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Monica BILLIO, Ludovic CALES, Dominique GUEGAN

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A Performance Measure of Zero-Dollar Long/Short Equally Weighted Portfolios ☆

Monica BILLIO[a], Ludovic CALÈS[*,a,b], Dominique GUÉGAN[b]

[a] *University Ca' Foscari of Venezia - Department of Economics*
Fondamenta San Giobbe - Cannaregio 873 - 30121 Venice, Italy
[b] *Paris School of Economics - MSE-CES - University Paris-1 Panthéon-Sorbonne*
106, avenue de l'hôpital - 75013 Paris, France

Abstract

Sharpe-like ratios have been traditionally used to measure the performances of portfolio managers. However, they suffer two intricate drawbacks (1) they are relative to a peer's performance and (2) the best score is generally assumed to correspond to a "good" portfolio allocation, with no guarantee on the goodness of this allocation. In this paper, we propose a new measure to quantify the goodness of an allocation and we show how to estimate this measure in the case of the strategy used to track the momentum effect, namely the Zero-Dollar Long/Short Equally Weighted (LSEW) investment strategy. Finally, we show how to use this measure to timely close the positions of an invested portfolio.

Key words: Portfolio Management, Performance Measure, Generalized Hyperbolic Distribution
JEL Classification: C13, C44, C46

1. Introduction

The last two decades have seen an explosive growth of the asset management industry. During this period, the analysis of investment performance became an important area of research in quantitative finance. This research, which is axed on Sharpe-like ratios proposed in the 60's [Sharpe (1966), Treynor (1965), Jensen (1968)], has developed the notion of performance as a reward counter-balanced by some risk . The main innovations focused on the definition and modeling of risk [Shadwick and Keating (2002), Darolles et al. (2009)]. Practically, the performance of a portfolio manager, over a given period, is usually computed as the ratio of his excess return over a risk measure [Grinblatt et al. (1994)]. The managers are then ranked according to these ratios, and the manager providing the highest and steadiest returns receives the best score. These measures are convenient because they require no assumption on the strategy of the portfolio managers. However, they suffer two intricate drawbacks. These measures are relative to a peer's performance and irrelevant if no peer is found. We generally assume that the best score corresponds to a "good" portfolio allocation, with no guarantee on the goodness of this allocation.

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*Corresponding author. Phone: +39 348.891.3647 - Fax: +39 041.234.9176

Email addresses: `billio@unive.it` (Monica BILLIO), `ludovic.cales@unive.it` (Ludovic CALÈS), `dguegan@univ-paris1.fr` (Dominique GUÉGAN)

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In this paper, we propose a new measure of performance which quantifies the quality of an allocation. For a given investment strategy, this measure provides the percentage of investable portfolios which are outperformed by the portfolio in consideration. Thus, it quantifies the ability of the portfolio manager to choose his portfolio and allows for a control of his allocation. While the classical measures make no assumption on the manager's strategy, in counterpart, they require the comparison of two peer's performances to be interpreted. On the contrary, this new measure provides a non-relative measure of performance when the investment strategy is specified. We study in details the computation and the properties of the measure for the Zero-Dollar Long/Short Equally Weighted (LSEW) strategy especially in the case of a very large number of assets. Using examples, we calibrate the measure for a portfolio of 10 assets, showing its ability to provide interesting and efficient information about allocations. The results obtained are extendible to large markets using the methodology described in the paper. The assumptions under which we work are quite reasonable and flexible. In particular, we consider the general framework of returns characterized by generalized hyperbolic distributions, Barndorff-Nielsen et al. (1977).

The paper is organized as follows. In Section 2, we introduce the new measure of performance and recall the LSEW investment strategy. We study and compute this measure under fair assumptions. In Section 3, assuming that the returns are characterized by generalized hyperbolic distributions, we detail the influence of their parameters on the performance measure. Section 4 is devoted to three applications. The first one investigates the relevance of the assumptions. The second one shows how to close the positions of a LSEW portfolio using the methodology developed in the previous sections. The last one monitors a LSEW portfolio in real time. Section 5 concludes.

2. Estimation of LSEW Portfolio Performance

In this section, we first introduce the new measure of portfolio performance. Next, we specify the framework inside which our problematic is developed. Finally, we propose a way to compute this performance measure.

2.1. Framework and Definitions

We consider a portfolio manager whose investment policy defines a finite set of portfolios. To provide an objective measure of his allocation performance, we compare the return of his portfolio with the returns of all other investable portfolios. If his portfolio outperforms $S\%$ of all portfolios, we say that it scores S, $S \in [0, 1]$. This score S will be the measure of the manager performance that we investigate in details. Such a measure is interesting because it is independent of the market conditions and it does not need to be compared to a peer's portfolio performance.

In this paper, we investigate this new measure for the LSEW investment strategy. This strategy consists in investing in portfolios which are long/short (i.e. include both long and short positions), zero-dollar (the value of the long positions is equal to the value of the short positions) and equally weighted (each position has the same value in absolute value). In addition, the leverage of these portfolios is fixed to 2:1[1].

[1]The notation 2:1 means that the amount of capital backing the portfolio represents 50% of the portfolio value. It is the minimum amount required under the U.S. Regulation (namely Regulation T). As a consequence, the sum of the absolute values of the weights of the portfolio equals 2.

This strategy is particularly interesting because it is the one used to track the momentum effect in most of the literature [Jeegadeesh and Titman (1993), Rouwenhorst (1998), Chan et al. (2000), Okunev and White (2003), Kazemi et al. (2009) and Billio et al. (2009) among others]. This LSEW strategy is also the base of pair trading [Gatev et al. (1999)].

We denote Γ the set of the investable portfolios γ induced by this strategy. In a market of n assets, we represent a portfolio as a weight vector, i.e. $\gamma = (\gamma(1), \ldots, \gamma(n))'$ where $\gamma(i)$ is the weight associated with asset i, $i = 1, \ldots, n$, and γ' is the transpose of γ. For instance, in a market of 4 assets (A, B, C, D), there are 6 LSEW portfolios. We represent them in Table 1. Note that, in a market of n assets, there are $|\Gamma| = \frac{n!}{\left(\frac{n}{2}!\right)^2}$ LSEW portfolios[2]. So, the number of portfolios increases exponentially with n .

