $\xi \notin N_{F,\Phi}(\xi)$.

In the special case in which $\Phi(\xi, I) \subseteq \Sigma$ is a singleton for all $\xi \in \Sigma$ and $I \in F(\xi)$, we have that the number of admissible solutions of the neighborhood $N_{F,\Phi}(\xi)$ equals the number of time intervals of $F(\xi)$.

In view of the above definition, choosing an admissible solution in the neighbourhood $N_{F,\Phi}(\xi)$ of a solution ξ means modifying the solution ξ in an interval $I \in F(\xi)$.

3.2 The basic steepest descent algorithm

Once we have fixed the mappings F and Φ and we have chosen an initial admissible solution, we obtain a local search steepest descent [16] algorithm by moving, at each step, from the current solution ξ to the *best* solution in its neighbourhood $N_{F,\Phi}(\xi)$. We observe that, if $\Phi(\xi, I)$ is a singleton for all $\xi \in \Sigma$ and $I \in F(\xi)$, then the local search steepest descent algorithm requires, at each step, first to determine the admissible solution associated with each interval $I \in F(\xi)$ and evaluate the objective functional of problem MVF in it, then to move from the current solution ξ to the *best* admissible solution associated with an interval in $F(\xi)$.

Basic steepest descent algorithm

Step 0: (initialization) construct a feasible solution $\xi^0 = (x^0, y^0, z^0, v^0, T^0)$

$n \leftarrow 0$

Step 1: determine the set of intervals $F(\xi^n)$

Step 2: $\eta^n \leftarrow \xi^n$

Step 3: **for all** $I \in F(\xi^n)$ **do**

Step 3.1: find the best solution $\eta \in \Phi(\xi^n, I)$

Step 3.2: **if** η is better than η^n, **then** $\eta^n \leftarrow \eta$

Step 4: **if** η^n is better than ξ^n, **then** $\xi^{n+1} \leftarrow \eta^n$

else stop

Step 5: $n \leftarrow n + 1$

Step 6: **go to** Step 1

In order to determine completely a descent algorithm one has to specify, in addition to the mappings F and Φ:

i) how to construct a feasible solution ξ^0 (Step 0),

ii) how to find a "best solution" η (Step 3.1).

We observe that the characteristics of the optimization problem of Step 3.1 depend on the characteristics of the set $\Phi(\xi^n, I)$ and in particular that there may not exist a best solution in $\Phi(\xi^n, I)$ and in that case Step 3.1 is undefined. We are particularly interested in having a mapping Φ which induces optimization problems, which are less difficult than the original problem MVF.

Now, assuming that a best solution in $\Phi(\xi^n, I)$ exists in all cases, we observe that after executing the Steps 1–4 for a given n, with η^n better than ξ^n, the value of the objective functional at the improved admissible solution ξ^{n+1} is greater than the value at the previous solution ξ^n. All feasible controls may be used to initialize the algorithm and in general the choice of the initial feasible solution affects the algorithm convergence.

3.3 First improvement rules

In addition to the steepest descent algorithm which implements the *best improvement pivoting rule,* we may consider a class of algorithms which implement the *first improvement* pivoting rule [2] and [18]. They consist in searching new solutions in the neighbourhood of the current one until one is found which presents an improved value of the objective functional. The use of such first improvement approach can modify the previous basic algorithm in several ways.

In view of the particular representation of the neighbourhoods that we have used in the basic steepest descent algorithm, we may propose two different versions of first improvement algorithms, by substituting Steps 2 to 4 as follows:

Version I

Step 2: $L \leftarrow F(\xi^n)$

Step 3: choose an interval $I \in L$

Step 3.1: $L \leftarrow L \setminus \{I\}$

Step 3.2: find the best solution $\eta^n \in \Phi(\xi^n, I)$

Step 4: **if** η^n is better than ξ^n

then begin

$\xi^{n+1} \leftarrow \eta^n$

go to Step 5

end

else if $L \neq \emptyset$ **then go to** Step 3

else stop

Version II

Step 2: $L \leftarrow F(\xi^n)$

Step 3: choose an interval $I \in L$

Step 3.1: $L \leftarrow L \setminus \{I\}$

Step 4: **if** there exists $\eta^n \in \Phi(\xi^n, I)$ such that η^n is better than ξ^n,

then begin

$\xi^{n+1} \leftarrow \eta^n$

go to Step 5

end

else if $L \neq \emptyset$ **then go to** Step 3

else stop

The resulting local search algorithms explore, at each iteration of Steps 3–4, a subset of the neighbourhood of the current solution. In particular, in Version II an arbitrary solution $\eta \in \Phi(\xi^n, I)$ is sought, which is better than the current solution ξ^n and is not necessarily the best one. In Version I, on the other hand, the best solution in $\Phi(\xi^n, I)$ is chosen, for some $I \in L$, but in general this solution will not be the best solution in $N_{F,\Phi}(\xi) = \bigcup_{I \in L} \Phi(\xi^n, I)$, as it was required by the steepest descent algorithm.

