BAYESIAN INFERENCE FOR MIXTURES OF STABLE DISTRIBUTIONS

Roberto Casarin

Dept. of Economics
Università Ca' Foscari
Cannareggio, 873-30121 Venezia
(e-mail: r.casarin@unive.it)

and

CEREMADE**
University Paris IX-Dauphine
75775 Paris Cédex 05

ABSTRACT. In many different fields such as hydrology, telecommunications, physics of condensed matter and finance, the gaussian model results unsatisfactory and reveals difficulties in fitting data with skewness, heavy tails and multimodality. The use of stable distributions allows for modelling skewness and heavy tails but gives rise to inferential problems related to the estimation of the stable distribution's parameters. The aim of this work is to generalise the stable distribution framework by introducing a model that accounts also for multimodality. In particular we introduce a stable mixture model and a suitable reparameterisation of the mixture, which allow us to make inference on the mixture parameters. We use a full Bayesian approach and MCMC simulation techniques for the estimation of the posterior distribution.

1 INTRODUCTION

Stable distribution has been introduced, in many different fileds, as a generalisation of the Gaussian model because it allows for infinite variance, skewness and heavy tails. For a summary of the properties of the stable distributions see Samorodnitsky and Taqqu (1994). In finance, the first studies on the hypothesis of stable distributed stock prices can be attributed to Mandelbrot (1963), Fama (1965). For other references see Casarin (2003). A recent work treating the use of stable distributions in finance is due to Rachev and Mittnik (2000). Early financial studies, suggest to use mixtures of distributions in order to modelling the financial markets heterogeneity, see for example Fieltz and Rozelle (1983). Different estimation methods for stable distributions have been proposed in the literature. See Buckle (1995) for a full Bayesian approach, while Casarin (2003) provide a brief review of other estimation methods.

The first aim of our work is to propose a stable distributions mixture model in order to capture the multimodality, which is present, for example, in financial data. The second goal of the work is to provide some inferential tools for stable distributions mixtures. We adopt the data augmentation principle (see Robert (1996)) in a

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full Bayesian approach, using MCMC simulation techniques in order to estimate the parameters. As pointed out by Stephens (1997) the Bayesian framework avoids some theoretical and numerical difficulties related to the maximum likelihood approach.

The structure of the work is as follows. *Section* 2 presents the Bayesian model and the Gibbs sampler for a stable distribution. *Section* 3 describes the Bayesian model for stable mixtures and the missing data structure. Moreover the Gibbs sampler for stable mixture is developed in the case where the number of components is fixed. *Section* 4 provides some simulation results and concluding remarks.

2 Bayesian Inference for Stable Distributions

In order to make inference on the parameters of a stable distribution in a Bayesian approach it is necessary to specify a *hierarchical model* (see Fig. 1) on the parameters of the distribution. The resulting posterior distribution of the Bayesian model cannot be calculated analytically, thus it is necessary to chose a numerical approximation method. Monte Carlo simulation techniques provide an appealing solution to the problem because, in high dimensional spaces, they are more efficient than traditional numerical integration methods and furthermore they require the densities involved in the posterior to be known only up to a normalising constant (see Casella and Robert (1999) for an introduction to MCMC methods and to convergence control techniques).

In the following we describe the Gibbs sampler for stable distributions (see Buckle (1995)). The stable density is obtained by integrating the bivariate density of the pair (x,y), with respect the auxiliary variable y:

$$f(x,y|\alpha,\beta,\sigma,\delta) = \frac{\alpha}{|\alpha-1|}\exp\left\{-|\frac{z}{\tau_{\alpha,\beta}(y)}|^{\alpha/(\alpha-1)}\right\}\left|\frac{z}{\tau_{\alpha,\beta}(y)}\right|^{\alpha/(\alpha-1)}\frac{1}{|z|} \quad (1)$$