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
Asset A	$1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$
Asset B	$1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$	$-1/2$
Asset C	$-1/2$	$-1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$
Asset D	$-1/2$	$1/2$	$1/2$	$-1/2$	$1/2$	$1/2$

Table 1: Set of the LSEW Portfolios in a market of 4 assets

2.2. Performance measure for the LSEW Strategy

We propose now a way to compute the performance measure S. Given an invested portfolio $\gamma \in \Gamma$, if $N(\gamma)$ is the number of portfolios outperformed by γ, then the performance S associated with γ is

$$S(\gamma) = \frac{N(\gamma)}{|\Gamma|} \tag{1}$$

The computation of $S(\gamma)$ requires the identification of all investable portfolios outperformed by γ. As soon as $|\Gamma|$ is large, this computation is not direct. To deal with this issue, we introduce the relevant theoretical framework.

We consider a market of n assets whose returns $\mathbf{X} = (X_1, \ldots, X_n)'$ have the joint density f. The marginal density of X_i, $i \in \{1, \ldots, n\}$ is denoted f_i , and the vector of order statistics induced by $\mathbf{X}$ is $\mathbf{X_{(n)}} = \left(X_{(1)}, X_{(2)}, ..., X_{(n)}\right)'$. Let be a portfolio $\gamma \in \Gamma$, it returns $\gamma'\mathbf{X}$, then for any realization $\mathbf{x} = (x_1, x_2, ..., x_n)'$, $\mathbf{x}_{(n)}$ being a permutation of the elements of $\mathbf{x}$, it exists a portfolio $\tilde{\gamma} \in \Gamma$ such that

$$\gamma'\mathbf{x} = \tilde{\gamma}'\mathbf{x_{(n)}}$$

In the following, we denote g the density of $\gamma'\mathbf{X}$ and $u_{\tilde{\gamma}}$ the density of $\tilde{\gamma}'\mathbf{X_{(n)}}$. Inside Γ, there exists an optimal portfolio γ_o which provides the highest return for a given realization $\mathbf{x}$, [Billio et al. (2009)]. This optimal portfolio is long the $\frac{n}{2}$ assets which perform the best and is short the $\frac{n}{2}$

[2]For instance, 10 assets lead to a set of 252 portfolios, 20 assets to $184,756$ portfolios and 30 assets to $1.55\ 10^8$ portfolios. So, considering 30 assets would require 4.33 Go of memory to stock the set of portfolios.

assets which perform the worst. Its order statistic representation is

$$\tilde{\gamma}_o(i) = \begin{cases} -2/n & \text{, if} \quad i \leq n/2 \\ 2/n & \text{, if} \quad i > n/2 \end{cases}$$

and its return is equal to

$$\gamma_o'\mathbf{x} = \tilde{\gamma_o}'\mathbf{x_{(n)}} \tag{2}$$

It is helpful to remark that the return of any portfolio $\gamma \in \Gamma$ can be expressed relatively to the return of the optimal portfolio $\tilde{\gamma}_o$. This means that there exists a parameter $k \in [-1, 1]$ such that:

$$\gamma'\mathbf{x} = k\tilde{\gamma_o}'\mathbf{x_{(n)}} \tag{3}$$

By definition, the optimal portfolio γ_o scores $S = 1$.

Coming back to the computation of S, we use the parameter k introduced in (3) which can be associated to any portfolio $\gamma \in \Gamma$. Thus, to obtain an approximation of $S(\gamma_i)$ for a given portfolio γ_i, we approximate the number of portfolios $N(\gamma_i)$ by the expected number of portfolios returning less than k_i times the return of the optimal portfolio. We denote this expected number $\bar{N}(k_i)$:

$$\begin{aligned}
\bar{N}(k_i) &= E\left(\left|\left\{\gamma \in \Gamma | \gamma'\mathbf{X} \leq k_i\tilde{\gamma_o}'\mathbf{X_{(n)}}\right\}\right|\right) \\
&= \sum_{\gamma \in \Gamma} P(\gamma'\mathbf{X} \leq k_i\tilde{\gamma_o}'\mathbf{X_{(n)}}) \\
&= \sum_{\gamma \in \Gamma} \left(\sum_{\tilde{\gamma} \in \Gamma} P\left(\tilde{\gamma}'\mathbf{X_{(n)}} \leq k_i\tilde{\gamma}_o'\mathbf{X_{(n)}}\right) P\left(\gamma'\mathbf{X} = \tilde{\gamma}'\mathbf{X_{(n)}}\right) \right) \\
&= \left(\sum_{\tilde{\gamma} \in \Gamma} P\left(\tilde{\gamma}'\mathbf{X_{(n)}} \leq k_i\tilde{\gamma}_o'\mathbf{X_{(n)}}\right) \right) \left(\sum_{\gamma \in \Gamma} P\left(\gamma'\mathbf{X} = \tilde{\gamma}'\mathbf{X_{(n)}}\right) \right)
\end{aligned} \tag{4}$$

Observing that $\sum_{\gamma \in \Gamma} P\left(\gamma'\mathbf{X} = \tilde{\gamma}'\mathbf{X_{(n)}}\right) = 1$, we obtain

$$\begin{aligned}
\bar{N}(k_i) = \sum_{\tilde{\gamma} \in \Gamma} P\left(\tilde{\gamma}'\mathbf{X_{(n)}} \leq k_i\tilde{\gamma}_o'\mathbf{X_{(n)}}\right) &= \sum_{\tilde{\gamma} \in \Gamma} P\left((\tilde{\gamma}' - k_i\tilde{\gamma}_o')\mathbf{X_{(n)}} \leq 0\right) \\
&= \sum_{\gamma \in \Gamma} P\left((\gamma' - k_i\tilde{\gamma}_o')\mathbf{X_{(n)}} \leq 0\right)
\end{aligned} \tag{5}$$

If we denote f_{γ,k_i} the density of $(\gamma' - k_i\tilde{\gamma_o}')\mathbf{X_{(n)}}$, the relationship (5) becomes:

$$\bar{N}(k_i) = \sum_{\gamma \in \Gamma} \int_{-\infty}^{0} f_{\gamma,k_i}(y)\, dy \tag{6}$$

Plugging relationship (6) in equation (1) provides an approximation of the score for any portfolio

γ_i returning k_i times the return of the optimal portfolio γ_o:

$$\bar{S}(k_i) = \frac{\bar{N}(k_i)}{|\Gamma|} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{-\infty}^{0} f_{\gamma,k_i}(y)\, dy \tag{7}$$

As soon as the number of assets is large, the enumeration of the portfolios of Γ is laborious and the computation of (7) remains difficult. To achieve this computation, we introduce a technical assumption on the returns and provide the resulting expression of $\bar{S}$ in the next proposition.

