A Bayesian Stochastic Correlation Model for Exchange Rates

Roberto Casarin and Marco Tronzano and Domenico Sartore

Abstract This paper applies a Bayesian multivariate stochastic correlation model to the detection of correlation regimes in exchange rates. We follow a MCMC approach to parameter and latent variable estimation and provide evidence of significant differences between volatility and correlation dynamics.

Key words: Bayesian Inference, Multivariate Stochastic Volatility, Markov-switching, Stochastic Correlation

1 Introduction

Modelling and forecasting contagion between financial markets are crucial and challenging issues in financial management. The time-variations in the financial return volatilities and in the correlations between returns are two of the most relevant features of the contagion dynamics. The seminal dynamic volatility paper of [4] has generated to main streams of literature: GARCH models and Stochastic Volatility (SV) models (e.g., see [10] and [7]). Earlier contributions in these areas focused on univariate time series modelling. Subsequently, the attention has shifted to multivariate models with dynamic covariances (e.g., see [8] and [5] for multivariate GARCH and [6] for multivariate SV).

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The aim of this paper is twofold. First, we provide a joint estimation of the return mean, volatility and correlation of exchange rates. Secondly, we investigate the presence of independent shifts in the volatility and correlation dynamics. In this sense we extend the empirical findings for exchange rates due to [9], [1] and [11].

The structure of the paper is as follows. Section 2 introduces the stochastic correlation model. Section 3 describes briefly the Bayesian inference approach used. Section 4 presents the results for three daily exchange rate series. Section 5 concludes.

2 A Markov-switching Stochastic Correlation Model

Let $\mathbf{y}_t = (y_{1t}, \ldots, y_{mt})' \in \mathbb{R}^m$ be a vector-valued time series, representing the log-differences in the spot exchange rates, $\mathbf{h}_t = (h_{1t}, \ldots, h_{mt})' \in \mathbb{R}^m$ the log-volatility process, $\Sigma_t \in \mathbb{R}^m \times \mathbb{R}^m$ the time-varying covariance matrix, and $s_{1,t} \in \{0, 1\}$ a two-states Markov chain. We consider here a special case of the stochastic correlation model (*MSSC*) given in [3]

$$\mathbf{y}_t = \mathbf{a}_{00} + \mathbf{a}_{01} s_{1,t} + (A_{10} + A_{11} s_{1,t})\,\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim \mathcal{N}_m(\mathbf{0}, \Sigma_t) \quad (1)$$

$$\mathbf{h}_t = \mathbf{b}_{00} + \mathbf{b}_{01} s_{1,t} + (B_{10} + B_{11} s_{1,t})\,\mathbf{h}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \mathcal{N}_m(\mathbf{0}, \Sigma_\eta) \quad (2)$$

with $\boldsymbol{\varepsilon}_t \perp \boldsymbol{\eta}_s \;\forall\, s, t$, and $\mathcal{N}_m(\boldsymbol{\mu}, \Sigma)$ the m-variate normal distribution, with mean $\boldsymbol{\mu}$ and covariance matrix Σ, and $\mathbf{a}_{00}, \mathbf{a}_{01}$, A_{10}, A_{11}, $\mathbf{b}_{00}$, $\mathbf{b}_{01}$, $\mathbf{b}_{10}$ and $\mathbf{b}_{11}$ parameters to be estimated. The probability law governing $s_{1,t}$ is $s_{1,t} \sim \mathbb{P}(s_{1,t} = j | s_{1,t-1} = i) = p_{1,ij}$, with $p_{1,ij}$, $ij, \in \{0, 1\}$. As regards the conditional covariance matrix Σ_t, we use the decomposition (see [2]):

$$\Sigma_t = \Lambda_t \Omega_t \Lambda_t, \quad (3)$$

with $\Lambda_t = \mathrm{diag}\{\exp\{h_{1t}/2\}, \ldots, \exp\{h_{kt}/2\}\}$, a diagonal matrix with the log-volatilities on the main diagonal and $\Omega_t = \tilde{Q}_t^{-1} Q_t \tilde{Q}_t^{-1}$ the stochastic correlation matrix with $\tilde{Q}_t = (\mathrm{diag}\{\mathrm{vecd}\, Q_t\})^{1/2}$ and $Q_t^{-1} \sim \mathcal{W}_m(\nu, S_{t-1})$ where:

$$S_{t-1} = \frac{1}{\nu} Q_{t-1}^{-d/2} \bar{Q}_t Q_{t-1}^{-d/2},\; \bar{Q}_t = \left[\lambda_{s_{2,t}} \bar{D}_{s_{1,t}} + (1 - \lambda_{s_{2,t}}) I_m\right],$$

$$\bar{D}_{s_{1,t}} = \sum_{k=0,1} \mathbb{I}_{\{k\}}(s_{1,t}) \bar{D}_k$$

and $\bar{D}_k$, $k \in \{0, 1\}$, is a sequence of positive definite matrices, which capture the long-term dependence structure between series in the different regimes, and d is a scalar parameter. The correlation-switching process $s_{2,t} \in \{0, 1\}$ has transition probability: $s_{2,t} \sim \mathbb{P}(s_{2,t} = j | s_{2,t-1} = i) = p_{2,ij}$.

