

On Retrospective Insurance Premium

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Abstract

In this paper we focus on the expected retrospective premium under equivalent representations. Some general properties related to extreme premia (minimum and maximum) are presented. The particular case of a risk with translated gamma distribution is discussed.

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1 Introduction

In insurance practice, premiums with a random component are often applied. As an example, in reinsurance contracts, when the reinsurer makes a profit above a certain threshold, he generally pays the insurer a commission on the profit. In this way, the effective cost of the reinsurance coverage of the direct insurer has a random component: given the fixed ex ante reinsurance premium, also the reimbursements received by the reinsurer have to be taken into account. Another example of a random insurance and reinsurance premium

is that expressed under the retrospective premium principle. This calculation method is widely used in liability insurance and worker compensation in the United States.

Retrospectively rated insurance is an insurance policy with a premium that adjusts according to the losses experienced by the insured company, rather than according to industry-wide loss experience. An initial premium is charged and adjustments are performed periodically after the policy has expired. This method serves as an incentive to the insured company to control its losses since the price of the policy is likely to decrease if the insured is able to limit risk exposure. The premium can be adjusted within a certain range of values, and it is subject to a minimum and a maximum amount.

In actuarial literature not so many papers analyzed this widespread practice from the analytical point of view (see [3, 5, 7, 8, 9]). In this paper, after a description of the retrospective rating plan (Section 2), we focus on the expected retrospective premium (Section 3) and, through some equivalent representations, we are able to prove some of its properties. The Section 4 analyzes the expected retrospective premium in the particular case of a risk with translated gamma distribution.

Section 5 contains some observations and research suggestions.

2 Retrospective rating plan

With the aim of taking into account the insured's current experience of loss, the insurance practice proposes a rating plan in which the premium can be adjusted, but with the condition that it still remains between a minimum and a maximum value. Among the different literature proposals (see [3, 8]), we make the choice of referring to the following definition of retrospective premium for a non-negative risk Y , with $E[Y] < \infty$:

$$\Pi(Y) = \min \{ \max \{ (B_Y + LY)T, G_Y \}, H_Y \} \quad (1)$$

where:

$L \geq 0$ is the loss conversion factor that covers the loss adjustment expenses;

$T > 1$ is the tax multiplier including premium tax;

B_Y , G_Y and H_Y are respectively, the basic, the minimum and the maximum premium, all are non-negative.

In literature, B_Y , G_Y , H_Y , L and T are called *rating parameters*; they are related to each other, in fact by definition (1) it is $G_Y \leq \Pi(Y) \leq H_Y$.

To avoid the case of a deterministic premium $\Pi(Y)$, from now on it is assumed $G_Y < H_Y$ and $L > 0$.

The basic premium is generally obtained as the minimum amount B_Y such that the expected retrospective premium $E[\Pi(Y)]$ assumes a not negative fixed value K not exceeding the maximum premium H_Y : $E[\Pi(Y)] = K$ where $0 \leq K \leq H_Y$.

In the recent paper [3], the Authors consider the retrospective premium as a reinsurance premium and set some hypothesis on the rating parameters: they assume $G_Y = 0$, $H_Y = (1 + \theta)E[Y]$, and they set the basic premium B_Y equal to the minimal solution of the following equation: $E[\Pi(Y)] = (1 + k)E[Y]$, ($0 < k < \theta$). The insurers total risk exposure is shown to be increasing in parameters L and θ in the sense of convex order (Theorem 3.1 in [3]).

In this paper we consider a different formalization of the retrospective premium $\Pi(Y)$ by referring to y_m and y_M where $y_m \leq y_M$. In this way, $\Pi(Y)$ is defined as:

$$\Pi(Y) = \begin{cases} G_Y & \text{for } Y \leq y_m \\ (B_Y + LY)T & \text{for } y_m < Y < y_M \\ H_Y & \text{for } Y \geq y_M \end{cases} \quad (2)$$

The quantities y_m and y_M play the role of *largest loss* that results in the minimum premium G_Y and of *smallest loss* that results in the maximum premium H_Y , respectively (see [8]). They are related to the rating parameters in this way:

$$G_Y = (B_Y + Ly_m)T \quad H_Y = (B_Y + Ly_M)T \quad (3)$$

so that the retrospective premium can be restated as

$$\Pi(Y) = G_Y + LT[\min(Y, y_M) - \min(Y, y_m)]. \quad (4)$$

Note that from (3), in order to have $y_m \geq 0$ (and $y_M \geq 0$), necessarily it is $G_Y \geq BT$.

