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Abstract
When a rich variety of available goods is classified into a smaller number of tradable commodities, we have a market economy that relies on a simplified representation. We study how the classification of goods into commodities affects the allocation of scarce resources among agents. We also consider a notion of decentralized equilibrium that is achieved merely by an appropriate classification of the goods, regardless of the presence of a price-based mechanism.

Keywords
goods and tradable commodities, classification externality, norm-based equilibrium

JEL Codes
D02, D51, D62

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1 Introduction

The intuitive notion of commodity is "somewhat ambiguous (should two apples of different sizes be considered two units of the same commodity?)". Economic theory relies on the idealized concept of an Arrow-Debreu commodity, whose description is "objective, quantifiable and universally agreed upon" and may include physical characteristics, geographic and temporal location, or other less obvious items.

It is apparent that "the less crude the categorization of commodities becomes, the more scope there is for agents to trade, and the greater the set of imaginable allocations". The Arrow-Debreu commodities are identified so precisely that "further refinements cannot yield imaginable allocations which increase the satisfaction of the agents". (Quotes are from Geanakoplos, 1989.)

It may be highly unpractical for an economy (if not impossible) to discover or to agree on the full set of Arrow-Debreu commodities. This concern is usually muted, especially when technical reasons call for simplifying assumptions. For instance, right after building up the idea that "a commodity is a good or a service completely specified physically, temporally, and spatially", Debreu (1959, p. 32) abruptly informs the reader that "it is assumed that there is only a *finite* number of distinguishable commodities" (emphasis added).

This paper unmutes the concern and considers exchange economies where a continuum of goods has been reduced to a finite number of tradable commodities, by means of a *classification.* Here is an example. The people of a village cultivate a vineyard on the banks of a long river. The quality of the grapes depends on the position of the vines: one obtains different types of wine by pressing grapes grown on different vines. By custom, the villagers blend their grapes and produce at most two types of wine. Their production technology separates the grapes growing on the lowlands (L) from those growing on the hills (H). The position θ of the boundary between L and H is a convention that defines the classification of grapes into two wines.

We view a market economy as an institution that binds agents' demands to the extant classification. This paper studies and compares competitive allocations generated via different classifications for the same endowment of goods. Following Richter and Rubinstein (2015), the competitive equilibrium may be seen as "a method of creating harmony in an interactive situation with [...] self-interested agents." We investigate both standard price-based competitive allocations and an alternate mechanism (without prices) where harmony is achieved through the selection of an appropriate classification.

Continuing the example, imagine that, after the elders choose a position θ, the yields are blended into wine L and wine H and distributed among the families of the village. Each family is entitled to receive their allotted number of bottles, but can demand the type of wine they prefer. When it is not possible to satisfy everyone's requests, there is a conflict. Assuming the elders use prices to resolve conflicts and allocate wine, the selection of θ influences the final allocation by which the conflict is eliminated. In the absence of prices, the mere choice of an appropriate θ may be enough to avoid conflict.

Our model ensures that the set of price-based competitive allocations depends continuously on the choice of the classification, and therefore agents receive similar payoffs in economies using similar classifications. The main insight is that the choice of a classification induces novel and significant externalities across consumers' demands. Therefore, when assessing competitive allocations, it is important to be aware of the underlying classification.[1] In particular, one might interpret our classification as an abstract public good, i.e. as a parameter that affects agents' preferences and endowments; see Mas-Colell (1980) and Diamantaras and Gilles (1996).

We show that, in equilibrium, the relative prices of two commodities depend on the classification of other (distinct) commodities; agents' consumption may change even if the relative prices do not move. Moreover, if one can modify the extant classification, a competitive allocation may no longer be Pareto-optimal. On the other hand, refining a classification by introducing new commodities may not be Pareto-improving or may even reduce utilitarian welfare. A related claim that increasing trading opportunities may not be Pareto-improving is already in Hart (1975, Section 6), where it is shown that opening a new market after agents have already traded their initial endowments may damage some consumers.

Section 5 of the paper defines an allocation mechanism where the mere choice of an appropriate classification suffices to achieve an equilibrium. This fits into the program initiated in Richter and Rubinstein (2015) and fully spelled out in Richter and Rubinstein (2020): a society may seek harmony through decentralized "price-like" institutions enforcing a set of social norms that constrain agents' choices. Instead of calling on prices and budgets, equilibrium may be achieved by enforcing an appropriate classification of goods into commodities.

Our model is consistent with the seminal insight in Lancaster (1966) that the carriers of value are not goods, but their characteristics. The classification determines which bundles of characteristics can be traded as a commodity. Following Mas-Colell (1975) and Jones (1984)'s model of economies with commodity differentiation, we assume a measure space for the characteristics and define commodities as measurable sets of characteristics. We focus on the classification and assume that every good in the endowment is part of a tradable commodity. Diamantaras and Gilles (2003) use a general equilibrium model to study the costly institutional arrangements that support the introduction of new tradable commodities.

We assume that the classifications are given and do not analyze how they can be selected. As in our model, Sprumont (2004) considers a measure space of Lancasterian characteristics and defines commodities through partitions but, differently from this paper, he provides an axiomatic "subjective" approach to the definition of commodities, using agents' preferences to reveal subsets of characteristics that are sufficiently homogeneous and distinct to deserve being singled out.

[1] The widespread diffusion of protected designation of origins attests to the importance of classification.

2 The model

We consider a society endowed with a continuum of goods to be distributed over a finite set of n agents. The agents have their own preferences on consumption and hold compatible claims on the society's endowment. It is infeasible (or impractical) to trade over a continuum, but the society can classify goods and reduce them to a finite number of commodities. The adoption of a classification by the society defines which commodities are tradable in the economy: a commodity is tradable if and only if it is present in the classification. The classification is shared by all agents and binds their choices: an agent can demand only bundles of tradable commodities.

Goods and bundles. The unit interval $\mathcal{I}$ represents the space of goods' characteristics. Each $t \in \mathcal{I}$ is interpreted as a complete description of the relevant attributes of a certain good. Every measurable subset $F \subseteq \mathcal{I}$ defines a selection of goods with specific characteristics, called a *type*. (In the sequel, any subset of $\mathcal{I}$ is understood to be measurable with respect to the Borel σ-algebra.) The distribution of goods is described by a non-negative measure ω on $\mathcal{I}$ normalized to $\omega(\mathcal{I}) = 1$, so that $\omega(F)$ is the total amount of goods with characteristics in F. Every positive, ω-integrable function g defines a *bundle of goods*, where $\int_F g \, d\omega$ is the amount of goods of type F that is considered in the bundle g.

Agents. There are n agents. Their preferences on bundles of goods are represented by utility functions on $L^1_+(\omega)$ that are linear, norm-continuous and normalized so that the bundle constantly equal to 1 gives utility 1. This implies that every agent i is associated with a measure ν_i on $\mathcal{I}$ such that $\int g \, d\nu_i$ is the utility that i receives from consuming a bundle g. Furthermore, each ν_i is absolutely continuous with respect to ω and normalized to $\nu_i(\mathcal{I}) = 1$. We call ν_i *agent i's evaluation measure*.

Every agent i has a claim $\kappa_i > 0$ on the society's endowment, and we assume $\sum_{i=1}^n \kappa_i = 1$.

Classification of goods. A *classification* is a partition of $\mathcal{I}$ into a finite number of intervals of positive ω-measure. We imagine that every cell of the classification combines its elements under the same label, and agents recognize its content as one of the tradable commodities. If the classification π has n cells, there are n tradable commodities in the economy. Given a classification π, the bundles of tradable commodities correspond to the functions that are measurable with respect to the partition π. A π-*bundle* of commodities is a vector $x = (x_B) \in \mathbb{R}^\pi_+$ that specifies a quantity for each tradable commodity, so that x corresponds to the simple function $\sum_{B\in\pi} \frac{x_B}{\omega(B)} \mathbb{1}_B$.

Consumers can demand only bundles of tradable commodities. Thus, a generic outcome of the model combines a labelling of tradable commodities and an allocation of goods compatible with it. We represent it as a configuration $\langle \pi, (x^i) \rangle$, where π is the shared classification and each $x^i \in \mathbb{R}^\pi_+$ is the π-bundle selected by agent i. We say that the profile $\langle \pi, (x^i) \rangle$ is *feasible* if the total amount of goods distributed does not exceed their initial availability, i.e. if $\sum_i x^i_B = \omega(B)$ for every $B \in \pi$.

The exchange economy defined by a classification. A classification π is the basis for an exchange economy $\mathcal{E}(\pi)$, where the commodity space is the set of π-bundles $\mathbb{R}^\pi_+$. Every agent is endowed with the vector $e^i = (\kappa_i \omega(B)) \in \mathbb{R}^\pi_+$ and ranks the π-bundles according to the linear function:

$$V_i(\pi, x) = \sum_{B \in \pi} \frac{x_B}{\omega(B)} \nu_i(B)$$

which is simply the restriction of his primitive utility function to the set of π-bundles. Because $\nu_i(B)$ measures agent i's evaluation for all the goods of type B, the ratio $\frac{x_B}{\omega(B)} \nu_i(B)$ gives i's utility from consuming x_B units of commodity B.