We observe that Step 3.2 in Version I is not well defined in the case that the best solution in the set $\Phi(\xi^n, I)$ does not exist. On the contrary all the steps in Version II are well defined.

4. Interval subproblems

Here we focus on the special kind of (F, Φ)–neighbourhood $N_{F,\Phi}(\xi)$ which is determined by the least restrictive mapping $\Phi : \bigcup_{\xi \in \Sigma} \{\xi\} \times F(\xi) \to \mathcal{P}(\Sigma)$, i.e. we consider $\Phi(\xi, I)$ as the set of all solutions $\eta = (x', y', z', v', T') \in \Sigma$ such that

$$\eta|_{[0,\min\{T,T'\}]\setminus I} = \xi|_{[0,\min\{T,T'\}]\setminus I}.$$

In this case, the problem of finding the best solution $\eta \in \Phi(\xi, I)$ is equivalent to the optimal control problem of maximizing the objective functional of the MVF problem on the interval I with suitable boundary conditions. We will call *maximum depth algorithm* the steepest descent algorithm which is associated with the above definition of Φ.

Now, let us consider the subproblem

$$P(t', I, X', Y', Z', X'', Y'', Z''), \tag{14}$$

associated with the time interval $[t', t'']$ (where the endtime t'' is variable in I), which is the problem of maximizing the objective functional

$$\int_{t'}^{t''} B(y(t), v(t))dt, \tag{15}$$

subject to the motion equations (1.1)–(1.3), the assumptions i)–iv) of Section 2.1, the control variable restriction (7) and the following initial and terminal conditions for the state variables:

$$x(t') \in X', \quad x(t'') \in X'', \tag{16.1}$$

$$y(t') \in Y', \quad y(t'') \in Y'', \tag{16.2}$$

$$z(t') \in Z', \quad z(t'') \in Z''. \tag{16.3}$$

In the following we will write s in place of the singleton $\{s\}$, for all $s \in \Re$, moreover we will denote by $\Re_+$ the set of all nonnegative real numbers, $\Re_+ = [0, +\infty)$.

We can state the steepest descent algorithm with reference to the actual neighbourhood structure, i.e. the maximum depth algorithm, and using the notation just introduced, as follows:

Maximum depth algorithm

Step 0: (initialization) construct a feasible solution $\xi^0 = (x^0, y^0, z^0, v^0, T^0)$

$n \leftarrow 0$

Step 1: determine the set of intervals $F(\xi^n) = F(x^n, y^n, z^n, v^n, T^n)$

Step 2: $\eta^n \leftarrow \xi^n$

Step 3: **for all** $I = [t', t''] \in F(\xi^n)$ **do**

Step 3.1: **if** $t'' < T^n$

then let η_I be an optimal solution to the interval subproblem

$P(t', t'', x^n(t'), y^n(t'), z^n(t'), x^n(t''), y^n(t''), z^n(t''))$

and let T' be the final time of η_I $(T' = t'')$

else let η_I be an optimal solution to the interval subproblem

$P(t', [t', \overline{T}], x^n(t'), y^n(t'), z^n(t'), \Re_+, \Re_+, \Re_+)$

and let T' be the final time of η_I $(T' \in [t', \overline{T}])$

Step 3.2: **if** η_I is better than $\xi^n|_I$

then begin

$$\eta(t) = \xi^n(t), \quad t \in [0, t'] \cup [t'', T^n]$$
$$\eta(t) = \eta_I(t), \quad t \in [t', T']$$

if $t'' < T^n$ **then** set final time of η as $T_\eta \leftarrow T^n$

else set final time of η as $T_\eta \leftarrow T'$

if η is better than η^n **then** $\eta^n \leftarrow \eta$

end

Step 4: **if** η^n is better than ξ^n, (i.e. if η^n has been modified)

then $\xi^{n+1} \leftarrow \eta^n$

else stop

Step 5: $n \leftarrow n + 1$

Step 6: **go to** Step 1

As already observed for the basic steepest descent algorithm, Step 3.1 of the maximum depth algorithm may not be well defined as the relevant optimal control subproblem may not have any optimal solution.