3 Expected Retrospective Premium

We analyze the expected retrospective premium by referring to the limited expected value function (l.e.v.f.) M_Y :

$$M_Y(u) = E[\min(Y, u)] = \int_0^u (1 - F_Y(y))dy, \quad (u \geq 0) \quad (5)$$

where F_Y denotes the distribution function di Y . The expected retrospective premium $E[\Pi(Y)]$ can be written as

$$E[\Pi(Y)] = G_Y + LT(M_Y(y_M) - M_Y(y_m)). \quad (6)$$

Given the basic premium B_Y and the rating parameters L and T , in this paper we study the expected premium (6) as a function $\Psi(y_m, y_M)$ of y_m and y_M :

$$\Psi(y_m, y_M) = B_Y T + LT \left(y_m + \int_{y_m}^{y_M} (1 - F_Y(y))dy \right). \quad (7)$$

Observe that the inequality $0 \leq \int_{y_m}^{y_M} (1 - F_Y(y)) dy \leq y_M - y_m$ and (3) imply

$$G_Y = B_Y T + LT y_m \leq \Psi(y_m, y_M) \leq B_Y T + LT y_M = H_Y \quad (8)$$

where the minimum and maximum premia, G_Y and H_Y respectively, are image values of Ψ : $G_Y = \Psi(y_m, y_m)$ and $H_Y = \Psi(y_M, y_M)$. The particular case of a minimum premium proportional to the basic premium through T (i.e. $G_Y = B_Y T$), corresponds to

$$\Psi(0, y_M) = B_Y T + LT \int_0^{y_M} (1 - F_Y(y)) dy. \quad (9)$$

Moreover, given that $\lim_{u \rightarrow \infty} M_Y(u) = E[Y]$, it is

$$\lim_{y_M \rightarrow \infty} \Psi(y_m, y_M) = B_Y T + LT (y_m + E[Y] - M_Y(y_m)) \quad (10)$$

where $E[Y] - M_Y(y_m) = E[\max(Y - y_m, 0)]$.

If we assume that the distribution function F_Y is continuous, then the function $\Psi(y_m, y_M)$ turns out to be differentiable with

$$\frac{\partial}{\partial y_m} \Psi(y_m, y_M) = LT F_Y(y_m) \quad \frac{\partial}{\partial y_M} \Psi(y_m, y_M) = LT (1 - F_Y(y_M)). \quad (11)$$

Moreover, $\Psi(y_m, y_M)$ results to be increasing as a function of the variable y_m or as a function of y_M .

The difference between the expected premium $\Psi(y_m, y_M)$ and the minimum premium $G_Y = \Psi(y_m, y_m)$ is

$$\Psi(y_m, y_M) - B_Y T - LT y_m = LT \int_{y_m}^{y_M} (1 - F_Y(y)) dy; \quad (12)$$

it is a decreasing function of y_m , given y_M (i.e., the maximum premium H_Y); it is an increasing function of y_M , given y_m (i.e., the minimum premium G_Y). Moreover, this difference is bounded from above: in fact, given y_m , the limit (10) and the monotonic behaviour ensure the inequality

$$\Psi(y_m, y_M) - B_Y T - LT y_m \leq LT (E[Y] - M_Y(y_m)). \quad (13)$$

The difference between the maximum premium $H_Y = \Psi(y_M, y_M)$ and the expected premium $\Psi(y_m, y_M)$ is

$$B_Y T + LT y_M - \Psi(y_m, y_M) = LT \left(y_M - y_m - \int_{y_m}^{y_M} (1 - F_Y(y)) dy \right); \quad (14)$$

given y_m , it turns out to be an increasing function of y_M , while given y_M , it is a decreasing function of y_m (by (11)). The difference is bounded from above, given y_M : by decreasing monotonicity and setting $y_m = 0$ it is

$$B_Y T + LT y_M - \Psi(y_m, y_M) \leq LT (y_M - M_Y(y_M)). \quad (15)$$

The expected retrospective premium $\Psi(y_m, y_M)$ admits another representation by referring to the inverse function¹ F_Y^{-1} :

$$F_Y^{-1}(p) = \inf\{x \in \mathfrak{R} \mid F_Y(x) \geq p\} = \sup\{x \in \mathfrak{R} \mid F_Y(x) < p\}. \tag{16}$$

The expected retrospective premium $\Psi(y_m, y_M)$ admits the following equivalent characterization

$$\Psi(y_m, y_M) = \begin{cases} B_Y T + LT(y_m + M_Y(y_M) - M_Y(y_m)) & \text{for } y_m < F_Y^{-1}(1) \\ B_Y T + LT y_m & \text{for } y_m \geq F_Y^{-1}(1) \end{cases}. \tag{17}$$

Note, in fact, that if $y_m \geq F_Y^{-1}(1)$, then $M_Y(y_m) = M_Y(y_M) = E[Y]$.