A competitive equilibrium in the economy $\mathcal{E}(\pi)$ is a pair $\langle p, (x^i) \rangle$ formed by a price vector $p \in \mathbb{R}^\pi_+$ and an allocation (x^i) such that: *(i)* $p \cdot x^i \leq \kappa_i \sum_B p_B \omega(B)$ for all i and *(ii)* if $V_i(\pi, y) > V_i(\pi, x^i)$ for some π-bundle y then $p \cdot y > \kappa_i \sum_B p_B \omega(B)$. We say that the configuration $\langle \pi, (x^i) \rangle$ is *competitive* if (x^i) corresponds to a competitive equilibrium in $\mathcal{E}(\pi)$.

In short, imposing a classification π reduces the economic interaction to a finite dimensional economy $\mathcal{E}(\pi)$ where agents have linear preferences and there is no self-sufficient set of agents (Gale 1976). It follows that: *(i)* there exists a competitive equilibrium in $\mathcal{E}(\pi)$; *(ii)* every two competitive prices in $\mathcal{E}(\pi)$ are proportional with each other; and *(iii)* agents receive the same utility under any two competitive allocations in $\mathcal{E}(\pi)$.

3 Classification matters

Standard models of exchange take the notion of commodity as primitive. We depart from this assumption and consider alternative classifications for the same set of goods. This section studies the competitive outcomes reachable under different classifications with the same number of commodities.

Our first set of results shows that similar classifications lead to similar exchange economies, and hence to similar sets of competitive allocations. To formalize this intuition we write $\Pi(k)$ for the set of classifications formed by at most k intervals, and we endow it with a suitable topological structure. Let $\sigma(\pi)$ denote the σ-algebra generated by a classification π and set:

$$\delta_\omega(\pi, \pi') = \sup_{B \in \sigma(\pi)} \inf_{B' \in \sigma(\pi')} \omega(B \triangle B').$$

The function $d_\omega(\pi, \pi') = \inf\{\delta_\omega(\pi, \pi'), \delta_\omega(\pi', \pi)\}$ defines a pseudo-metric on the set of classifications; see Boylan (1971). Indeed, d_ω is the Hausdorff pseudo-metric induced by the set-theoretical distance, mapping every $B, B' \subseteq \mathcal{I}$ into $\omega(B \triangle B')$; see Aliprantis and Border (2006), Section 13.5.

Proposition 1. $(\Pi(k), d_\omega)$ *is a compact space where two classifications have zero-distance if and only if they coincide up to negligible sets.*

For readability, all proofs are relegated to Appendix A.3.

We can now formalize the intuitive notion that the set of competitive outcomes in the society depends continuously on the choice of the classification. Let $\mathcal{W}(\pi)$ denote the set of all the competitive equilibria in the economy $\mathcal{E}(\pi)$ and recall that, by the observations above, $\mathcal{W}(\pi)$ is non-empty.

Theorem 2. *The competitive equilibrium-correspondence* $\mathcal{W}$ *is upper-hemicontinuous on the compact space* $(\Pi(k), d_\omega)$*.*

A corollary of Theorem 2 is the intuition that, if goods are allocated through a price-based mechanism, then agents receive similar payoffs in economies defined by similar classifications. Define the function $\Psi_i(\pi)$ that assigns to every classification π the utility that agent i receives in a competitive equilibrium of the economy $\mathcal{E}(\pi)$. The function Ψ_i is well defined because any two competitive equilibria in $\mathcal{E}(\pi)$ generate the same utility profile. The intuition can be stated as follows.

Corollary 3. *The function* Ψ_i *is continuous on the compact space* $(\Pi(k), d_\omega)$*.*

3.1 Pareto-optimality and welfare theorems

We turn to studying the efficiency of competitive outcomes. We say that a configuration $\langle \pi, (x^i) \rangle$ *Pareto-dominates* another configuration $\langle \rho, (y^i) \rangle$ if $V_i(\pi, x^i) \geq V_i(\rho, y^i)$ for all i, with a strict inequality for at least an agent i. Given a set of feasible configurations $\mathcal{F}$, a configuration $\langle \pi, (x^i) \rangle \in \mathcal{F}$ is *Pareto-optimal* in $\mathcal{F}$ if there is no configuration in $\mathcal{F}$ that Pareto-dominates it.

It is a simple observation that any competitive configuration is Pareto-optimal among those based on the same classification, i.e. that if $\langle \pi, (x^i) \rangle$ is competitive then no feasible configuration of the type $\langle \pi, (y^i) \rangle$ Pareto-dominates it. This is a consequence of the first Welfare Theorem applied to the exchange economy $\mathcal{E}(\pi)$, because (x^i) is a competitive allocation within $\mathcal{E}(\pi)$.

A more interesting scenario opens when we compare allocations based on different classifications: then it is no longer true that any competitive configuration is Pareto-optimal. The following example shows a simple society where a competitive configuration is (strictly) Pareto-dominated by another competitive one.

Example 1. *Consider a society where* ω *is the Lebesgue measure and there are three agents who all have the same claims (i.e.,* $\kappa_i = \frac{1}{3}$ *for* $i = 1, 2, 3$*). Agent* 1*'s preferences are described by the evaluation measure* ν_1*; agents* 2 *and* 3 *have identical preferences described by the evaluation measure* ν_2*. We assume*

$$\nu_1(F) = 3\omega\left(F \cap \left[0, \frac{1}{3}\right]\right) \quad \text{and} \quad \nu_2(F) = 3\omega\left(F \cap \left[\frac{1}{3}, \frac{1}{2}\right)\right) + \omega\left(F \cap \left[\frac{1}{2}, 1\right]\right);$$

that is, agent 1 *cares only about goods in the first third of the interval (and is indifferent over them), while agents* 2 *and* 3 *value any good in* $\left[\frac{1}{3}, \frac{1}{2}\right)$ *thrice as much as those in* $\left[\frac{1}{2}, 1\right]$*.*

Let π be the classification formed by the intervals $A = \left[0, \frac{1}{2}\right)$ and $B = \left[\frac{1}{2}, 1\right]$ and consider the economy $\mathcal{E}(\pi)$. Agent 1's evaluations for the two tradable commodities are respectively 1 and 0, and thus he demands only the first commodity. Agents 2 and 3 have identical evaluations for the commodities and demand whatever is cheaper. A competitive equilibrium has identical prices for the commodities: it assigns the π-bundle $x^1 = \left(\frac{1}{3}, 0\right)$ to agent 1, and the bundles $x^2 = x^3 = \left(\frac{1}{12}, \frac{1}{4}\right)$ to agents 2 and 3. At the competitive configuration $\langle \pi, (x^i) \rangle$ the agents have utilities:

$$V_1(\pi, x^1) = \frac{2}{3}, \quad V_2(\pi, x^2) = \frac{1}{3}, \quad V_3(\pi, x^3) = \frac{1}{3}.$$

Consider an alternative classification $\rho = \{A', B'\}$ where $A' = \left[0, \frac{1}{3}\right)$ and $B' = \left[\frac{1}{3}, 1\right]$. In the economy $\mathcal{E}(\rho)$, agent 1 demands only commodity A' and agents 2 and 3 demand only commodity B'. A competitive equilibrium has identical prices for the commodities: it assigns the ρ-bundle $y^1 = \left(\frac{1}{3}, 0\right)$ to agent 1, and the bundles $y^2 = y^3 = \left(0, \frac{1}{3}\right)$ to agents 2 and 3. In this case, agents have utilities:

$$V_1(\rho, y^1) = 1, \quad V_2(\rho, y^2) = \frac{1}{2}, \quad V_3(\rho, y^3) = \frac{1}{2}.$$

Clearly, the competitive configuration $\langle \rho, (y^i) \rangle$ Pareto-dominates the competitive configuration $\langle \pi, (x^i) \rangle$.

The non-optimality of competitive allocations follows because the classification π affects agents' preferences within the economy $\mathcal{E}(\pi)$. In other words, the classification underlying an economy creates an externality, and thus the first Welfare Theorem does not hold across different classifications. This is reminiscent of the case of an exchange economy where agents' preferences are price-dependent and the standard Welfare Theorems may fail.

Nonetheless, using the continuity of agents' preferences on classifications, we can prove the existence of competitive configurations that are Pareto-optimal.

Theorem 4. *The set of competitive configurations based on classifications in $\Pi(k)$ has a Pareto-optimal configuration.*

This result states that, for any k, there exists a Pareto-optimal configuration among those based on no more than k commodities. The upper bound makes it possible to to use the compactness of $\Pi(k)$ in the proof. In general, if there are no bounds on the number of commodities forming a classification, there may not exist a Pareto-optimal configuration, even within the set of the competitive configurations. Appendix A.1 shows a pathological case in which every competitive configuration can be Pareto-improved by a classification based on a strictly larger number of commodities.

Perhaps surprisingly, while the first Welfare Theorem fails for competitive configurations based over different classifications, it is possible to prove a version of the second Welfare Theorem for competitive configurations.

Proposition 5. *If $\langle \pi, (x^i) \rangle$ is a feasible Pareto-optimal configuration such that $x^i_B > 0$ for every i and every B in π, then one can redefine agents' claims so that $\langle \pi, (x^i) \rangle$ is competitive.*

Proposition 5 ensures that a feasible interior configuration that is Pareto-optimal can be cast as a competitive configuration after suitably modifying agents' claims. This aligns well with the classical version of the second Welfare Theorem, by which any interior Pareto-optimal allocation in an exchange economy is competitive for some suitable initial distribution of resources.