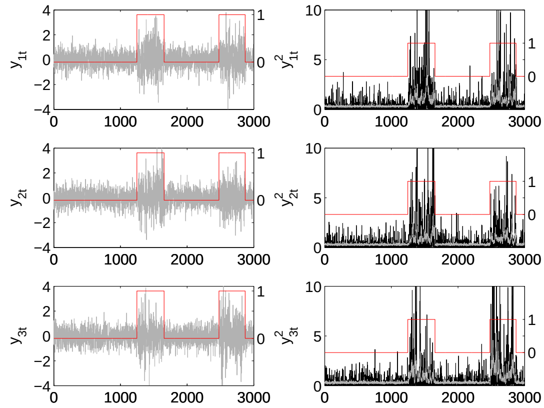

Fig. 1 Simulated trajectory of the MSSC model. Left: the observable variables (gray lines, left axis). Right: square of the observable variables (black lines, left axis). In all plots, the Markov-switching process, s_{1t}, (stepwise, right axis).

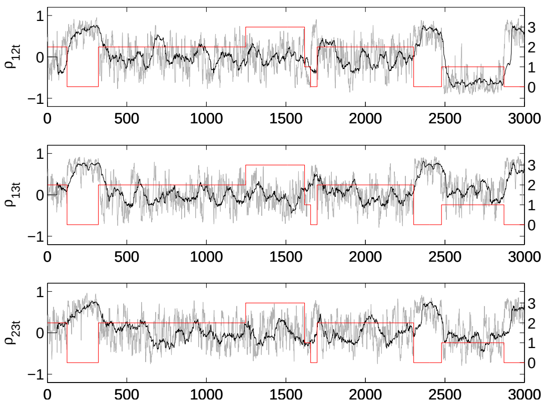

Fig. 2 Simulated trajectory of the MSSC model. In each plot the stochastic correlation process (gray line, left axis), and the sequential estimation of the empirical correlation (black line, left axis). In all plots, the Markov-switching process $s_t = s_{1,t} + 2s_{2,t}$ (stepwise line, right axis) is generated by the composition of $s_{1,t}$ with $s_{2,t}$.

A simulated trajectory of this process is given in Fig. 1-2, where we set $\mathbf{b}_{00} = -1.1\boldsymbol{\iota}$, $\mathbf{b}_{01} = 1.06\boldsymbol{\iota}$, $\mathbf{B}_{10} = 0.12I_3$, $\mathbf{B}_{11} = 0.5I_3$, $\Sigma_\eta = \mathrm{diag}((0.03, 0.06, 0.08)')$, where $\boldsymbol{\iota} = (1, 1, 1)'$. For the correlation process we set

$$\bar{D}_0 = \begin{pmatrix} 1.01 & -0.09 & -0.11 \\ -0.09 & 1.14 & -0.08 \\ -0.11 & -0.09 & 1.10 \end{pmatrix}, \ \bar{D}_1 = \begin{pmatrix} 1.01 & 0.11 & -0.01 \\ 0.11 & 1.14 & 0.02 \\ -0.01 & 0.02 & 1.02 \end{pmatrix}$$

$\lambda_0 = 1$, $\lambda_1 = 0.02$, $\nu = 28$ and $d = 0.9$. For the two switching processes we consider: $p_{1,11} = 0.90$, $p_{1,22} = 0.94$, $p_{2,11} = 0.97$ and $p_{2,22} = 0.97$. The trajectories of the three variables exhibit volatility clustering, and the square of the observables represents an effective graphical tool to detect the presence of volatility regimes (see Fig. 1). Fig. 2 shows that a recursive estimation of the correlation can be useful for detecting correlation changes.