A_0 : The asset returns are exchangeable[3].

Proposition 1: Let $\mathbf{X}$ be an exchangeable random vector. Denote $\mathbf{X}_{(\mathbf{n})}$ its corresponding vector of order statistics, γ_o the optimal portfolio and γ a portfolio returning k times the return of γ_o, then the approximated score $\bar{S}$ for a portfolio γ is equal to

$$\bar{S}(k) = \int_{-\infty}^{0} (g * h_k)(y)\ dy \tag{8}$$

where $*$ stands for the convolution product; g is the density of $\gamma'\mathbf{X}$ and h_k the density of $-k\tilde{\gamma_o}'\mathbf{X}_{(\mathbf{n})}$, where $\tilde{\gamma_o}$ is the ordered representation of the optimal portfolio γ_o.

Proof: The proof of this proposition is postponed in Appendix A.

In practice, the computation of $\bar{S}$ using the expression (8) requires to determine the density g corresponding to a linear combination of n random variables, the density h_k corresponding to the linear combination of n order statistics and the convolution product between g and h_k. For the computation of h_k, we use the methodology developed by Arellano-Valle and Genton (2007). Nevertheless, their result is difficult to apply as soon as n is large. In that case Monte Carlo simulations are appropriate. We illustrate now our approach with an example.

Let consider a market of 10 assets - inducing $|\Gamma| = 252$ LSEW portfolios - whose returns are independent and identically distributed (i.i.d.), and follow a normal distribution with mean 0 and variance 0.01. Then, the density g is the sum of 10 independent Gaussian densities, and we copmute the density h_k using Monte Carlo simulations, computing $\gamma'\mathbf{x} - k\tilde{\gamma_o}'\mathbf{x}_{(\mathbf{n})}$ for each realization $\mathbf{x}$, with $\gamma \in \Gamma$ and $\tilde{\gamma_o}$ the optimal portfolio obtained by ranking the 10 returns. In Figure 1, we represent $\bar{S}$ as a function of k. We remark that the score of a portfolio γ_i providing $k_i = 60\%$ of the return of the optimal portfolio is $\bar{S} = 92\%$. This means that only 8% of the LSEW portfolios provide an higher return than γ_i, on average.

[3] We recall that a sequence of random variables is exchangeable if, for any permutation of these random variables, the joint probability distribution of the rearranged sequence is the same as the joint probability of the original sequence, Arellano-Valle and Genton (2007). In particular, a sequence of i.i.d. random variables is exchangeable.

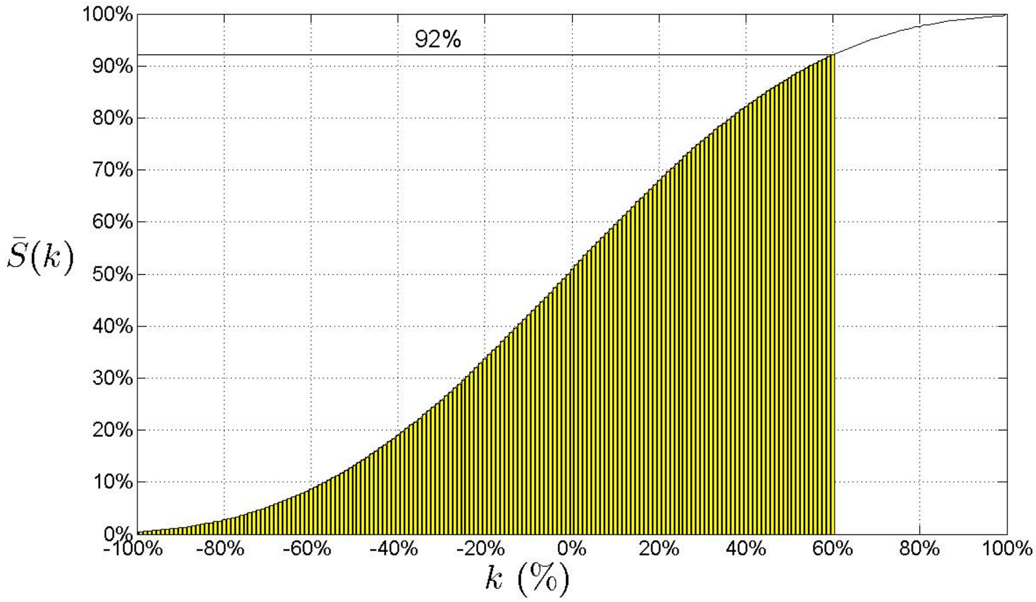

Figure 1: $\bar{S}(k)$ in a market of 10 assets whose returns are i.i.d. $N(0, 1\%)$

3. The determinants of the score

Financial asset returns are well known to have distributions which are asymmetric and leptokurtic. Thus, it is important to be able to compute $\bar{S}$ when the asset returns are modeled by distributions more complex than the Gaussian one. Here, we assume that the observations $\mathbf{X} = (X_1, \ldots, X_n)$ are characterized by a multivariate generalized hyperbolic distribution, which is among the most general distributions used in finance [Eberlein et al. (1995), Prause (1999) and Fajardo et al. (2009) among others], and we identify the distribution's parameters affecting $\bar{S}$.

A multivariate generalized hyperbolic distributions $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$ can be represented as a normal mean-variance mixture [Barndorff-Nielsen et al. (1982)], and is characterized by six parameters: the mean $\mu \in R^d$, the variance-covariance matrix $\Sigma \in R^{d \times d}$, the skewness parameter, $\kappa \in R^d$, and the shape parameters λ, χ and ψ. In the following, we use this very flexible class of distributions to characterize the assets on a market since it contains a lot of well known distributions (Laplace, Student-t, normal inverse Gaussian, inverse Gaussian, etc.). We introduce now a new assumption which permits to extend the results of the Proposition 1.