3.2 The relative scarcity of commodities

In a competitive equilibrium, one may interpret the ratio of the prices of two commodities as an index of relative scarcity: given preferences and endowments, the greater is the ratio, the higher is the value attributed to the first commodity. We argue that this ratio is not an intrinsic property of the two commodities, because it depends on how other distinct commodities have been classified.

The next example keeps two commodities fixed and studies how the ratio of their prices varies as we change the rest of the classification. Even if agents' evaluations of the two commodities remain constant, the ratio of their prices ranges over an interval that can be made arbitrarily large. This implies that knowing the ratio of the prices of two commodities is meaningless without a full description of the whole classification.

Example 2. *Consider a society where ω is the Lebesgue measure and there are $2n$ agents who all have identical claims. There are two types of agents, forming groups of equal size. The preferences of agents in groups 1 and 2 are respectively described by the evaluation measures:*

$$\nu_1(F) = 2\omega\left(F \cap \left[0, \frac{1}{2}\right]\right) \qquad \text{and} \qquad \nu_2(F) = 2\omega\left(F \cap \left[\frac{1}{2}, 1\right]\right).$$

For every $t \in (0,1)$, let π_t be the classification formed by the four intervals:

$$A = \left[0, \frac{1}{4}\right), \quad B_t = \left[\frac{1}{4}, \frac{1+2t}{4}\right), \quad C_t = \left[\frac{1+2t}{4}, \frac{3}{4}\right), \quad D = \left[\frac{3}{4}, 1\right].$$

The two commodities A and D are acknowledged as tradable in any classification π_t, while the other two tradable commodities B_t and C_t depend on the choice of t. We claim that the ratio of equilibrium prices for the two commodities A and D depends on the threshold t.

For every t, let p_t be a competitive price system in $\mathcal{E}(\pi_t)$ and let $\varphi(t)$ denote the ratio $p_t(A)/p_t(D)$. Note that the function φ does not depend on how the p_t's are chosen, because two competitive prices in $\mathcal{E}(\pi_t)$ must be proportional to each other. Assuming $p_t(D) = 1$ for all t, we compute the equilibrium prices case by case.

Suppose $t \leq \frac{1}{2}$. Then commodities A and B_t are desirable only for agents in group 1, C_t is desirable for agents of both groups, and D is desirable only for agents *in group 2. Computations show that in equilibrium agents from group 1 demand commodities A, B_t and C_t as long as $t \leq \frac{1}{6}$, and demand only commodities A and B_t if $t > \frac{1}{6}$. On the other hand, agents from group 2 demand positive amounts of commodities C_t and D. Because $p_t(D) = 1$, the resulting equilibrium prices are:*

$$p_t(A) = \begin{cases} \frac{1}{1-2t}, & \text{if } t \leq \frac{1}{6}, \\ \frac{2}{1+2t}, & \text{if } t > \frac{1}{6} \end{cases}, \quad p_t(B_t) = \begin{cases} \frac{1}{1-2t}, & \text{if } t \leq \frac{1}{6}, \\ \frac{2}{1+2t}, & \text{if } t > \frac{1}{6} \end{cases}, \quad p_t(C_t) = \frac{1}{2(1-t)}.$$

Suppose instead $t \geq \frac{1}{2}$. The situation is symmetric to the above. In equilibrium, agents from group 1 demand commodities A and B_t, while those from group 2 demand only commodities C_t and D if $t \leq \frac{5}{6}$, and may add commodity B_t when $t \geq \frac{5}{6}$. The resulting equilibrium prices are:

$$p_t(A) = \begin{cases} \frac{3-2t}{2}, & \text{if } t \leq \frac{5}{6}, \\ 2t-1, & \text{if } t > \frac{5}{6} \end{cases}, \quad p_t(B_t) = \begin{cases} \frac{3-2t}{4t}, & \text{if } t \leq \frac{5}{6}, \\ \frac{2t-1}{2t}, & \text{if } t > \frac{5}{6} \end{cases}, \quad p_t(C_t) = 1.$$

The function $\varphi(t)$ coincides with $p_t(A)$: it is increasing for $t \leq \frac{1}{6}$, decreasing for $\frac{1}{6} < t < \frac{5}{6}$, and increasing again for $t \geq \frac{5}{6}$. Its graph is plotted in Figure 1. It attains every value between $\frac{3}{2}$ (when $t = \frac{1}{6}$) and $\frac{2}{3}$ (when $t = \frac{5}{6}$). The index

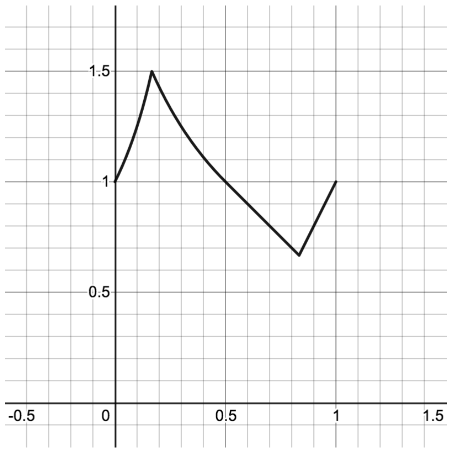

Figure 1: Graph of the ratio of the equilibrium prices of commodities A and D.

of relative scarcity for commodities A and D depends on the whole classification of tradable commodities. Moreover, for every threshold t there exists s such that $\varphi(t) = 1/\varphi(s)$, so that $p_t(A)/p_t(D) = p_s(D)/p_s(A)$. We conclude that whatever holds about the equilibrium value of A relative to D in a given classification is specularly true about D relative to A under a different classification.

Note that changes in the threshold t affect the utilities that agents receive in equilibrium, and hence their relative welfare within the society (in spite of their identical claims). Precisely, the ratio of the utilities that the two groups of agents receive in equilibrium varies significantly with t: if $u^i(t)$ is the utility received by agents of group $i = 1, 2$ in the economy $\mathcal{E}(\pi_t)$, computations show that the function $\xi(t) = u^1(t)/u^2(t)$ has the same graph as the function φ plotted in Figure 1.

In Example 2, the ratio of the prices of two commodities A and D ranges over a closed interval as we change the classification and keep A and D fixed. One can modify the example and make the range of the interval arbitrarily large. However, because both agents's preferences and the function $\varphi(t)$ are continuous and the set of classifications we consider is compact, the range must remain bounded.

4 Refining classifications

Refining a classification introduces new tradable commodities by splitting an interval into two or more sub-intervals: goods that were bundled in the original classification are acknowledged as distinct commodities. Formally, we say that a classification ρ *refines* a classification π (and write $\rho \succ \pi$) if the latter belongs to the σ-algebra generated by ρ. When $\rho \succ \pi$, every π-bundle corresponds to a unique ρ-bundle and every feasible exchange in $\mathcal{E}(\pi)$ can also be realized within $\mathcal{E}(\rho)$. But the set of bundles that are tradable in ρ is larger, giving agents more trading possibilities.

An immediate consequence of refining the classification underlying an economy is that allocations that were Pareto-optimal under the initial classification may cease to be so. Given two classifications $\rho \succ \pi$, any allocation f in the economy $\mathcal{E}(\pi)$ has an equivalent allocation in $\mathcal{E}(\rho)$, but not vice versa. Therefore, the Pareto-optimal allocation f in $\mathcal{E}(\pi)$ may be dominated by some other allocation in $\mathcal{E}(\rho)$ that has no correspondent in $\mathcal{E}(\pi)$. A similar argument holds for core-allocations, which are the allocations that no coalition can improve upon: given a core-allocation f in $\mathcal{E}(\pi)$, a group of agents may find a profitable way to reallocate their endowments among themselves in $\mathcal{E}(\rho)$, but not in $\mathcal{E}(\pi)$. Core allocations in $\mathcal{E}(\pi)$ may not remain so after refining a classification.

The rest of this section compares the competitive outcomes that can be achieved by increasing the number of commodities and refining the classification underlying the economy. Throughout, we assume that agents act competitively, and therefore goods are assigned only via competitive allocations consistent with the given classification. The agents' levels of utility are uniquely determined by the set of tradable commodities under consideration.

4.1 Refinements may switch trading positions

The introduction of a new commodity may change drastically the individual trading positions. An agent who only buys commodity A in a given economy may switch to selling her entire endowment of A when the underlying classification is refined. This suggests that a social planner who knows only agents' demands for a given classification (but not their true preferences on goods) cannot predict how they would trade under a finer classification.

The next example compares an economy based on three commodities A, B and C against one that refines the classification by splitting C into two distinct commodities. There is an agent who consumes only commodity A in the first economy but consumes only B in the second one. This occurs even though neither

A or B are directly affected by the refinement and even if their relative prices remain identical.