$$(x,y) \in (-\infty,0)\times(-1/2,l_{\alpha,\beta})\cup(0,\infty)\times(l_{\alpha,\beta},1/2)$$

where $z=\frac{x-\delta}{\sigma}$, while the functions $\tau_{\alpha,\beta}(y)$, $\eta_{\alpha,\beta}$ and $l_{\alpha,\beta}$ are defined in Buckle (1995). Previous elements allow to perform simulation based Bayesian inference on the parameters of the stable distribution. The Bayesian model for stable distributions is described through the *Directed Acyclic Graph* (DAG) in Figure 1.a. Parameters are estimated by simulating from the complete posterior distribution and by averaging simulated values. Suppose to observe n realizations $\mathbf{x}=(x_1,\dots,x_n)$ from a stable distribution $S_\alpha(\beta,\sigma,\delta)$ and simulate a vector of auxiliary variables $\mathbf{y}=(y_1,\dots,y_n)$, then the Gibbs sampler is defined through the following steps:

(i) *Update the completing variable:*

$$\pi(y_i|\alpha,\beta,\delta,\sigma,z_i) \propto \exp\left\{1-\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\right\}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}, \quad i=1,\dots,n \quad (2)$$

(ii) *Simulate from the complete full conditional distributions:*

$$\pi(\alpha|\beta,\delta,\sigma,\mathbf{x},\mathbf{y}) \propto \frac{\alpha^n}{|\alpha-1|^n}\exp\left\{-\sum_{i=1}^{n}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\right\}\prod_{i=1}^{n}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\pi(\alpha) \quad (3)$$

$$\pi(\beta|\alpha,\delta,\sigma,\mathbf{x},\mathbf{y}) \propto \exp\left\{-\sum_{i=1}^{n}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\right\}\prod_{i=1}^{n}\left|\frac{1}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\pi(\beta) \quad (4)$$

$$\pi(\delta|\alpha,\beta,\sigma,\mathbf{x},\mathbf{y}) \propto \exp\left\{-\sum_{i=1}^{n}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\right\}\prod_{i=1}^{n}\left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\frac{\alpha}{(\alpha-1)}}\frac{1}{|x_i-\delta|}\pi(\delta) \quad (5)$$

$$\pi(\sigma|\alpha,\beta,\sigma,\mathbf{x},\mathbf{y}) \propto \left(\frac{1}{\sigma^{\alpha/(\alpha-1)}}\right)^n \exp\left\{-\frac{1}{\sigma^{\alpha/(\alpha-1)}}\sum_{i=1}^{n}\left|\frac{(x_i-\delta)}{\tau_{\alpha,\beta}(y_i)}\right|^{\alpha/(\alpha-1)}\right\}\pi(\sigma) \quad (6)$$

where $\pi(\alpha)$, $\pi(\beta)$, $\pi(\delta)$, $\pi(\sigma)$ are the prior distributions on the parameters and $\mathbf{y}$ is a vector of auxiliary variables $(y_1,\dots,y_n)$. The simulation strategies for the steps of the Gibbs sampler and the joint prior distribution are given in Casarin (2003).

3 Bayesian Inference for Mixtures of Stable Distributions

In this section we extend the Bayesian framework, introduced in the previous section, to the mixtures of stable distributions. A way to simultaneously modelling heavy tails, skewness, and multimodality, is to introduce stable mixtures. Moreover stable mixtures are appealing also because they have normal mixtures as special case, which are a widely studied topic, see for example Stephens (1997), Richardson and Green (1997). We assume that the stable mixture model $m(x|\theta,p)$ has a known and finite number, L, of components. Let $f(x|\alpha_l,\beta_l,\delta_l,\sigma_l)$ be the l-th stable distribution in the mixture, then:

$$m(x|\theta,p) = \sum_{l=1}^{L} p_l f(x|\theta_l) \quad \textit{with} \quad \sum_{l=1}^{L} p_l = 1, \quad p_l \geq 0, \quad l = 1,\dots,L \quad (7)$$

where $\theta_l = (\alpha_l,\beta_l,\delta_l,\sigma_l)$, $l = 1,\dots,L$ are the parameter vector and $\theta = (\theta_1,\dots,\theta_L)$.