A_1 : The asset returns are characterized by a multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$.

Under the assumptions (A_0) and (A_1), the vector $\mathbf{X}$ is an exchangeable random vector characterized by a multivariate generalized hyperbolic distribution , and $\Sigma = \sigma^2 [(1-\rho)I_n + \rho \mathbf{1_n}\mathbf{1_n}']$ where σ is the variance of $\mathbf{X}$ and ρ is the correlation between X_i and X_j, $i, j \in \{1, \ldots, n\}$, [Arellano-Valle and Genton (2007)].

Proposition 2: Let $\mathbf{X}$ be an exchangeable random vector distributed according to a multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$, $\mathbf{X_{(n)}}$ being the random vector of its order statistics and γ_o the optimal portfolio, then

$$\bar{S}(k) = \int_{-\infty}^{0} v_k(y)\, dy \tag{9}$$

with v_k the density function of $Z - k\tilde{\gamma}_o{}'\mathbf{U_{(n)}}$ where Z is an elliptically contoured random variable $EC_1\left(0, \frac{4}{n}, \phi^{(1)}\right)$, and $\mathbf{U_{(n)}}$ is the vector of order statistics induced by $\mathbf{U_n} \sim EC_n\left(\mathbf{0}, I_n, \phi^{(n)}\right)$ with the density generator $\phi^{(m)}$ given by

$$\phi^{(m)}(u) = C_m \frac{K_{\lambda-\frac{m}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{m}{2}-\lambda}} \tag{10}$$

with C_m a normalizing constant, and K_ν the modified Bessel function of the third kind.

Proof: The proof of this proposition is postponed in Appendix B.

We remark that the score $\bar{S}$ depends only on the shape parameters λ, χ and ψ. Therefore, in the case of Gaussian i.i.d. returns as presented in Figure 1, we can observe that $\bar{S}(k)$ is impacted neither by the mean of the returns nor by its variance.

4. Applications and Empirical Validation

We provide three applications showing the interest of our methodology. The first one investigates the impact of assumptions (A_0) and (A_1); the second one proposes a new exit strategy for managers willing to close their positions and the third one illustrates the usefulness of this measure for monitoring portfolios in real time.

4.1. Empirical relevance of the assumptions (A_0) and (A_1)

Let consider a market whose returns follow an arbitrary random vector $\mathbf{X}$. In order to verify that the assumptions (A_0) and (A_1) are not too strong to be relevant, we compare the score $\bar{S}$ computed assuming (A_0) and (A_1) and the score $\breve{S}(k)$ computed as the average percentage of portfolios returning less than k times the return of the optimal portfolio, using the relationship (1). Practically, to obtain $\breve{S}(k)$, we need to enumerate all the LSEW portfolios. In our example, we restrict ourselves to a market of 10 assets, corresponding to 252 LSEW portfolios. The market is composed by the 10 Datastream sectorial world indices[4], with their monthly returns, from January 1975 to May 2008. To compute $\bar{S}$, we assume that the asset returns are stationary, exchangeable and characterized by a generalized hyperbolic (GH) distribution. Here, we fit the assets' returns with a Normal Inverse Gaussian (NIG) distribution $(\lambda = -0.5)$[5]. In order to illustrate the accuracy of our choice, we propose in Figure 2 the Q-Q plots corresponding to the adjustments of a Gaussian distribution and a NIG one over the assets' returns empirical distribution function.

[4]WORLD-DS Oil & Gas, Basic Mats, Industrials, Consumer Gds, Health Care, Consumer Svs, Telecom, Utilities, Financials, Technology

[5]The estimation has been performed using the Matlab package developed by Saket Sathe. It is available on-line in the Matlab©Central web site: http://www.mathworks.com

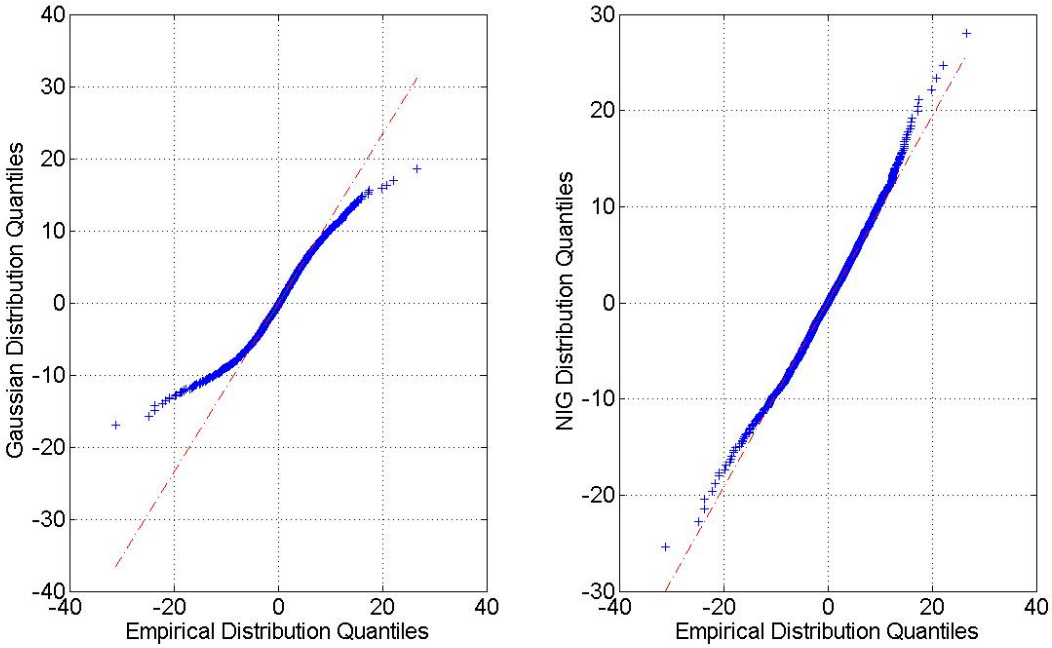

Figure 2: Q-Q plots of the assets returns

The Q-Q plots clearly show the superiority of the fit obtained using the NIG distribution. In Table 2, we present the p-values of the Kolmogorov-Smirnov test considering our empirical sample of 4010 returns (10 (assets) $\times$ 401 (months)). Under the null hypothesis, we first assume that the empirical sample is drawn from the Gaussian distribution, and next from the NIG distribution.