Moreover we observe that at the n–th iteration of the algorithm we may know that the current solution cannot be improved in some subintervals of $[0, T^n]$, so that only the intervals of a subset of $F(\xi^n)$ need to be considered in

Step 3. We call "improvable" an interval $I \in F(\xi^n)$ until we find out that the restriction $\xi^n|_I$ is an optimal solution to the interval subproblem associated to it, i.e. either to problem $P(t', t'', x^n(t'), y^n(t'), z^n(t'), x^n(t''), y^n(t''), z^n(t''))$ or to problem $P(t', [t', \overline{T}], x^n(t'), y^n(t'), z^n(t'), \Re_+, \Re_+, \Re_+)$.

4.1 Interval optimality

The use of the maximum depth algorithm is justified by taking into account the fact that whenever the current solution is an optimal solution, the algorithm stops. This can be seen by using Bellman's optimality principle, which, for an autonomous problem like MVF, states that any portion of an optimal path is optimal [16, pp. 168–169]. In the context of the museum visitors flow problem we obtain that an optimal solution, with final time T^*, is characterized as follows, in any time interval $[t', t''] \subseteq [0, T^*]$.

Theorem 1 Let $\xi^* = (x^*, y^*, z^*, v^*, T^*)$ be an optimal solution to the MVF problem and let $[t', t''] \subseteq [0, T^*]$. Then the restriction of ξ^* to the interval $[t', t'']$ is an optimal solution to the interval subproblem

$$P(t', t'', x^*(t'), y^*(t'), z^*(t'), x^*(t''), y^*(t''), z^*(t'')).$$

Moreover, the restriction of ξ^* to the interval $[t', T^*]$ is also an optimal solution to the interval subproblem

$$P(t', [t', \overline{T}], x^*(t'), y^*(t'), z^*(t'), \Re_+, \Re_+, \Re_+).$$

An analogous justification holds for the basic steepest descent algorithm, although, depending on the definition of $\Phi(\xi, I)$, the optimization problem of Step 3.1 may not be an optimal control problem.

5. Cycle based neighbourhood structure

A crucial question for the realization of the maximum depth algorithm is the definition of the mapping $F : \Sigma \to \mathcal{P}(\mathcal{I})$, which is a part of the definition of the (F, Φ)–neighbourhood $N_{F,\Phi}(\xi)$. A special choice of subintervals of $[0, \overline{T}]$

is the set of the cycles of the current solution, which we have defined in Section 2.2. As the set of the cycles of $\xi = (x, y, z, v, T)$ is finite, let $F(\xi)$ be the set of the intervals $[t', t''] \subseteq [0, T]$ such that either $[t', t'']$ is a cycle of ξ, or it is the last epoch (i.e. $[t', t'']$ is an epoch and $t'' = T$). We call the resulting (F, Φ)–neighbourhood of a feasible solution ξ, a *cycle based neighbourhood* of ξ.

A first important consequence of using the cycle based neighbourhood structure is that all the elements (intervals) of $F(\xi^n)$ but at most three are also elements of $F(\xi^{n+1})$.

A second consequence is related with the observation following the statement of the maximum depth algorithm in Section 4, i.e. that if $I \in F(\xi^n) \cap F(\xi^{n+1})$ and I is not improvable for ξ^n, then it is not improvable for ξ^{n+1} either.

Now, let the solution ξ^n, with final time T^n, be modified in an interval $I^n = [t', t''] \in F(\xi^n)$ at Step 3 of the n–th iteration and let such modified solution be defined as ξ^{n+1}, with final time T^{n+1}, at Step 4; then the set of the improvable intervals of ξ^n is transformed into that of ξ^{n+1} as follows.

First we observe that either $I^n \in F(\xi^{n+1})$ but is not improvable for ξ^{n+1}, or $I^n \notin F(\xi^{n+1})$. Moreover an interval $I \in F(\xi^{n+1})$ is not improvable for the solution ξ^{n+1} if either $I \subseteq I^n$ (whenever $t'' < T^n$) or $I \subseteq [t', T^{n+1}]$ (whenever $t'' = T^n$).