Example 3. *Consider a society where ω is the Lebesgue measure and there are 4 agents with different claims: agents 1 and 2 have claim $\frac{1}{3}$, while 3 and 4 have claim $\frac{1}{6}$. Individual preferences are respectively described by the evaluation measures:*

$$\nu_1(F) = \frac{3}{2}\omega\left(F \cap \left[0, \frac{2}{3}\right)\right), \qquad \nu_2(F) = \frac{3}{2}\omega\left(F \setminus \left[\frac{1}{3}, \frac{2}{3}\right)\right),$$

$$\nu_3(F) = \frac{5}{3}\omega\left(F \cap \left[\frac{1}{3}, \frac{2}{3}\right)\right) + \frac{8}{3}\omega\left(F \cap \left[\frac{2}{3}, \frac{5}{6}\right)\right),$$

$$\nu_4(F) = \frac{5}{3}\omega\left(F \cap \left[\frac{1}{3}, \frac{2}{3}\right)\right) + \frac{8}{3}\omega\left(F \cap \left[\frac{5}{6}, 1\right]\right).$$

Let $\pi = \{A, B, C\}$ be the classification defined by $A = \left[0, \frac{1}{3}\right)$, $B = \left[\frac{1}{3}, \frac{2}{3}\right)$ and $C = \left[\frac{2}{3}, 1\right]$.

When all commodities carry the same price, agent 1 is indifferent between A and B, agent 2 between A and C, while 3 and 4 prefer B over the other commodities. Therefore, there is only one equilibrium in $\mathcal{E}(\pi)$ where all prices are identical and the agents consume the bundles

$$x^1 = \left(\frac{1}{3}, 0, 0\right), \ x^2 = \left(0, 0, \frac{1}{3}\right), \ x^3 = x^4 = \left(0, \frac{1}{6}, 0\right).$$

Consider the finer classification $\rho = \{A, B, C^1, C^2\}$ obtained by splitting C into two intervals $C^1 = \left[\frac{2}{3}, \frac{5}{6}\right)$ and $C^2 = \left[\frac{5}{6}, 1\right]$. This refinement of π allows 3 and 4 to differentiate between which parts of C they like more: 3 prefers C_1 while 4 prefers C_2, while in the earlier classification this difference in preferences was muted. On the other hand, 1 and 2 value C as much as C_1 or C_2.

The only equilibrium in $\mathcal{E}(\rho)$ assigns identical prices to all commodities and gives to each agent the bundle:

$$x^1 = \left(0, \frac{1}{3}, 0, 0\right), \ x^2 = \left(\frac{1}{3}, 0, 0, 0\right), \ x^3 = \left(0, 0, \frac{1}{6}, 0\right), \quad x^4 = \left(0, 0, 0, \frac{1}{6}\right).$$

In the economy $\mathcal{E}(\pi)$ agent 1 sells all her endowments of B and C to buy A, while in $\mathcal{E}(\rho)$ she buys only commodity B. Nevertheless, the commodities A and B and their relative prices are the same in both economies.

An adaptation of the argument in Example 2 shows that refining a classification may change the relative scarcity of commodities, and hence the equilibrium allocation. The current example says more, proving that agents' consumption of some commodities may change even when their relative prices do not.

4.2 Refinements may not be Pareto-improving

In many situations, increasing the number of commodities allows every agent to achieve higher levels of utility. As a way of illustration, if π is the trivial classification $\{\mathcal{I}\}$ then every refinement of π gives agents the opportunity of demanding more sophisticated bundles while leaving them the possibility of consuming as much as they did under π. Thus, every agent considers any refinement at least as good as π.

However, it is possible that introducing a new tradable commodity gives some agents more market power and damages some others. The next example considers a classification π with the following property: for every $\rho \succ \pi$ formed by adding a new commodity to π there is an agent who strictly prefers every competitive allocation in $\mathcal{E}(\pi)$ to any competitive allocation in $\mathcal{E}(\rho)$. This shows that adding a new commodity to π is not a Pareto-improvement for the society, and indeed it damages at least one agent.

Example 4. *Consider an economy where ω coincides with the Lebesgue measure. There are 4 agents with identical claims and with preferences derived from the following evaluation measures:*

$$\nu_1(F) = 2\omega\left(F \setminus \left[\frac{1}{4}, \frac{3}{4}\right]\right), \quad \nu_2(F) = 2\omega\left(F \cap \left[\frac{1}{4}, \frac{3}{4}\right]\right),$$

$$\nu_3(F) = 2\omega\left(\left[0, \frac{1}{2}\right]\right), \quad \nu_4(F) = 2\omega\left(\left[\frac{1}{2}, 1\right]\right).$$

Let π be the classification formed by the two intervals $A = \left[0, \frac{1}{2}\right]$ and $B = \left(\frac{1}{2}, 1\right]$. In the exchange economy $\mathcal{E}(\pi)$ agents 3 cares only about commodity A, agent 4 only about B, and agents 1 and 2 are indifferent between them. An equilibrium is achieved when the two commodities have the same price and agents demand, for example, the π-bundles:

$$x^1 = x^3 = \left(\frac{1}{4}, 0\right), \; x^2 = x^4 = \left(0, \frac{1}{4}\right).$$

We claim that for every refinement ρ of π formed by 3 tradable commodities there is an agent that strictly prefers (x^i) to any competitive allocation in $\mathcal{E}(\rho)$. Precisely, we assume that ρ is obtained by splitting A into two commodities A_1 and A_2 and we prove that, in equilibrium, agent 3 cannot afford $\frac{1}{4}$ units of goods of type A_1 or A_2, implying that 3 receives a strictly lower utility under ρ. The same strategy shows that if ρ is obtained by splitting B then agent 4 strictly prefers π to ρ.

Assume $t \in \left(0, \frac{1}{2}\right)$ such that $\omega(A_1) = t$ and $\omega(A_2) = \frac{1}{2} - t$. Let p be a competitive price in $\mathcal{E}(\rho)$ normalized so that $p(B) = 1$ and let w be agent 3's wealth at p. We assume that $p(A_1) \le p(A_2)$ (the other case is treated identically) so that agent 3 demands exactly:

$$\frac{w}{p(A_1)} = \frac{1}{4}\left[t + \frac{p(A_2)}{p(A_1)}\left(\frac{1}{2} - t\right) + \frac{1}{2p(A_1)}\right]$$

units of commodity A_1.

Let us assume by contradiction that $w/p(A_1)$ is greater than $\frac{1}{4}$. There are two possible cases:

- *if* $p(A_1) = p(A_2) \leq 1$ *then each of the agents* 1, 2 *and* 3 *demands* $\frac{1}{4}$ *units of commodity* A_1 *or* A_2. *This creates an excess of demand and* p *cannot be an equilibrium price. On the other hand, if* $p(A_1) = p(A_2) > 1$ *then* $w/p(A_1)$ *is strictly less than* $\frac{1}{4}$.

- *If* $p(A_1) < p(A_2)$ *then agents* 1 *and* 3 *demand commodity* A_1 *instead of* A_2. *Therefore,* $p(A_2) \leq 2$, *or no agents would demand commodity* A_2. *At the same time, it must be that* $p(A_1) \geq \frac{1}{2t}$ *or agent* 1 *would demand only commodity* A_1, *leaving* 3 *with strictly less than* $\frac{1}{4}$ *units of commodity* A_1. *Combining these two inequalities we obtain:*

$$\frac{w}{p(A_1)} = \frac{1}{4}\left[t + \frac{p(A_2)}{p(A_1)}\left(\frac{1}{2} - t\right) + \frac{1}{2p(A_1)}\right] \leq \frac{1}{4}\left[t + 2t(1-2t) + t\right] = t - t^2$$

which is strictly smaller than $\frac{1}{4}$ *for every* $t < \frac{1}{2}$.

The argument in Example 4 is based only on refinements of π formed by 3 intervals. If we allow for richer classifications, then we could find refinements of π that are strictly preferred to π by every agent in the society. As a way of illustration, let ρ be formed by the intervals:

$$A = \left[0, \frac{1}{4} - \varepsilon\right), \quad B = \left[\frac{1}{4} - \varepsilon, \frac{1}{2}\right], \quad C = \left(\frac{1}{2}, \frac{3}{4} - \varepsilon\right], \quad D = \left(\frac{3}{4} - \varepsilon, 1\right]$$

with $\varepsilon \in \left(0, \frac{1}{4}\right)$. For ε sufficiently small, an equilibrium in $\mathcal{E}(\rho)$ is achieved when all commodities have identical prices with agent 1 consuming the whole of commodity A, agent 3 the whole of commodity B, agent 2 commodity C, and 4 commodity D. This leaves every agent with a utility strictly larger than the one they received with the allocation (x^i).

The example in the next paragraph refines Example 4 by describing an economy where adding new commodities (in any number) to the initial classification damages at least an agent. This is obtained by assuming an atom in the space of goods' characteristics, so that some tradable commodities cannot be split into smaller parts.

4.3 Refinements may reduce the utilitarian welfare

The *utilitarian social welfare* associated to a classification π is the sum of the utilities that agents receive in any competitive equilibrium of $\mathcal{E}(\pi)$. In Example 4, adding new commodities to the initial classification damages some agents but increases the utilitarian social welfare. The next example describes a society where every refinement of the starting classification gives a strictly lower utilitarian social welfare: hence, adding new commodities (in any number) may reduce the sum of agents' utilities, and the initial classification is preferred to any refinement both from a Paretian and a utilitarian point of view.