	Gaussian dist.	NIG dist.
p-value	$2.0175\ 10^{-5}$	0.7846

Table 2: Kolmogorov-Smirnov test p-values

The test validates the choice of the NIG distribution for the returns (p-value higher that 0.05). Nevertheless, in the following we provide the scores $\bar{S}$ issued from the Gaussian hypothesis denoted $\bar{S}_N$, and from the NIG hypothesis denoted $\bar{S}_{NIG}$. Both scores are computed using Monte Carlo simulations using the 4010 returns. In Figure 3, we represent $\bar{S}_{NIG}$ with the blue line, $\breve{S}$ with the red line, and $\bar{S}_N$ with the black dot line.

We observe that $\bar{S}_{NIG}$ (blue line) and $\breve{S}$ (red line) coincide. The blue line covers the red one almost everywhere. Thus, it seems that the assumptions (A_0) and (A_1) used to compute $\bar{S}_{NIG}(k)$ do not create any relevant bias in the computation of the score. When we assume that the data set comes from a Gaussian random vector - which is invalidated in Table 2 - we observe a difference between $\bar{S}_N$ (black dashed line) and $\breve{S}$ (red line). The score $\bar{S}_N$ underestimates $\breve{S}$ for negative k and overestimates it for positive k.

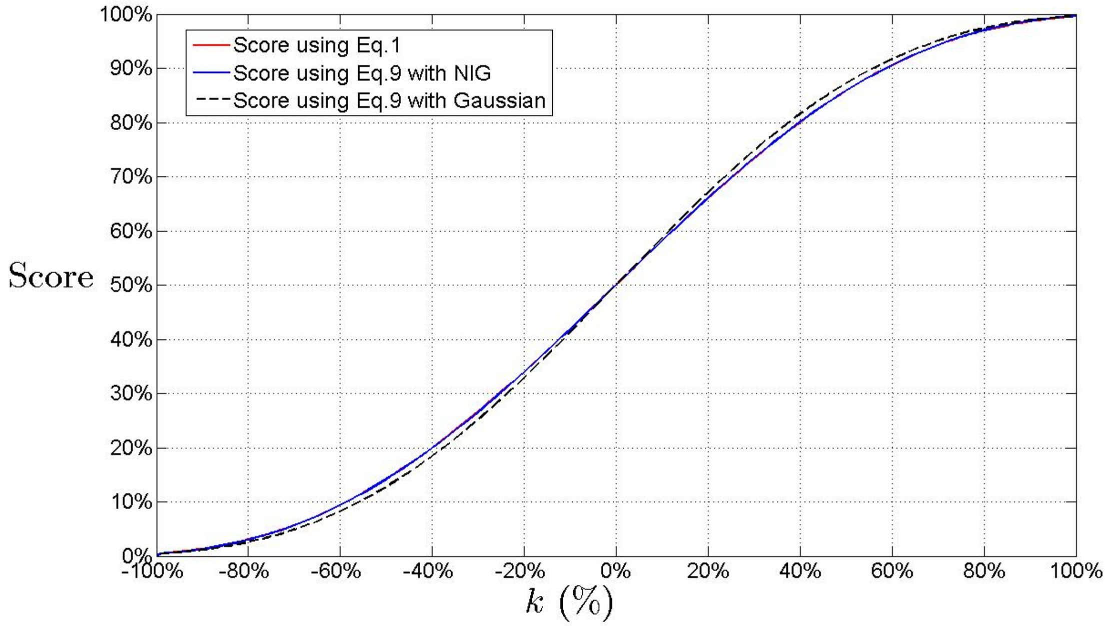

Figure 3: Scores $\breve{S}$, $\bar{S}_{NIG}$ and $\bar{S}_N$ functions of k

4.2. Application to exit positions

An interesting application of the measure S is to appreciate the opportunity to close positions. Indeed, this new measure quantifies the goodness of an allocation for given market conditions. We consider a manager whose portfolio is invested, and we assume that, due to fluctuating market conditions, the knowledge of his portfolio's return is not enough to decide to close his positions. Suppose now that the portfolio provides a high score, $S = 90\%$, then its return is among the highest possible ones for a given time and given market conditions. Roughly speaking, the manager has performed the most it was possible to perform. Consequently, a reasonable decision is to close the positions and try to do his best over the next period.

As an illustration, we consider the following LSEW portfolio γ invested on the 10 Datastream world sectorial indices, and in Table 3 we report the weights of this portfolio.

Oil&Gas	Basic Mat.	Industry	Consumer Gds	Health Care
-0,2	-0,2	-0,2	-0,2	0,2
Consumer Svs	Telecom	Utilities	Finance	Techno
0,2	-0,2	0,2	0,2	0,2

Table 3: Invested Portfolio γ

Suppose it has been invested the 31/07/2009 at the closing. Then, the portfolio manager can follow the score of his portfolio over the next days. We give the evolution of the score $S(\gamma)$ in Figure 4 from July 31, 2009 to September 30, 2009. We observe that the first days the score of the portfolio is poor: indeed it is starting below 30%. However after few days, it performs particularly well because it is above 80%. Finally, after 30 days it drops to the median score (around 50%). Note that thanks to the symmetry of the LSEW strategy a score below 50% corresponds to a negative return and respectively a score above 50% corresponds to a positive return. Clearly, it would have been timely for the manager to close his positions between the days 10 and 25.