Finally, if the interval $I \in F(\xi^n)$ is a cycle of ξ^n and $I \cap I^n$ is an epoch of ξ^n, then in general $I \notin F(\xi^{n+1})$.

On the other hand an interval $I \in F(\xi^{n+1})$, $I \subseteq [0, T^n]$, which is a non–final cycle of ξ^{n+1}, is improvable for the solution ξ^{n+1} if $I \cap I^n$ is an epoch of ξ^{n+1}.

In conclusion, if we denote by L the set of the improvable intervals for the solution ξ^n, then the set of the improvable intervals for the successive solution ξ^{n+1} is given by $L \setminus A^-(I^n, \xi^n) \cup A^+(I^n, \xi^{n+1})$, where

$$\begin{aligned} A^-(I^n, \xi^n) &= \{I^n\} \cup \{I \mid I \text{ cycle of } \xi^n \text{ and } I \cap I^n \text{ epoch of } \xi^n\}, \\ A^+(I^n, \xi^{n+1}) &= \{I \subseteq [0, T^{n+1}] \mid I \text{ non–final cycle of } \xi^{n+1} \\ &\qquad\qquad \text{and } I \cap I^n \text{ epoch of } \xi^{n+1}\}. \end{aligned} \tag{17}$$

Using the cycle based neighbourhood structure and the observations just presented, we obtain the following *cycle based maximum depth algorithm.*

Cycle based maximum depth algorithm

Step 0: construct a feasible solution $\xi^0 = (x^0, y^0, z^0, v^0, T^0)$, $\quad n \leftarrow 0$

Step 1: $L \leftarrow F(\xi^0)$ (list of *improvable cycles* and last epoch)

Step 2: $\eta^n \leftarrow \xi^n$

Step 3: **for all** $I = [t', t''] \in L$ **do**

Step 3.1: **if** $t'' < T^n$

then let η_I be an optimal solution to the cycle subproblem $P(t', t'', x^n(t'), y^n(t'), z^n(t'), x^n(t''), y^n(t''), z^n(t''))$ and let T' be the final time of η_I ($T' = t''$)

else let η_I be an optimal solution to the interval subproblem $P(t', [t', \overline{T}], x^n(t'), y^n(t'), z^n(t'), \Re_+, \Re_+, \Re_+)$ and let T' be the final time of η_I ($T' \in [t', \overline{T}]$)

Step 3.2: **if** η_I is better than $\xi^n|_I$

then begin

$\eta(t) = \xi^n(t), \quad t \in [0, t'] \cup [t'', T^n]$

$\eta(t) = \eta_I(t), \quad t \in [t', T']$

if $t'' < T^n$ **then** set final time of η as $T_\eta \leftarrow T^n$

else set final time of η as $T_\eta \leftarrow T'$

if η is better than η^n **then** $\eta^n \leftarrow \eta, \quad I' \leftarrow I$

end

else $L \leftarrow L \setminus \{I\}$

Step 4: **if** η^n is better than ξ^n, (i.e. if η^n has been modified)

then $\xi^{n+1} \leftarrow \eta^n, \quad L \leftarrow L \setminus A^-(I', \xi^n) \cup A^+(I', \xi^{n+1})$

else stop

Step 5: $n \leftarrow n + 1$

Step 6: **if** $L \neq \emptyset$ **then go to** Step 2

else stop

We observe that, after executing Steps 3 and 4 on the cycle $I' = [t', t'']$, with $t'' < T^n$, the new (improved) solution ξ^{n+1} may as well have one internal transition time in the interval I' as more than one. In the first case, the interval $I' = [t', t'']$ is still a cycle of the solution ξ^{n+1}. In the second case, there are $1 + 2k$ transition times in (t', t''), with $k \geq 1$, so that ξ^{n+1} has more than one cycle in $[t', t'']$. In both cases, the interval I' is not present in the updated list of improvable intervals, because of Steps 3.2 and 4.

When the improvement procedure is executed on the interval $I' = [t', t''] = [t', T^n]$, which is either the last cycle or the last epoch of the current solution ξ^n, then the new improved solution ξ^{n+1} whose final time is T' may as well have one internal transition time in the interval $[t', T']$, as none or more than one.