Example 5. *Let* λ *be the standard Lebesgue measure on* $\mathcal{I}$ *and* $\delta_{\{1\}}$ *denote the Dirac measure associated with the point* 1, *i.e. the function defined by* $\delta_{\{1\}}(F) = 1$

if $1 \in F$ *and* $\delta_{\{1\}}(F) = 0$ *otherwise. We consider a society where there are* $2n$ *agents with identical claims and the measure* ω *is given by:*

$$\omega(F) = \frac{1}{2}\left(\lambda(F) + \delta_{\{1\}}(F)\right).$$

There are only two types of agents, forming groups of equal size. Agents in type 1 *and* 2 *have preferences derived respectively from the evaluation measures:*

$$\nu_1(F) = \lambda(F) \quad \text{and} \quad \nu_2(F) = \frac{1}{4}\lambda\left(F \cap \left[0, \frac{1}{2}\right]\right) + \frac{3}{4}\lambda\left(F \cap \left[\frac{1}{2}, 1\right]\right) + \frac{1}{2}\delta(F).$$

Intuitively, agents of type 1 *value all types of goods identically, while those of type* 2 *care more about goods in* $\left[\frac{1}{2}, 1\right)$ *and give a special importance to those labelled with* 1.

Let π *be the classification formed by the commodities* $A = [0, 1)$ *and* $B = \{1\}$. *Within the economy* $\mathcal{E}(\pi)$ *a competitive equilibrium is reached when* A *and* B *have the same prices, with every agent from group* 1 *consuming* $\frac{1}{2n}$ *units of commodity* A *and every agent from group* 2 *consuming* $\frac{1}{2n}$ *units of* B.

We want to prove that, if $\rho \succ \pi$, *then every competitive allocation in* $\mathcal{E}(\rho)$ *assigns a positive amount of goods of type* A *to agents in group* 2. *Because the utility received from goods of type* A *is higher for agents in group* 1, *we conclude that the sum of agents' utilities in* $\mathcal{E}(\rho)$ *must be strictly lower than in* $\mathcal{E}(\pi)$.

Suppose by contradiction that this is not the case, i.e. that there exists a refinement ρ *of* π *and a competitive allocation in* $\mathcal{E}(\rho)$ *such that agents in group* 1 *consume all goods of type* A *and those in group* 2 *all goods of type* B. *Because* B *is an atom,* ρ *can refine* π *only by splitting* A *into smaller intervals and leaving* B *intact. We write* $\rho = \{A_1, \ldots, A_m, B\}$ *where* $i < j$ *implies* $s < t$ *for all* $s \in A_i$ *and* $t \in A_j$. *By assumption, in equilibrium agents from group* 1 *demand all commodities* $A_1, \ldots, A_m$ *and so these must have all equal prices (or agents of group* 1 *would demand only the cheapest ones). At the same time,* A_m *must cost strictly more than* B, *or agents in group* 2 *would rather demand* A_m *than* B. *It follows that the average price of the commodities* A_j*'s is is strictly greater than that of* B, *implying that each agent in group* 2 *can demand more than* $\frac{1}{2n}$ *units of commodity* B. *This leads to an excess of demand of commodity* B, *which contradicts the assumption that prices are competitive.*

Example 5 compares two competitive configurations: in the first one, agents in group 2 have a large demand set that allows agents in group 1 to select their best option. In the second configuration, the introduction of a new commodity allows agents in group 2 to demand more sophisticated bundles and to compete for some goods that were previously consumed by agents of group 1, even if the latter have a higher evaluation of these goods. This causes the utilitarian welfare to be lower.

The example has a significant share of goods represented by an atom, that cannot be split into smaller commodities. The only possible refinements damage agents of type 1, putting those of type 2 in a favorable position. This suggests that agents with higher evaluations of atoms may benefit more from the introduction of new commodities. Appendix A.2 elaborates on Example 5 and obtains slightly weaker results under the additional assumption that all commodities can be refined, i.e. that the measure ω is atomless.

The last two examples are possibility results, but their import does not hold in general. The example in Appendix A.1 can be used to define a sequence of competitive configurations $\langle \pi_n, (x^i_n) \rangle$ in which every π_{n+1} refines π_n and every configuration is strictly preferred to the previous one both from a Paretian and a utilitarian point of view.

4.4 The optimal number of commodities

Limitations on the number of commodities may be driven by economical considerations. Even when technically possible, operating within economies with many commodities is costly both for the agents, who have to process more information, and for the market institutions, that have to handle more elaborated transactions. A social planner may prefer a simpler environment if the social cost of increasing the number of commodities is higher than what agents gain from the richer trading possibilities.

The socially optimal number of commodities depends on how the planner ranks the possible configurations, on agents' preferences and on the cost of introducing new commodities. This paragraph studies the simple case in which the configuration is chosen as to maximize the utilitarian social welfare and the cost of operating in the market is proportional in the number of commodities acknowledged by the classification.

Formally, let $USW(k)$ be the maximum utilitarian social welfare that can be achieved using *at most* k commodities. In defining USW we consider any configuration $\langle \pi, (x^i) \rangle$ with $|\pi| \leq k$ and do not require it to be competitive (i.e. that (x^i) is a competitive allocation in $\mathcal{E}(\pi)$). The social cost of operating with k commodities is ck, with $c \in (0, 1)$. Thus, the problem faced by the planner is formalized as follows:

$$\max_{k \geq 1} USW(k) - ck. \tag{1}$$

There is always a solution to the problem (1) because the function USW is bounded from above. To see this, recall that none of the n agents can receive a utility higher than 1, which corresponds to consuming all the goods available. Thus, the optimal utilitarian social welfare with any number of commodities lies between 1, which corresponds to giving all goods to one agent, and n, which could be obtained only if every agent could consume all goods. We conclude that there exists a k^* that solves problem (1) and that $1 - c \leq k^* \leq \frac{n-1}{c} + 1$.

Without restrictions on agents' preferences, it is not possible to give any sharper estimation of the optimal value k^*. As a way of illustration, observe that if all agents have evaluation measures that coincides with ω then $USW(k) = 1$ for all k, meaning that $k^* = 1$. On the other hand, the next example shows that k^* could be $\frac{n-1}{c} + 1$.

Example 6. *Let $c \in (0, 1)$ be such that $\frac{n-1}{c} + 1 \in \mathbb{N}$. Consider an economy where ω coincides with the Lebesgue measure and n agents. The evaluation measure of* *each i is:*

$$\nu_i(F) = (1-c)\omega\left(\left[0, \frac{1}{n}\right]\right) + c\,\omega\left(\left(\frac{i-1}{n}, \frac{i}{n}\right]\right).$$

We claim that $USW(k)$ is constantly equal to $1-c$ for every $k \in \{1, \dots, \frac{n-1}{c}+1\}$ because for every k smaller than $\frac{n-1}{c}+1$ the classification formed by k intervals that maximizes the utilitarian social welfare is given by the intervals

$$A_1 = \left[0, \frac{1}{n}\right],\ A_2 = \left(\frac{1}{n}, \frac{2}{n}\right], \dots, A_k = \left(\frac{k-1}{n}, 1\right].$$

5 Equilibrium without prices

Typically, a decentralized allocation mechanism rests on a set of social norms that constrain agents' consumption possibilities. In equilibrium, agents have the right to demand their preferred bundle among those conformant with the active social norm, and their demands are mutually compatible; see Richter and Rubinstein (2020). A competitive configuration, for example, is a price-based equilibrium where individuals demand only bundles that are tradable and meet specific budget constraints.

This section defines an allocation mechanism where the mere choice of an appropriate classification suffices to achieve an equilibrium. To this end, we compartmentalize the effect of the classification on individual demands by introducing a simple social norm of rationing that is not price-based: the quantity of goods that an agents can demand cannot exceed his claim. Thus, given a classification π, a π-bundle is *admissible for agent i* if it belongs to the set:

$$A_i(\pi) = \left\{ y \in \mathbb{R}_i^\pi : \sum_{B \in \pi} y_B \leq \kappa_i \right\}.$$

Agent i's *π-demand set* is the collection of π-bundles that maximize i's utility among those that are admissible for i, i.e. the set:

$$D_i(\pi) = \{x \in A_i(\pi) : V_i(\pi, x) \geq V_i(\pi, y) \text{ for every } y \in A_i(\pi)\}.$$

A configuration $\langle \pi, (x^i) \rangle$ is an *equilibrium* if $x^i \in D_i(\pi)$ for every i. We say that π *supports* the equilibrium.

Compare the standard price-based notion of equilibrium with the one just defined. The first one relies on the assumption that agents can exchange goods in different quantities. All consumers may trade up a large amount of some commodity for a smaller portion of another one they like more, when the prices allow it. This is problematic when agents are not allowed to bargain over the quantity of goods they receive. On the contrary, our notion of equilibrium let agents exchange only bundles that contain the same total amount of goods. The following is a simple example of a situation where the total amount of commodities for each agent is fixed, so that our equilibrium notion is preferable to the standard price-based one.

Example 7. *A College study program offers n courses organized in two teaching periods. The schedule must satisfy three criteria: lessons cannot overlap, the total number of teaching hours is constant, and the classes of a course are evenly distributed within each of the assigned periods. Every professor must teach one course of the same length and has preferences on the period and the concentration of his classes. The school board chooses how to divide the academic year in two terms, and then asks each professor to indicate in which of the two terms she prefers to teach. We ask whether there exists a division in two terms that satisfies every teacher's requests.*

We imagine that each professor is assigned a bundle (x, y) where x and y are the number of hours she has to teach in the first and in the second term, respectively. By viewing each teaching period as a different commodity, the scheduling problem can be approached as an allocation problem. In this framework, an equilibrium would be a division of the school year in two teaching periods that ensures that every teacher is free to choose how to divide his classes between the two terms.