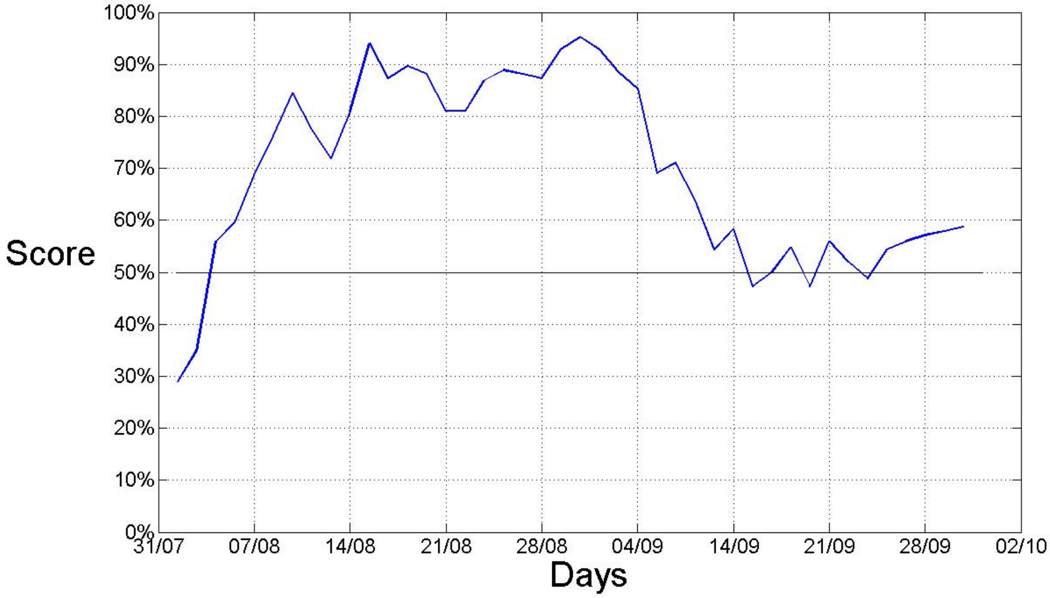

Figure 4: Score of a LSEW portfolio invested the 31/07/2009, at the closing, over the 42 following days

4.3. Monitoring with the Estimated Score

Given a portfolio γ invested at date $t = 0$ we compare its scores $(\bar{S}_t)_{t>0}$ defined in (9) with its scores $(S_t)_{t>0}$ defined in (1) at the dates $t \geq 1$. We assume that the assets' log-returns are governed by a strictly stationary process $(\mathbf{X_t})_{t>0}$ characterized by a NIG distribution. Given the realizations $(\mathbf{x_t})_{t>0}$, the log return of the portfolio γ is approximatively $\gamma' \left(\sum_{i\in\{1,..,t\},t\geq 1} \mathbf{x_i}\right)$ at each time t. From relationship (3) we compute the sequence $(k_t)_{t>0}$ associated with γ at the dates t, $t > 0$. The stationarity property implies that $\bar{S}_t(k_t) = \bar{S}(k_t)$ for all t. Therefore, the scores $(\bar{S}(k_t))_{t>0}$ derive from the previous sub-sections and the scores $(S_t)_{t>0}$ are directly computed as the percentage numbers of outperformed portfolios for the given realizations. To illustrate our purpose, we use the same portfolio and data set introduced in Sub-section 4-4.2. The values of $(\bar{S}(k_t))_{t>0}$ and $(S_t)_{t>0}$ are reported in Figure 5. We observe that $\bar{S}$ correctly fits S.

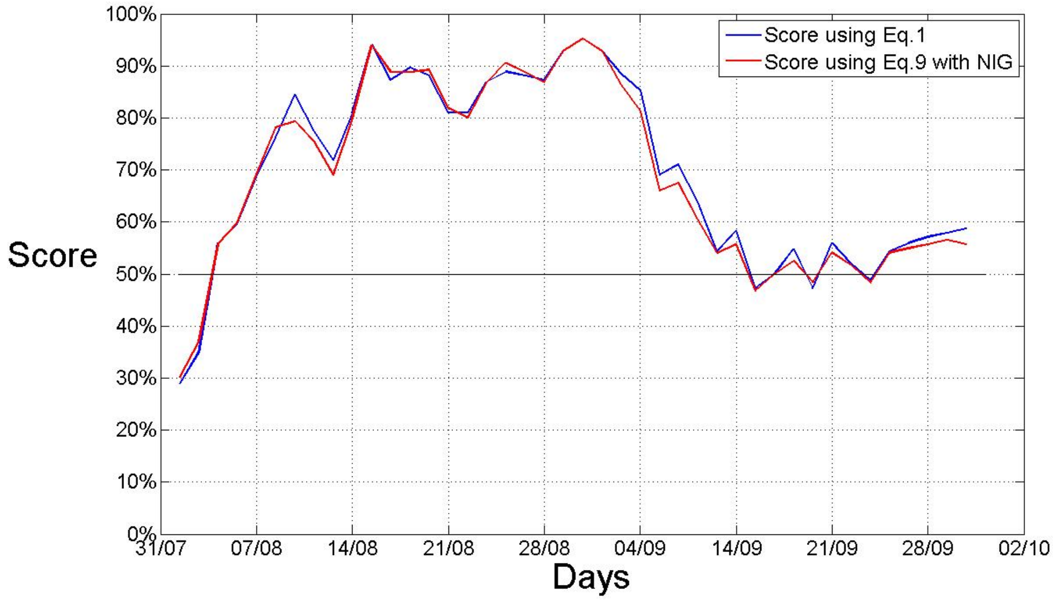

Figure 5: S_t and $\bar{S}_t$ of the LSEW portfolio γ invested the 31/07/2009, at the closing, over the 42 following trading days

Now, we provide a quantitative criteria for the estimation of S based on the mean absolute error, computing the errors produced by $\bar{S}$ and $\breve{S}$. As $\bar{S}(k)$ is invariant over time, we focus on one-step ($t \leq 1$) daily periods , over the whole sample, from 01/02/1973 to 09/24/2009 (N=9262 observations), and we compute the error for each portfolio $\gamma \in \Gamma$. We obtain

$$\bar{E} = \frac{1}{N|\Gamma|} \sum_{\gamma \in \Gamma, n \in \{1,\ldots,N\}} |\bar{S}(k_n) - S_n(\gamma)| = 2.90\%$$

and

$$\breve{E} = \frac{1}{N|\Gamma|} \sum_{\gamma \in \Gamma, n \in \{1,\ldots,N\}} |\breve{S}(k_n) - S_n(\gamma)| = 2.22\%$$

The two errors are competitive and justify the relevance of the assumptions (A_0) and (A_1).