The following theorem sheds some light on the relation between possible solutions to which the cycle based algorithm converges and optimal solutions of the MVF problem, by considering the features of the optimal solutions of interval subproblems for all intervals.

Theorem 2 Let $\xi^* = (x^*, y^*, z^*, v^*, T^*)$ be a solution to which the cycle based maximum depth algorithm converges. If there exists an interval $I = [t', t''] \subseteq [0, T^*]$, such that the optimal solution ξ^*_I of the interval subproblem $P(t', t'', x^*(t'), y^*(t'), z^*(t'), x^*(t''), y^*(t''), z^*(t''))$ (or alternatively $P(t', [t', \overline{T}], x^*(t'), y^*(t'), z^*(t'), \Re, \Re, \Re)$, if $t'' = T^*$) differs from the restriction $\xi^*|_I$ of the solution ξ^* to the interval I, then the interval I is neither a cycle nor a subset of a cycle of ξ^*.

Proof Let us first consider $t'' < T^*$ and assume that there exists an interval $I = [t', t''] \subset [0, T^*]$, such that the optimal solution of the interval subproblem $P^* = P(t', t'', x^*(t'), y^*(t'), z^*(t'), x^*(t''), y^*(t''), z^*(t''))$ differs from the restriction $\xi^*|_I$ of the solution ξ^* to the interval I.

If the interval I is a cycle of the solution ξ^*, then a contradiction follows directly from the assumption that ξ^* is a solution to which the cycle based maximum depth algorithm converges.

If the interval I is not a cycle of ξ^*, let $C = [t^*_{i-1}, t^*_{i+1}]$ be the cycle of the solution ξ^* such that $I \subset C$. In this case we consider the new solution $\hat{\xi}$ obtained by substituting the optimal solution of the interval subproblem P^*, to the restriction $\xi^*|_I$ in ξ^* and we consider the restriction $\hat{\xi}|_C$ of the new solution

$\hat{\xi}$ to the cycle C. We obtain that $\hat{\xi}|_C$ differs from $\xi^*|_C$ only in I. Moreover the objective functional of the subproblem attains a higher value in $\hat{\xi}|_C$ than in $\xi^*|_C$. Then ξ^* is improvable in the cycle C and we have a contradiction.

The case $t'' = T^*$ is similar to the previous one with the exception that we have to use problem $P(t', [0, \overline{T}], x^*(t'), y^*(t'), z^*(t'), \Re, \Re, \Re)$ as the interval subproblem P^*. Moreover, the cycle under consideration is the last cycle $C = [t^*_{n-2}, t^*_n], \quad \text{where } t^*_n = T^*.$ ■

Although a cycle subproblem is a nondifferentiable optimal control problem, it is nevertheless simpler than the original MVF problem because of the presence of a unique nondifferentiable point which is the transition time internal to the cycle. On the other hand, if we consider a first improvement algorithm with the cycle based neighbourhood structure, by modifying Steps 3 and 4 of the cycle based maximum depth algorithm along the lines of "Version II" of Section 3.3, then we can obtain a simplification by requiring that η_I is a feasible solution of the relevant control problem which is better than the current one. Of course, care is needed in order to have a significant set of feasible solutions among which to choose in the relaxed version of the algorithm.

6. Conclusions

The aim of this paper is twofold. On one hand, we have introduced the cultural marketing problem of determining optimal advertising policies for a museum institution. We have formulated the "museum visitors flow problem" as a nonlinear and nondifferentiable optimal control problem with three state and one control variables. On the other hand we have developed a special family of local search algorithms for the solution of the problem, presenting the basic steepest descent algorithm and two first improvement variants of it. The local search approach consists in improving iteratively an initial admissible solution to the museum visitors flow problem, by exploring a suitable neighbourhood of the current solution at each iteration. The key issue of our analysis is the special definition of neighbourhood of an admissible solution, which is based on some relevant information on the solution. The steepest descent algorithm requires to solve a set of special optimal control problems at each iteration and such problems may present the same difficulties as the original one. Then attention has to be devoted mainly to first improvement algorithms which allow one to consider less complicated problems. The observation that the evolution of the system is determined by piecewise linear differential equations may be useful to determine special first improvement rules.

Finally we observe that the initialization process affects the solution to the museum visitors flow problem which we may obtain from the execution of a local search algorithm. The number of iterations required by the algorithm will depend on it too. Then it will be important to deal with the problem of finding "good" initial admissible solutions.

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