In order to perform Bayesian inference two steps of completion are needed. First, the auxiliary variable, y, is introduced in order to obtain an integral representation of the mixture distribution. A second step of completion is needed to reduce the complexity problem, which arises in the inference for mixtures. The completing variable (or *allocation variable*), $\nu = \{\nu_1,\dots,\nu_L\}$ is defined as follow: $\nu_l = 1$ if $x \sim f(x,y|\theta_l)$

and 0 otherwise. It is used to select the mixture component and is not observable. The resulting missing data structure can be estimated by following a simulation based approach. Note that the demarginalized mixture model is:

$$m(x,\nu|\theta,p) = \prod_{l=1}^{L}\left(p_l \int_{-1/2}^{1/2} f(x,y|\theta_l)\,dy\right)^{\nu_l}, \qquad \sum_{l=1}^{L}\nu_l = 1 \tag{8}$$

This completion strategy is now quite popular in Bayesian inference for mixtures (see Robert (2001), Robert and Casella (1999)). For a discussion of the numerical and identifiability problems in mixtures inference see Richardson and Green (1997) and Celeux *et al.* (2000).

The Bayesian model for inference on stable mixtures is represented through the DAG in Figure 1.b. As suggested in the literature on gaussian mixtures, we assume a multinomial prior for the completing variable: $\nu \sim \mathcal{M}_L(1, p_1, \ldots, p_L)$ and the standard conjugate Dirichlet prior: $(p_1, \ldots, p_L) \sim \mathcal{D}_L(\delta_1, \ldots, \delta_L)$, for the parameters of the discrete part of the mixture distribution.

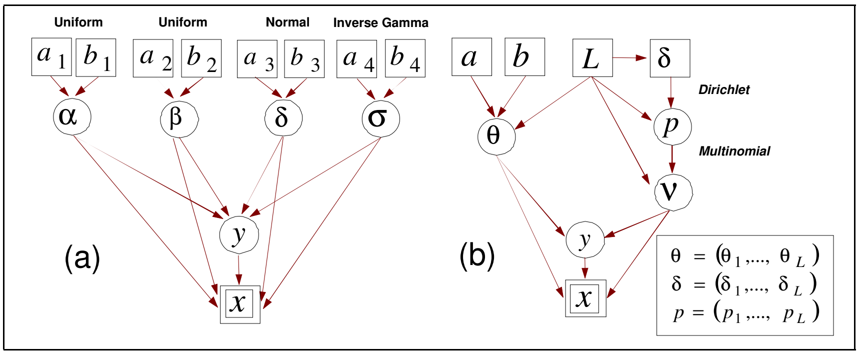

Figure 1. DAG of the Bayesian hierarchical model for inference on stable mixtures. It exhibits the hierarchical structure of priors, and hyperparameters. A single box around a quantity indicates that it is a known constant, a double box indicates the variable is observed and a circle indicates the random variable is not observable. The directed arrows show the dependence structure of the model. Note that the completing variable ν is not observable. Thus, two levels of completion, y and ν, are needed for a stable mixture model.

Given n independent values, $\mathbf{x} = (x_1, \ldots, x_n)$, from a stable mixture the complete posterior distribution of the Bayesian mixture model is:

$$\pi(\theta, p|\mathbf{x},\mathbf{y},\nu) \propto \prod_{i=1}^{n}\left\{\prod_{l=1}^{L}(p_l f(x_i,y_i|\theta_l))^{\nu_{il}}\,\pi(\nu_i)\right\}\pi(\theta)\pi(p). \tag{9}$$