Note that under stationarity condition and a correct choice for the distribution of the returns $\mathbf{X}$, the score $\bar{S}$ is obtained only through the computation of the parameter k. Thus, this score is suitable for real time applications as opposed to S which requires a complete enumeration for each realization.

5. Conclusion

This paper proposes a new methodology to quantify the goodness of an allocation and provides an estimate of this quantification in the case of the Zero-Dollar Long/Short Equally Weighted investment strategy. This work can be viewed as a complementary contribution to the debate concerning the performance measure of portfolio managers. Most of the previous works require a peer system to appreciate a manager's performance, and this approach permits to be free of this constraint. Consequently, it releases new information which enables a manager to appreciate the opportunity to close his positions. It also provides a nice and simple way to value the performance of an invested portfolio in real time.

A. Proof of Proposition 1

Let $\mathbf{X}$ be an absolutely continuous exchangeable random vector, $\mathbf{X_{(n)}}$ be the corresponding random vector of its order statistics. Let be $\gamma_i \in \Gamma$ any portfolio, γ_o the optimal portfolio and g the density of $\gamma_i'\mathbf{X}$, then we have

$$P(\gamma_i'\mathbf{X} = y) = \sum_{\gamma \in \Gamma} P(\gamma'\mathbf{X_{(n)}} = y) P(\gamma_i'\mathbf{X} = \gamma'\mathbf{X_{(n)}}) \tag{11}$$

As $\mathbf{X}$ is an exchangeable random vector, then γ has the same probability to be the representation of γ_i in terms of order statistics, thus

$$P(\gamma_i'\mathbf{X} = \gamma'\mathbf{X_{(n)}}) = \frac{1}{|\Gamma|} \tag{12}$$

Plugging relationship (11) in expression (12) leads to

$$P(\gamma_i'\mathbf{X} = y) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} P(\gamma'\mathbf{X_{(n)}} = y) \tag{13}$$

Denoting u_γ the density function of $\gamma'\mathbf{X_{(n)}}$, we remark that

$$g = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} u_\gamma \tag{14}$$

From (7), we know that if the portfolio γ_i returns k_i times the return of the optimal portfolio γ_o, we have

$$\bar{S}(k_i) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \int_{-\infty}^{0} f_{\gamma,k_i}(y)\, dy \tag{15}$$

where f_{γ,k_i} is the density function of $(\gamma - k_i\tilde{\gamma}_o)'\mathbf{X_{(n)}}$. Denoting h_{k_i} the density of $-k_i\tilde{\gamma}_o'\mathbf{X_{(n)}}$, we obtain

$$\bar{S}(k_i) = \frac{1}{|\Gamma|} \int_{-\infty}^{0} \sum_{\gamma \in \Gamma} (u_\gamma * h_{k_i})(y)\, dy \tag{16}$$

Using the property of distributivity of the convolution product, the relationship (16) can be rewritten as follows:

$$\bar{S}(k_i) = \frac{1}{|\Gamma|}\int_{-\infty}^{0}\left(\left(\sum_{\gamma\in\Gamma}u_\gamma\right) * h_{k_i}\right)(y)\,dy \tag{17}$$

Now, from (14), we have $\sum_{\gamma\in\Gamma}u_\gamma = |\Gamma|g$, and the relationship (17) becomes:

$$\begin{aligned}\bar{S}(k) &= \frac{1}{|\Gamma|}\int_{-\infty}^{0}\left(|\Gamma|g * h_{k_i}\right)(y)\,dy \\ &= \int_{-\infty}^{0}\left(g * h_{k_i}\right)(y)\,dy \end{aligned}\tag{18}$$

The proof of Proposition 1 is complete.

B. Proof of Proposition 2

Let $\mathbf{X} = (X_1, X_2, ..., X_n)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$, $\mathbf{X_{(n)}}$ be the random vector of its order statistics, $\gamma \in \Gamma$ be a portfolio and γ_o the optimal portfolio.

In Proposition 1, we established that $\bar{S}$ depends on $(g * h_k)$. Here, we need to study separately g, the distribution of $\gamma'\mathbf{X}$, and h_k, the distribution of $\tilde{\gamma_o}'\mathbf{X_{(n)}}$. We begin with the study of g in Corollary 1.

Corollary 1: Let $\mathbf{X} = (X_1, X_2, ..., X_n)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$ and $\gamma \in \Gamma$ be any LSEW portfolio, then $\gamma'\mathbf{X}$ is distributed according to an elliptically contoured distribution such that

$$\gamma\mathbf{X} \sim EC_1\left(0, \sigma^2(1-\rho)\frac{4}{n}, \phi^{(1)}\right) \tag{19}$$

where the density generator $\phi^{(1)}$ is given by

$$\phi^{(1)}(u) = C_1\frac{K_{\lambda-\frac{1}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{1}{2}-\lambda}} \tag{20}$$

with C_1 a normalizing constant and K_ν the modified Bessel function of the third kind.

■

Proof of Corollary 1:

From McNeil et al. (2005), we know that the generalized hyperbolic distributions are closed under linear transformation. So, if $\mathbf{X} \sim GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$ and $\mathbf{Y} = \gamma'\mathbf{X}$ where $\gamma \in R^n$, then

$$\mathbf{Y} \sim GH_1\left(\lambda, \chi, \psi, \gamma'\mu, \gamma'\Sigma\gamma', \gamma'\kappa\right) \tag{21}$$

In our case, we have

- γ is a LSEW portfolio, so $\gamma'\mathbf{1_n} = 0$, thus $\gamma'\mu = 0$ and $\gamma'\kappa = 0$
- the random variables are exchangeable, so $\Sigma = \sigma^2 \left[(1-\rho)I_n + \rho\mathbf{1_n}\mathbf{1_n}'\right]$, where σ is the scale and ρ is the correlation

Consequently, $\gamma'\mathbf{X}$ is distributed as follows

$$\gamma'\mathbf{X} \sim GH_1\left(\lambda, \chi, \psi, 0, \sigma^2(1-\rho)\frac{4}{n}, 0\right) \tag{22}$$

i.e. $\gamma'\mathbf{X}$ follows a symmetric generalized hyperbolic distribution.