Remark. Because an equilibrium corresponds to a competitive configuration where all commodities come at the same price, the equilibrium allocation is trivially envy-free, in the sense that no agent will prefer the bundles assigned to any other consumer with the same claim. Furthermore, one may adapt the argument in Example 1 exhibiting two equilibria, one of which strictly Pareto-dominates the other, and show that our notion of equilibrium may not lead to Pareto-optimal configurations. It does not seem obvious how to establish the existence of a Pareto-optimal equilibrium.

5.1 Equilibrium existence

The coarse partition $\pi = \{\mathcal{I}\}$ always supports an equilibrium where each agent i receives a fraction k_i of the overall endowment. This section studies conditions under which for every number of commodities there is a classification supporting an equilibrium. We find that two conditions are sufficient to ensure this result: a) the measure ω has to be non-atomic, so that the endowment of goods is well distributed; and b) at least an agent i has a higher value when a commodity is defined by a smaller interval, exhibiting a strong preference for concentration. To formalize this last notion, we consider a larger class of preferences.

We associate every agent i with a set-function η_i defined on the subsets of $\mathcal{I}$, and called i's *evaluation function*. Intuitively, $\eta_i(B)$ is the utility that agent i receives from consuming commodity B; in particular, x units of commodity B yield utility $\frac{x}{\omega(B)}\eta_i(B)$. Given a classification π, the utility $V_i(\pi, x)$ of agent i for a π-bundle $x \in \mathbb{R}^{\pi}_{+}$ is the sum of the evaluations for each commodity weighted by the quantity received:

$$V_i(\pi, x) = \sum_{B \in \pi} \frac{x_B}{\omega(B)} \eta_i(B). \tag{2}$$

We assume that each evaluation function η_i satisfies the following conditions:

- η_i is *normalized*, i.e. $\eta_i(\emptyset) = 0$ and $\eta_i(\mathcal{I}) = 1$;

- η_i is *monotone*, i.e. $\eta_i(F) \leq \eta_i(G)$ whenever $F \subseteq G$ are subsets of $\mathcal{I}$;
- η_i is *submodular*, i.e. $\eta_i(F \cup G) + \eta_i(F \cap G) \leq \eta_i(F) + \eta_i(G)$ for every two $F, G \subseteq \mathcal{I}$;
- η_i is *absolutely continuous with respect to* ω, i.e. $\eta_i(F^n) \to 0$ whenever (F^n) is a monotone sequence of subsets of $\mathcal{I}$ such that $\omega(\bigcap F^n) = \emptyset$.

The submodularity of the function η_i implies decreasing marginal evaluations: for every $F, G, G' \subseteq \mathcal{I}$ such that $G \subset G'$, we have $\eta_i(F \cup G) - \eta_i(G) \geq \eta_i(F \cup G') - \eta_i(G')$, meaning that the marginal benefit of adding F to a portion G of $\mathcal{I}$ is decreasing in the content of G. The absolute continuity of η_i with respect to ω implies that any agent's evaluation for a vanishing quantity of goods decreases to 0. Note that the evaluation measures used above are just additive evaluation functions: Equation (2) is the natural extension of the functions V_i's generated by evaluation measures.

Another consequence of submodularity is that the ratio $\eta_i(F)/\omega(F)$ increases as $\omega(F)$ decreases. Intuitively, the average benefit from a type of goods increases as the type becomes "more concentrated". We say that agent i has a *strong preference for concentration* (*SPC* for short) if $\lim_n \eta_i(F^n)/\omega(F^n) = \infty$ for every sequence (F^n) of intervals such that $\omega(F^n) > 0$ for all $n \in \mathbb{N}$ and $\lim_n \omega(F^n) = 0$.

We can now formalize the main result.

Theorem 6. *If ω is non-atomic and at least an agent has a strong preference for concentration, then for every $k \in \mathbb{N}$ there exists an equilibrium supported by a classification with k intervals.*

The two hypotheses are: a) the measure ω is non-atomic, and b) there is an agent with SPC. The next two examples show that neither assumption can be easily dropped. A third example shows that they are not necessary.

Example 8 (A society where ω is non-atomic but no agent has SPC)**.** *There are n agents and ω is the Lebesgue measure. Every agent i has an evaluation function defined by:*

$$\eta_i(F) = \int_F u_i \, d\omega.$$

for some strictly increasing, integrable function u_i. Because each η_i is additive, no agent exhibits SPC. We claim that no classification based on $k \geq 2$ intervals can support an equilibrium.

Take any classification $\pi = (B_1, \ldots, B_k)$ and let $0 = \theta_0 < \theta_1 < \cdots < \theta_k = 1$ be such that θ_{j-1} and θ_j are the extreme points of the interval B_j. An agent i maximizes the utility $V_i(\pi, x)$ by demanding positive amounts only for the tradable commodities B_j for which the ratio

$$\frac{\eta_i(B_j)}{\omega(B_j)} = \frac{\int_{\theta_{j-1}}^{\theta_j} u_i \, d\omega}{(\theta_j - \theta_{j-1})}$$

is maximized. On the other hand, because u_i is an increasing function, the map $t \mapsto \int_0^t u_i \, d\omega$ is convex and so:

$$\frac{\int_{\theta_{k-1}}^{\theta_k} u_i \, d\omega}{(\theta_k - \theta_{k-1})} > \frac{\int_{\theta_{k-2}}^{\theta_{k-1}} u_i \, d\omega}{(\theta_{k-1} - \theta_{k-2})} > \cdots > \frac{\int_{\theta_0}^{\theta_1} u_i \, d\omega}{(\theta_1 - \theta_0)}.$$

Because every agent demands exclusively the same k-th tradable commodity, there is a positive excess of demand under any classification π with $k \geq 2$. We conclude that no such classification can support an equilibrium.

In Example 8 agents have additive evaluations: their demands are not affected by the width of the intervals in the classification π. This no longer holds if a consumer exhibits SPC, because that consumer is attracted to sufficiently smaller cells.

Example 9 (A society where every agent has SPC but ω is atomic)**.** *Consider an economy where half of the total amount of goods correspond to the point 0 and the other half correspond to 1. Using the notation for Dirac measures, the measure ω assigns to each $F \subseteq \mathcal{I}$ the value*

$$\omega(F) = \frac{1}{2}\delta_{\{0\}}(F) + \frac{1}{2}\delta_{\{1\}}(F)$$

There are n agents, who all have the same evaluation function $\eta_i(F) = \delta_{\{1\}}(F)$. By construction, ω has two atoms and every agent exhibits SPC. We claim that no classification based on $k \geq 2$ intervals can support an equilibrium.

Given any classification π, every agent prefers the cell B containing 1 over any other cell and therefore demands only this commodity. This implies a positive excess of demand for B, and the conclusion follows.

Example 9 shows how the presence of large chunks of identical goods can make agents' demands insensitive to changes in the classification. This cannot occur when the measure ω is non-atomic, because the amount of goods labelled with the same $t \in \mathcal{I}$ is negligible.

Our third example shows that the two sufficient assumptions in Theorem 6 are not necessary.

Example 10 (A society where ω is atomic and no agent has SPC, but equilibrium exists)**.** *There are three agents. The measure ω is defined by*

$$\omega(F) = \lambda\left(F \cap \left[0, \frac{2}{3}\right]\right) + \frac{1}{3}\delta_{\{1\}}(F),$$

where λ denotes the standard Lebesgue measure. Assume that the three agents have the following evaluation functions:

$$\eta_1(F) = \int_F 2t \, dt, \qquad \eta_2(F) = \eta_3(F) = \int_F 2(1-t) \, dt.$$

Note that ω has the atom $\{1\}$ and that no agent has SPC because all the evaluation functions are additive.

Consider the classification $\pi = \{[0, 2/3], (2/3, 1]\}$. Then agent 1 demands the π-bundle $x^1 = (0, 1)$, while agents 2 and 3 demand the π-bundle $x^2 = x^3 = (1/2, 0)$. Because $\langle \pi, (x^a) \rangle$ is an allocation, we conclude that π supports an equilibrium.

5.2 Further extensions

We can relax some assumptions on the model without compromising the main existence result of Theorem 6. We illustrate two extensions under which one can prove the existence of equilibria in slightly more general settings.

Measure space for the goods' characteristics. Our model assumes that the space of goods' characteristics is a totally ordered set and that commodities are defined as intervals. This can be relaxed to an abstract measure space for the goods' characteristics, where commodities are defined by measurable subsets. We sketch the main features of this more general approach.

Let a measurable space (X, Σ) denote the set of goods' characteristics. Similarly to our framework, every $t \in X$ corresponds to a complete description of a single good and each $F \in \Sigma$ is a *type of good*. A normalized measure $\omega \colon \Sigma \to [0, 1]$ describes the availability of goods. We define a *classification of goods* as a partition π of X formed by finitely many sets in Σ with positive ω-measure. The definitions of π-bundles, of agents' evaluations and of equilibrium are naturally adapted to this new setup.