3 Bayesian Inference

Define $\mathbf{y} = (\mathbf{y}_1', \ldots, \mathbf{y}_T')'$, and $\mathbf{z} = (\mathbf{h}, \mathbf{q}, \mathbf{s}_1, \mathbf{s}_2)$, with $\mathbf{h} = (\mathbf{h}_0', \ldots, \mathbf{h}_T')'$, $\mathbf{s}_k = (s_{k,0}', \ldots, s_{k,T}')'$, $k = 1, 2$, and $\mathbf{q} = (\text{vech}(Q_0)', \ldots, \text{vech}(Q_T)')'$. The complete-data likelihood function of the MSSC model is:

$$
\begin{aligned}
&\mathcal{L}(\mathbf{y}, \mathbf{z}|\boldsymbol{\theta}) = \qquad (4)\\
&\prod_{t=1}^{T} \left(\frac{1}{(2\pi)^{m/2}|\Sigma_t|^{1/2}} e^{-\frac{1}{2}\boldsymbol{\varepsilon}_t' \Sigma_t^{-1} \boldsymbol{\varepsilon}_t} \frac{1}{(2\pi)^{m/2}|\Sigma_\eta|^{1/2}} e^{-\frac{1}{2}\boldsymbol{\eta}_t' \Sigma_\eta^{-1} \boldsymbol{\eta}_t} \right.\\
&\cdot 2^{-\frac{m\nu}{2}} \Gamma_m(\nu/2)^{-1} |S_{t-1}|^{-\frac{\nu}{2}} e^{-\text{tr}\left(\frac{1}{2} S_{t-1}^{-1} Q_t^{-1}\right)} |Q_t^{-1}|^{\frac{\nu-m-1}{2}}\\
&\left. \cdot \prod_{k=1,2} \left(p_{k,00}^{1-s_{k,t}} (1-p_{k,00})^{s_{k,t}} \right)^{1-s_{k,t-1}} \left(p_{k,01}^{s_{k,t}} (1-p_{k,01})^{1-s_{k,t}} \right)^{s_{k,t-1}} \right),
\end{aligned}
$$

where $\Gamma_m(\nu/2)$ is the m-variate gamma function and $\boldsymbol{\theta} = (\mathbf{a}_{00}', \mathbf{a}_{01}', \text{vec}(A_{10})', \text{vec}(A_{11})', \mathbf{b}_{00}', \mathbf{b}_{01}', \text{vec}(B_{10})', \text{vec}(B_{11})', \text{vech}(\Sigma_\eta)', \nu, d, \lambda_1, \text{vech}(\bar{D}_0), \text{vech}(\bar{D}_1), p_{1,11}, p_{1,22}, p_{2,11}, p_{2,22})'$ is the parameter vector. We arrange $\boldsymbol{\theta}$ in four sub-vectors: $\boldsymbol{\theta}_1 = \text{vec}(\Psi)$, with $\Psi = (\psi_1, \ldots, \psi_m)$, which has in the columns the vectors $\psi_j = (a_{00,j}, a_{01,j}, (A_{10,j1}, \ldots, A_{10,jm}), (A_{11,j1}, \ldots, A_{11,jm}))'$, $j = 1, \ldots, m$; $\boldsymbol{\theta}_2 = (\boldsymbol{\phi}', \text{vech}(\Sigma_\eta)')'$, with $\boldsymbol{\phi} = \text{vec}(\Phi)$, where $\Phi = (\phi_1, \ldots, \phi_m)$ has in the columns the vectors $\phi_j = (b_{00,j}, b_{01,j}, (B_{10,j1}, \ldots, B_{10,jm}), (B_{11,j1}, \ldots, B_{11,jm}))'$, $j = 1, \ldots, m$; $\boldsymbol{\theta}_3 = (\nu, d, \lambda_1, \text{vech}(\bar{D}_0), \text{vech}(\bar{D}_1))'$; $\boldsymbol{\theta}_4 = (p_{1,00}, p_{1,11}, p_{2,00}, p_{2,11})'$. We specify the following prior distributions

$$
\begin{aligned}
&\boldsymbol{\theta}_1 \sim \mathcal{N}_{24}(\mathbf{0}, 10 I_{24}), \; \boldsymbol{\phi}|\Sigma_\eta \sim \mathcal{N}_{24}(\mathbf{0}, \Sigma_\eta \otimes 10 I_8), \; \Sigma_\eta^{-1} \sim \mathcal{W}_3(10, 4I_3)\\
&d \sim \mathcal{U}_{(-1,1)}, \; \lambda_1 \sim \mathcal{U}_{(0,1)}, \; \nu \sim \frac{1}{\Gamma(10)} (\nu-3)^{10-1} \exp\{-(\nu-3)\} \mathbb{I}_{(3,+\infty)}(\nu)\\
&\bar{D}_i^{-1} \sim \mathcal{W}_3(10, 0.1 I_3), \; p_{k,ii} \sim \mathcal{U}_{(0,1)}, \; k = 1, 2, \; i = 0, 1
\end{aligned}
$$

We apply the Gibbs sampler given in [3] for the posterior approximation.