Bayesian inference on the mixture parameters requires the calculation of the expected value from the posterior distribution. A closed form solution of this integration problem does not exist, thus numerical methods are needed. More precisely, the auxiliary variables can be replaced by simulated values and the simulated completed likelihood can be used for calculating the posterior distributions. Then the Gibbs sampler for mixtures of stable distributions allows us to simulate from the posterior distribution:

(i) *Simulate initial values:* $\nu_i^{(0)}, y_i^{(0)}, \quad i = 1, \ldots, n$ *and* $p^{(0)}$ *respectively from:*

$$\nu_i^{(0)} \sim \mathcal{M}_L(1, p_1, \ldots, p_L), \quad y_i^{(0)} \sim f(y_i|\theta, \nu, x_i), \quad p^{(0)} \sim \mathcal{D}_L(\delta, \ldots, \delta). \tag{10}$$

(ii) *Simulate from the full conditional posterior distributions:*

$$\pi(\theta_l|\theta_{-l}, p, \mathbf{x}, \mathbf{y}, \mathbf{v}) \propto \prod_{i=1}^{n} \{f(x_i, y_i|\theta_l)\, p_l\}^{\nu_{il}}\, \pi(\theta_l) \quad l = 1, \ldots, L \tag{11}$$

$$\pi(p_1, \ldots, p_L|\theta, \mathbf{x}, \mathbf{y}, \mathbf{v}) = \mathcal{D}(\delta + n_1(\nu), \ldots, \delta + n_L(\nu)) \tag{12}$$

(iii) *Update the completing variables:*

$$\pi(y_i|\theta, p, \mathbf{x}, \mathbf{y}_{-i}, \mathbf{v}) \propto \exp\left\{1 - \left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\alpha/(\alpha-1)}\right\} \left|\frac{z_i}{\tau_{\alpha,\beta}(y_i)}\right|^{\alpha/(\alpha-1)} \tag{13}$$

$$\pi(\nu_i|\theta, p, \mathbf{x}, \mathbf{y}, \mathbf{v}_{-i}) = \mathcal{M}_L(1, p_1^*, \ldots, p_L^*) \tag{14}$$

for $i = 1, \ldots, n$, where:

$$z = \frac{x - \delta}{\sigma}, \quad n_l(\nu) = \sum_{i=1}^{n} \nu_{il}, \quad p_l^* = \frac{p_l f(x_i, y_i|\theta_l)}{\sum_{l=1}^{L} f(x_i, y_i|\theta_l) p_l}, \quad l = 1, \ldots, L.$$

Observe that simulations from the conditional posterior distribution of Eq.(11) can be obtained by running the Gibbs sampler given in equations (3)-(6), conditionally to the value of the completing variable ν. For a detailed description of the Gibbs sampler see Casarin (2003).

4 Simulation Results and Conclusions

We verify the efficiency of the Gibbs sampler on synthetic data simulated from a two components stable mixture. For each component we assume uniform priors for α and β and informative priors for δ and σ (see Casarin (2003)). Results (see Tab. 4) are obtained on a PC with Intel *1063 MHz* processor, using routines implemented in C/C++.

In this paper we have described a method for performing Bayesian inference for mixture of stable distributions. The approach to the mixture models estimation is quite general and works well in our simulation studies, but a deeper analysis is needed both on synthetic data and on real data.

Table 1. Numerical results - Ergodic Averages over 15,000 Gibbs realisations.

	Dataset:	$0.5S_{1.7}(0.3,1,1)$	$+\ 0.5S_{1.3}(0.5,30,1)$		
Par.	True Value	Starting Value	Estimate$^{(*)}$	Std.Dev.	Acc. Rate
α_1	1.7	1.9	1.66	0.09	0.32
α_2	1.3	1.9	1.36	0.07	0.41
β_1	0.3	0.8	0.28	0.09	0.41
β_2	0.5	0.8	0.37	0.10	0.42
p_1	0.5	0.4	0.52	0.02	-

(*)*Time (sec):9249*

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