Even in this broader setting, there exists a non-trivial classification supporting an equilibrium if ω is non-atomic and at least an agent has SPC. In fact, one can define a specific family of classifications with properties so similar to those of classifications formed by intervals of $\mathcal{I}$ that the proofs are almost identical. The idea is to choose an increasing family of sets, then mimic a "moving-knife procedure" to define a class of partitions similar to those formed by intervals in $\mathcal{I}$.

Formally, let $\mathcal{C} = \{C_t : t \in \mathcal{I}\} \subseteq \Sigma$ be a monotone chain such that $\omega(C_t) = t$ for all $t \in \mathcal{I}$. Such a chain always exists by the non-atomicity of ω. A set J is a $\mathcal{C}$*-interval* if there exists $t < s$ in $\mathcal{I}$ such that $J = C_s \setminus C_t$. Let $\Pi^{\mathcal{C}}(k)$ be the set of classifications formed by at least a number $k \geq 2$ of $\mathcal{C}$-intervals. One may extend the proof of Theorem 6 with respect to $\mathcal{C}$-intervals in X instead of intervals in $\mathcal{I}$.

It is worth observing that in this more general setting the class of possible classifications is much larger than the one described by using $\mathcal{I}$ as space of goods' characteristics. Therefore, while it is easy to prove an equilibrium exists, some results from the other sections may no longer hold. As a way of illustration, the proof that there exists a configuration that is Pareto-optimal within the set of competitive configurations cannot be directly extended to this broader environment.

Weaker form of SPC. The assumption that at least an agent has SPC is restrictive, because it requires that there is an agent that will drastically change his choice whenever he is offered a sufficiently concentrated commodity. From a technical viewpoint, however, this assumption is used only to show that the aggregate demand correspondence meets some standard boundary conditions.

The assumption can be relaxed into a local requirement: if the interval defining a commodity is sufficiently small, then there is at least one consumer who prefers it to all the other commodities. Using the notation above, we formalize this as a condition of *distributed SPC*:

> If $\pi^n = (B_1^n, \dots, B_i^n, \dots B_k^n)$ is a sequence of classifications in $\Pi(k)$ and $\omega(B_i^n) \to 0$ as $n \to \infty$, then there exists an agent whose demand for commodity B_i^n goes to infinity as $n \to \infty$.

Under distributed SPC, the proof of Theorem 6 holds unchanged.

It may be useful to compare the import of SPC versus distributed SPC to appreciate the greater realism of this latter. Concerning the explanatory example in the introduction, SPC requires that there is an agent who, given any classification, might change his choice if he is offered another type of wine using a purer selection of grapes; distributed SPC requires only that, for any classification, there is some agent willing to. Similarly, concerning the example of the academic calendar, SPC requires that a professor would accept to teach in any moment of the year as long as his course is sufficiently concentrated, while distributed SPD is satisfied if, for any calendar, one can find a professor who is willing to.

A Appendix

A.1 Non-existence of Pareto-optimal configurations

The next example describes a society with two agents, where every configuration can be improved both from a Paretian and an utilitarian point of view with a suitable refinement of the underlying classification.

Example 11. *Consider an economy where ω is the Lebesgue measure and there are two agents with identical claims. Let $S \subset \mathcal{I}$ denote the Smith-Volterra-Cantor set (SVC set for short), which is a measurable set of size $\frac{1}{2}$ with the property that every non-null interval in $\mathcal{I}$ contains a non-null interval disjoint from S; see the ϵ-Cantor set in Aliprantis Burkinshaw (1981, p. 141). Agents' of each group have preferences derived from the following evaluation measures:*

$$\nu_1(F) = 2\omega(F \cap S), \quad \nu_2(F) = 2\omega(F \setminus S).$$

We claim that the only Pareto-optimal configurations assign the 0 bundle to all agents of group 1.

Let $\langle \pi, (x^i) \rangle$ be a configuration and let the interval $B \in \pi$ be a commodity such that $x_B^1 > 0$. By the properties of the SVC set S, there exists an interval $C \subseteq B$ such that $C \cap S = \emptyset$, and so $\nu_1(C) = 0$ and $\nu_2(C) = 2\omega(C)$. If we label C as a new commodity, we obtain a finer classification ρ under which one can transfer all goods of type C previously assigned to 1 to agent 2, while leaving the rest of the allocation unchanged. But this benefits agent 2 without causing harm to 1 (because her evaluation of C is null), proving that $\langle \pi, (x^i) \rangle$ is Pareto-dominated.

Formally, let B^1 and B^2 be the two (possibly, empty) intervals obtained by removing C from B. Let ρ be a refinement of the classification π, where the *commodity B has been replaced with B^1, B^2 and C. Consider a new allocation (y^i) in $\mathcal{E}(\rho)$ where the bundle assigned to agent 1 is*

$$y^1_A = \frac{\omega(A)}{\omega(B)} x^1_B \text{ if } A \in \{B^1, B^2\}, \quad y^1_C = 0, \quad y^1_A = x^1_A \text{ otherwise;}$$

and the bundle assigned to agent 2 is:

$$y^2_A = \frac{\omega(A)}{\omega(B)} x^2_B \text{ if } A \in \{B^1, B^2\}, \quad y^2_C = \frac{\omega(C)}{\omega(B)} x^2_B + \frac{\omega(C)}{\omega(B)} x^1_B, \quad y^2_A = x^2_A \text{ otherwise.}$$

Computations shows that (y^i) is a feasible allocation in $\mathcal{E}(\rho)$ that agent 1 finds equivalent to (x^i), while agent 2 strictly prefers it to (x^i). Standard arguments based on the continuity and monotonicity of the function $V_i(\rho, \cdot)$ prove that one can modify (y^i) into a new allocation that every agent strictly prefers to (x^i).

Clearly, Example 11 relies crucially on the assumption that agents' evaluations of goods are expressed through extremely elaborated subsets of $\mathcal{I}$ (such as the SVC set) while commodities can only be defined as intervals. If we allow commodities to be arbitrary subsets of $\mathcal{I}$, then the classification $\pi = \{S, S^c\}$ would generate a Pareto-optimal configuration where all goods of type S are assigned to agent 1 and the rest to agent 2. This suggests that the stronger the exogenous constraints on the classification of goods into commodities, the further agents may be from reaching optimal allocations.

A.2 Refinements may reduce the utilitarian welfare even without atoms.

It is possible to adapt Example 5 and obtain slightly weaker results under the additional assumption that all commodities can be refined, i.e. that the measure ω is atomless. The following example describes a society with two commodities A and B, where splitting A into smaller and smaller intervals keeps reducing the social welfare, while splitting B leaves it unchanged.

Example 12. *Let ω be the Lebesgue measure and let there be $2n$ agents with identical claims, arranged in two groups of equal size. Agents in type 1 and 2 have preferences derived respectively from the evaluation measures:*

$$\nu_1(F) = 2\omega\left(F \cap \left[0, \frac{1}{2}\right)\right), \text{ and}$$
$$\nu_2(F) = \frac{1}{2}\omega\left(F \cap \left[0, \frac{1}{4}\right)\right) + \frac{3}{2}\left(F \cap \left[\frac{1}{4}, \frac{1}{2}\right)\right) + \omega\left(F \cap \left[\frac{1}{2}, 1\right]\right).$$

Consider the classification $\pi = \{A, B\}$ where $A = \left[0, \frac{1}{2}\right)$ and $B = \left[\frac{1}{2}, 1\right]$. In $\mathcal{E}(\pi)$ an equilibrium occurs when the commodities have identical prices and each agent i consumes $\frac{1}{2n}$ units of commodity A or B if she is in group 1 or in group 2, respectively.

This situation is almost identical to Example 5, except for the fact that the atom in 5 is replaces by the refinable commodity B. We claim that no refinement of π can improve the utilitarian social welfare, although some may reduce it.

Let $\rho \succ \pi$ be a refinement of any size. If ρ is obtained by splitting B in two commodities B^1 and B^2, then agents of either group 2 remain indifferent between B^1 and B^2. The equilibrium remains essentially unaltered and the utilitarian welfare does not change. Differently, if ρ is obtained by splitting A in any number of commodities, then the same arguments in Example 5 show that the utilitarian welfare decreases.

We conclude that the refinements that do not cause a decrease in the utilitarian social welfare must split only commodity B in smaller commodities.

In Example 12, if one defines ρ by randomly adding a new threshold to the partition that defines the classification π, then the probability that the utilitarian welfare decreases is equal to the probability that the new threshold refines commodity B. When this threshold is chosen according to a uniform distribution on $\mathcal{I}$, there is a 50% chance that the utilitarian social welfare decreases. Therefore, a social planner that selects a sequence of refinements (π_n) at random will almost surely cause a loss in the utilitarian social welfare, as long as the thresholds defining each π_{n+1} are uniformly distributed on $\mathcal{I}$.

A.3 Proofs

Proposition 1. *$(\Pi(k), d_\omega)$ is a compact space where two classifications have zero-distance if and only if they coincide up to negligible sets.*

Proof. Let $\mathcal{J}$ be the set of all intervals in $\mathcal{I}$ and h the pseudo-metric $h(F, G) = \omega(F \triangle G)$. Then $d_\omega(\pi, \rho)$ is the Hausdorff distance between the algebras $\sigma(\pi)$ and $\sigma(\rho)$, seen as subsets of $(\mathcal{J}, h)$. Therefore, $d_\omega(\pi, \rho) = 0$ if and only if $\sigma(\pi)$ and $\sigma(\rho)$ have the same closure in $(\mathcal{J}, h)$ (see Lemma 3.72, Aliprantis and Border, 2006), i.e. if every $B \in \sigma(\pi)$ coincides with a $B' \in \sigma(\rho)$ up to null sets. Because $\sigma(\pi)$ and $\sigma(\rho)$ are finite, this holds if and only if the sets of generators for the two algebras (i.e. the classifications π and ρ) coincide up to null sets.