4 Exchange Rates Correlation Dynamics

We consider daily closing values for three exchange rates against the US$, namely Euro, Yen and Pound. We compute the percentage log-returns of the exchange rates and denote them as $y_{1,t}$, $y_{2,t}$ and $y_{3,t}$ for Euro, Yen and Pound, respectively (see left column of Figure 3). The presence of time-varying conditional volatility is evident from the squared return series (see right column of Figure 3). We fit the proposed Bayesian MSSC model on the exchanger rate dataset. The estimation results given in Fig. 4-5 show the presence of significant shifts in both volatilities and correlations. Moreover, the stepwise line in Fig. 5 indicates the presence of correlation-specific shifts (i.e., $\hat{s}_t \in \{2, 3\}$),

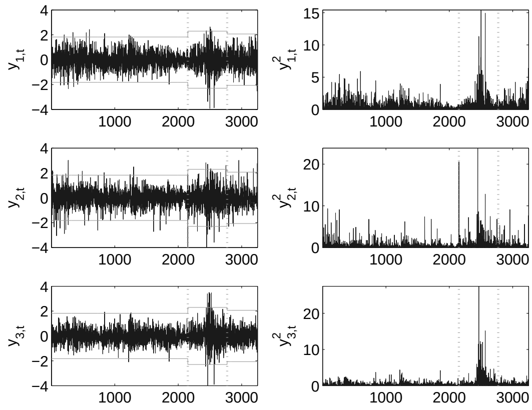

Fig. 3 Log-differences (left) and squared log-differences (right) of EUR-USD ($y_{1,t}$), YEN-USD ($y_{2,t}$) and GBP-USD ($y_{3,t}$) daily exchange rates for the period 01/01/1999-03/1/2011. The vertical dashed lines correspond to dates of beginning of the 2007 financial crisis (15/08/2007) and the beginning of the Greek's debt crisis (31/12/2009).

thus suggesting the coexistence of different configurations of risk (volatility) and contagion level (correlation) in the exchange rate markets analysed in this paper.

5 Conclusion

We apply a stochastic correlation model to detect changes in the correlations between exchange rates sampled at a daily frequency. We follow a Bayesian inference approach to parameter and latent variable estimation and apply a MCMC algorithm for posterior approximation. Our results document the coexistence of different volatility and correlation-specific regimes.

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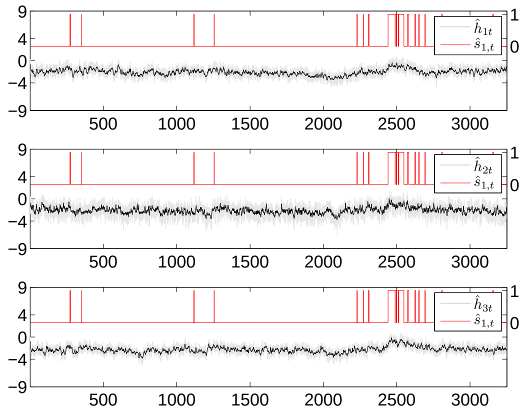

Fig. 4 Posterior means (solid lines, left axes) and 95% credibility regions (gray areas, left axes) of the log-volatility $\mathbf{h}_t$. Each figure of panel includes $\hat{s}_{1,t}$ (stepwise, right axes).

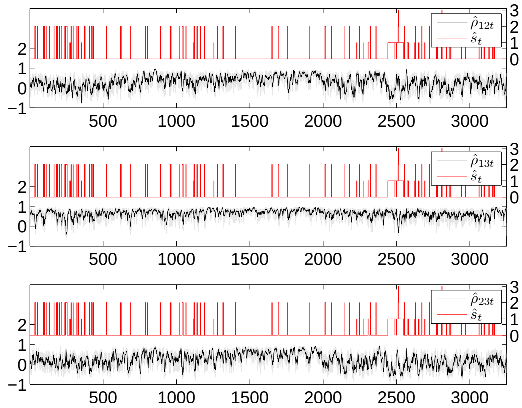

Fig. 5 Posterior means (solid lines, left axes) and 95% credibility regions (gray areas, left axes) of the correlation Ω_t. Each figure includes $\hat{s}_t = \hat{s}_{1,t} + 2\hat{s}_{2,t}$ (stepwise, right axes).

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