From Schmidt (2003) (p.54, definition 3.2.12), we know that the symmetric generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, 0)$ is the elliptically contoured distribution $EC_n(\mu, \Sigma, \phi)$ where the density generator $\phi^{(n)}$ is given by

$$\phi^{(n)}(u) = C_n \frac{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{n}{2}-\lambda}} \tag{23}$$

with C_n a normalizing constant defined in Schmidt (2003) (formula 5.3) and K_ν the modified Bessel function of the third kind.

So, in our case, we have

$$\gamma'\mathbf{X} \sim EC_1\left(0, \sigma^2(1-\rho)\frac{4}{n}, \phi^{(1)}\right) \tag{24}$$

where

$$\phi^{(1)}(u) = C_1 \frac{K_{\lambda-\frac{1}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{1}{2}-\lambda}} \tag{25}$$

The proof of Corollary 1 is complete.

□

Now, we investigate the distribution of $\tilde{\gamma_o}'\mathbf{X_{(n)}}$:

Corollary 2: Let $\mathbf{X} = (X_1, X_2, ..., X_n)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$, $\mathbf{X_{(n)}}$ be the random vector of its order statistics and $\tilde{\gamma_o} \in \Gamma$ be the order statistics representation of the optimal portfolio, then $\tilde{\gamma_o}'\mathbf{X_{(n)}}$ is distributed according to an elliptically contoured distribution such that

$$\tilde{\gamma_o}'\mathbf{X_{(n)}} =^d \sigma\sqrt{1-\rho}\,\tilde{\gamma_o}'\mathbf{U_{(n)}} \tag{26}$$

where $\rho \in [0,1)$ and $\mathbf{U_{(n)}}$ is the vector of order statistics induced by the spherically contoured random vector $\mathbf{U} \sim EC_n\left(\mathbf{0}, I_n, \phi^{(n)}\right)$ with $\phi^{(n)}$ given by

$$\phi^{(n)}(u) = C_n \frac{K_{\lambda-\frac{n}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{n}{2}-\lambda}} \tag{27}$$

with C_n a normalizing constant and K_ν the modified Bessel function of the third kind.

■

Proof of Corollary 2:

From Arellano-Valle and Genton (2007) (Corollary 1), we have

$$\tilde{\gamma_o}'\mathbf{X_{(n)}} =^d \left(\tilde{\gamma_o}'\mathbf{X}|\Delta\mathbf{X} \geq \mathbf{0}\right) \tag{28}$$

where Δ is such that $\mathbf{\Delta X} = (X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})'$. We note that $\Delta\Delta' = (\delta_{i,j})$, δ being the Kronecker product, with $\delta_{i,i} = 2$, $\delta_{i-1,i} = \delta_{i+1,i} = -1$ and $\delta_{i,j} = 0$ otherwise.

The generalized hyperbolic distributions are closed under linear transformation and $\mathbf{X}$ is an exchangeable random vector, so we have

$$\mathbf{\Delta X} \sim GH_{n-1}\left(\lambda, \chi, \psi, 0, \sigma^2(1-\rho)\Delta\Delta', 0\right)$$

Thus, from Schmidt (2003) as seen in Corollary 1, $\mathbf{\Delta X}$ follows an elliptically contoured distribution

$$\mathbf{\Delta X} \sim EC_{n-1}\left(0, \sigma^2(1-\rho)\Delta\Delta', \phi^{(n-1)}\right) \tag{29}$$

where

$$\phi^{(n-1)}(u) = C_{n-1} \frac{K_{\lambda-\frac{n-1}{2}}(\sqrt{\psi(\chi+u)})}{(\sqrt{\chi+u})^{\frac{n-1}{2}-\lambda}} \tag{30}$$

Since $\tilde{\gamma_o}$ is a LSEW portfolio, relationship (24) holds. So, from expression (24) and expression (29), we have

$$\begin{cases} \tilde{\gamma_o}'\mathbf{X} \sim EC_1\left(0, \sigma^2(1-\rho)\frac{4}{n}, \phi^{(1)}\right) \\ \mathbf{\Delta X} \sim EC_{n-1}\left(0, \sigma^2(1-\rho)\Delta\Delta', \phi^{(n-1)}\right) \end{cases}$$

which are the intermediary results obtained in the proof of Corollary 3 in Arellano-Valle and Genton (2007). Thus, Corollary 3 can be used here, and we extend it to generalized hyperbolic distributions. It follows

$$\tilde{\gamma_o}'\mathbf{X_{(n)}} =^d \sigma\sqrt{1-\rho}\,\tilde{\gamma_o}'\mathbf{U_{(n)}} \tag{31}$$

where $\mathbf{U_{(n)}}$ is the vector of order statistics induced by the spherically contoured random vector $\mathbf{U} \sim EC_n\left(\mathbf{0}, I_n, \phi^{(n)}\right)$ and $\rho \in [0,1)$.

The proof of Corollary 2 is complete.

□

Now, we prove Proposition 2. From Corollary 1 and denoting $Z \sim EC_1\left(0, \frac{4}{n}, \phi^{(1)}\right)$, we have

$$\gamma' \mathbf{X} =^d \sigma\sqrt{1-\rho}Z \tag{32}$$

Then, from Corollary 2 and relationship (32), we have

$$\gamma'\mathbf{X} - k\tilde{\gamma_o}'\mathbf{X_{(n)}} =^d \sigma\sqrt{1-\rho}\left(Z - k\tilde{\gamma_o}'\mathbf{U_{(n)}}\right) \tag{33}$$

Let denote v_k the density function of $Z - k\tilde{\gamma_o}'\mathbf{U_{(n)}}$. From (33), we have the following expression of $\bar{S}(k)$:

$$\bar{S}(k) = \int_{-\infty}^{0} (g * h_k)(y)\ dy = \int_{-\infty}^{0} v_k(y)\ dy \tag{34}$$

So, $\bar{S}(k)$ is independent of μ, σ, ρ and κ.

The proof of Proposition 2 is complete.

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