As per the compactness of $\Pi(k)$, if we identify intervals with zero-distance, then the function ω maps isometrically $\mathcal{J}_0 = \{F \in \mathcal{J} : 0 \in F\}$ into a closed and bounded subset of $\mathcal{I}$, so that $(\mathcal{J}_0, h)$ is itself compact. This implies that $\Pi(k)$ is compact, because it is the image of the compact product space $\mathcal{J}_0^{k-1}$ under the continuous function

$$\varphi(F_1, \ldots, F_{k-1}) = \{F_{i+1} \setminus F_i : \omega(F_{i+1}) > \omega(F_i) \text{ and } F_k = \mathcal{I}\}.$$

□

Theorem 2. *The competitive equilibrium-correspondence $\mathcal{W}$ is upper-hemicontinuous on the compact space $(\Pi(k), d_\omega)$.*

Proof. We associate every $\pi \in \Pi(k)$ with an auxiliary exchange economy $\tilde{\mathcal{E}}(\pi)$ with commodity space $\mathbb{R}^k_+$ and with the property that $\mathcal{W}(\pi)$ is the continuous image of the set $\tilde{\mathcal{W}}(\pi)$ of competitive outcomes in $\tilde{\mathcal{E}}(\pi)$. This way, it is sufficient to prove that the correspondence $\tilde{\mathcal{W}}$ is upper-hemicontinuous.

For any classification $\pi = (B_1, \dots, B_m)$ with $m \le k$, let $\tilde{\mathcal{E}}(\pi) = \mathcal{E}(\pi)$ if $m = k$. Otherwise, if $m < k$, for each agent i define his endowment in $\tilde{\mathcal{E}}(\pi)$ as

$$\tilde{e}^i(\pi) = \kappa^i \left(\omega(B_1), \dots, \omega(B_{m-1}), \frac{\omega(B_m)}{k-m}, \dots, \frac{\omega(B_m)}{k-m} \right)$$

and his utility from the vector $x \in \mathbb{R}^k_+$ as

$$\tilde{V}_i(\pi, x) = \sum_{j=1}^{m-1} \frac{x_j}{\omega(B_j)} \nu_i(B_j) + \sum_{j=m}^{k} \frac{x_j}{\omega(B_m)} \nu_i(B_m).$$

Because the economies $\tilde{\mathcal{E}}(\pi)$ satisfy all the assumptions of the main Theorem in Hildebrand and Mertens (1972), the equilibrium-set correspondence $\tilde{\mathcal{E}}(\pi) \mapsto \tilde{\mathcal{W}}(\pi)$ is upper-hemicontinuous. The map $\pi \mapsto \tilde{\mathcal{E}}(\pi)$ is also continuous, as $\tilde{e}^i(\pi)$ and $\tilde{V}^i(\pi, \cdot)$ change continuously with π. Then the whole correspondence $\tilde{\mathcal{W}}$ is upper-hemicontinuous.

To conclude the proof, notice that an allocation $(\tilde{x}^i)$ in $\tilde{\mathcal{E}}(\pi)$ is competitive if and only if the vectors $x^i = (\tilde{x}^i_1, \dots, \tilde{x}^i_{m-1}, \sum_{j=m}^k \tilde{x}^i_j)$ constitute a competitive allocation in $\mathcal{E}(\pi)$. Thus, $\mathcal{W}(\pi)$ is the continuous image of $\tilde{\mathcal{W}}(\pi)$. □

Corollary 3. *The function Ψ_i is continuous on the compact space $(\Pi(k), d_\omega)$.*

Proof. Because every two allocations in $\mathcal{W}(\pi)$ yield the same utility profiles, $\Psi_i(\pi)$ is the composition $V_i(\pi, \mathcal{W}(\pi))$ and its continuity derives from that of the functions V_i and $\mathcal{W}$. □

Theorem 4. *The set of competitive configurations based on classifications in $\Pi(k)$ has a Pareto-optimal configuration.*

Proof. Because the Ψ_i's are continuous functions on the compact space $(\Pi(k), d_\omega)$, there exists $\pi^* \in \Pi(k)$ that maximizes $\sum \Psi_i(\pi)$. Let $(x^i) \in \mathcal{W}(\pi^*)$ and observe that a competitive configuration $\langle \pi, (y^i) \rangle$ Pareto-dominates $\langle \pi^*, (x^i) \rangle$ if and only if $\sum V_i(\pi, y^i) > \sum V_i(\pi^*, x^i)$, violating the maximality of π^*. We conclude that $\langle \pi^*, (x^i) \rangle$ is Pareto-optimal. □

Proposition 5. *If $\langle \pi, (x^i) \rangle$ is a feasible Pareto-optimal configuration such that $x^i_B > 0$ for every i and every B in π, then one can redefine agents' claims so that $\langle \pi, (x^i) \rangle$ is competitive.*

Proof. The second Welfare Theorem applied to the allocations in the exchange economy $\mathcal{E}(\pi)$ implies that there exists a price system $p \in \mathbb{R}^\pi_+$ that supports the allocation (x^i). Then $\langle \pi, (x^i) \rangle$ is a competitive configuration once we redefine agents' claim as:

$$\kappa^i = \sum p_B \frac{x^i_B}{\omega(B)}.$$

□

Theorem 6. *The competitive equilibrium-correspondence $\mathcal{W}$ is upper-hemicontinuous on the compact space $(\Pi(k), d_\omega)$.*

Proof. Let $\Pi[k] \subset \Pi(k)$ denote the set of classifications formed by exactly k commodities, whereas $\Pi(k)$ is the set of classification formed by *at most* k commodities. We may rearrange each π in $\Pi[k]$ in a string of intervals $\pi = (F_1, \dots, F_k)$ where $i < j$ if and only if $s < t$ for every $s \in F_i$ and $t \in F_j$. This associates each classification π in $\Pi[k]$ with a vector $\omega(\pi) \in \mathbb{R}^k_+$ defined by:

$$\omega(\pi) = (\omega(F_1), \dots, \omega(F_k)) .$$

Since the measures of every cell in π is positive, $\omega(\pi)$ lies in the set Δ° of all strictly positive vectors in $\mathbb{R}^k_+$ whose components sum up to one.

When ω is atomless, for every $p \in \Delta^\circ$ there exists a classification $\pi_p \in \Pi(k)^*$ such that $\omega(\pi_p) = p$, and π_p is unique up to null sets. Thus, the map $\pi \mapsto \omega(\pi)$ identifies $\Pi[k]$ with Δ° up to null differences.

Standard arguments prove that all the following properties hold:

1. $\hat{A}\colon p \mapsto A(\pi_p)$ is a continuous correspondence with convex and compact values.

2. For every i, the function $\hat{V}_i\colon \Delta^\circ \times \mathbb{R}^k_+ \to \mathbb{R}_+$ defined by $\hat{V}_i(p, x) = V_i(\pi_p, x)$ is continuous and linear in the second coordinate. Therefore, $\hat{D}_i\colon p \mapsto D_i(\pi_p)$ is a upper-hemicontinuous correspondence with convex and compact values.

3. For every i, $p \in \Delta^\circ$ and $x \in \hat{D}_i(p)$ imply $p \cdot x = \kappa^i$.

4. Suppose that a sequence $(p^n) \subset \Delta^\circ$ converges to some $p^* \notin \Delta^\circ$ and let $I^* = \{j \le k : p^*_j = 0\}$. If an agent i^* has SPC and $x^n \in \hat{D}_{i^*}(p^n)$ for every n, then $\sum_{j \in I^*} x^n_j \to \infty$.

Define the excess of demand correspondence $Z\colon \Delta^\circ \twoheadrightarrow \mathbb{R}^k$ by

$$Z(p) = \left(\sum \hat{D}_i(p)\right) - p, \quad \text{for every } p \in \Delta^\circ.$$

Then the properties above imply: (i) Z is an upper-hemicontinuous correspondence with convex and compact values; (ii) Z is bounded from below; (iii) $p \cdot z = 0$ for every $p \in \Delta^\circ$ and $z \in Z(p)$; and (iv) $\lim_n \inf Z(p^n) = \infty$ whenever $(p^n) \subset \Delta^\circ$ is a sequence that converges to some $p \notin \Delta^\circ$. The Gale-Debreu-Nikaidô Lemma applies to Z and so there must be a p^* such that $0 \in Z(p^*)$.

Let $\pi_{p^*} \in \Pi[k]$ be such that $\omega(\pi_{p^*}) = p^*$. By construction, $D_i(\pi_{p^*}) = \hat{D}_i(p^*)$ for every i, which implies that $\omega(\pi_{p^*}) \in \sum_i D_i(\pi_{p^*})$. Thus we can select a π-bundle $x^i \in D_i(\pi_{p^*})$ for every i with the property that $\sum_i x^i_B = \omega(B)$ for every $B \in \pi$. But then $\langle \pi_{p^*}, (x^i) \rangle$ is an equilibrium. □

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