4 A numerical example

To study the distribution of the random loss Y we refer to the collective risk model (see [4]). Let μ_Y , σ_Y^2 and γ_Y denote the mean, variance and coefficient of skewness of Y , respectively. We assume Y has the same distribution of the random variable $Z+z_0$, where z_0 is constant and Z follows a gamma distribution with positive parameters α e β :

$$F_Z(z) = \frac{1}{\Gamma(\alpha)} \int_0^z x^{\alpha-1} \beta^\alpha e^{-\beta x} dx, \quad z \geq 0. \tag{18}$$

As it is well known, $F_Z(z) = \Gamma(\alpha; \beta z)$ where $\Gamma(\alpha; \beta z)$ is the incomplete gamma function, i.e.

$$\Gamma(\alpha; \beta z) = \frac{1}{\Gamma(\alpha)} \int_0^{\beta z} x^{\alpha-1} e^{-x} dx. \tag{19}$$

The parameters α , β and z_0 are chosen so that $Z + z_0$ and Y have the same first three moments: $\mu_Y = z_0 + \frac{\alpha}{\beta}$, $\sigma_Y^2 = \frac{\alpha}{\beta^2}$, $\gamma_Y = \frac{2}{\sqrt{\alpha}}$, therefore the parameters are linked to μ_Y , σ_Y^2 and γ_Y according to the rules:

$$\alpha = \frac{4}{\gamma_Y^2}, \quad \beta = \frac{2}{\gamma_Y \sigma_Y} \quad \text{and} \quad z_0 = \mu_Y - \frac{2\sigma_Y}{\gamma_Y}. \tag{20}$$

The l.e.v.f. M_Z of Z is a continuous increasing function (see [6])

$$M_Z(u) = \frac{\alpha}{\beta} \Gamma(\alpha + 1; \beta u) + u[1 - \Gamma(\alpha; \beta u)], \quad u \geq 0 \tag{21}$$

with first derivative $M'_Z(u) = 1 - \Gamma(\alpha; \beta u)$. Furthermore, given the assumptions on Y and Z and the properties of the l.e.v.f., we obtain the following results:

$$M_Y(u) = M_{Z+z_0}(u) = M_Z(u - z_0) + z_0, \quad u \geq z_0 \tag{22}$$

¹ $p \in [0, 1]$, and, by convention, $\inf \emptyset = +\infty$

$$\lim_{u \rightarrow z_0} M_Y(u) = M_Y(z_0) = z_0 \quad \lim_{u \rightarrow \infty} M_Y(u) = \frac{\alpha}{\beta} + z_0. \quad (23)$$

Therefore, referring to (7) and (22), the expected retrospective premium $\Psi(y_m, y_M)$ admits the following formulation:

$$\Psi(y_m, y_M) = B_Y T + LT(y_m + M_Z(y_M - z_0) - M_Z(y_m - z_0)) \quad (24)$$

where $y_M \geq y_m \geq z_0$ and

$$\begin{aligned} M_Z(y_M - z_0) - M_Z(y_m - z_0) &= \frac{1}{\beta \Gamma(\alpha)} \int_{\beta(y_m - z_0)}^{\beta(y_M - z_0)} x^\alpha e^{-x} dx \\ &+ y_M [1 - \Gamma(\alpha; \beta(y_M - z_0))] - y_m [1 - \Gamma(\alpha; \beta(y_m - z_0))] \\ &+ \frac{z_0}{\Gamma(\alpha)} \int_{\beta(y_m - z_0)}^{\beta(y_M - z_0)} x^{\alpha-1} e^{-x} dx. \end{aligned}$$

In this way, we obtain the following relations:

- i) $\Psi(z_0, y_M) = B_Y T + LT(z_0 + M_Z(y_M - z_0))$
- ii) $\Psi(y_m, z_0) = \Psi(z_0, z_0) = B_Y T + LT z_0$
- iii) $\lim_{y_m \rightarrow y_M} \Psi(y_m, y_M) = B_Y T + LT y_M$
- iv) $\lim_{y_M \rightarrow \infty} \Psi(y_m, y_M) = B_Y T + LT \left(y_m + \frac{\alpha}{\beta} - M_Z(y_m - z_0) \right)$

As we have seen, the differences between the expected retrospective premium $\Psi(y_m, y_M)$ and the extreme premia (i.e., G_Y and H_Y) are bounded. In this case, by (13) the difference $\Psi(y_m, y_M) - G_Y$ is bounded from above

$$\Psi(y_m, y_M) - B_Y T - LT y_m \leq LT \left(\frac{\alpha}{\beta} - M_Z(y_m - z_0) \right) \quad (25)$$

and by (15) the difference $H_Y - \Psi(y_m, y_M)$ is bounded from above

$$B_Y + LT y_M - \Psi(y_m, y_M) \leq LT (y_M - z_0 - M_Z(y_M - z_0)). \quad (26)$$

In Figures 1 and 2, the graphs of the function $\Psi(y_m, y_M)$ and of some level curves are plotted², with parameters $\alpha = 4$, $\beta = 0.004$, $B_Y = 400$, $T = 1.002$, $L = 0.9$, $z_0 = 9000$ and ranges $y_m \in [9050, 10000]$, $y_M \in [10000, 11000]$. Note that some of the properties discussed in Section 3 are highlighted in the images.

²Wolfram Research, Inc., Mathematica, Version 12.3, Champaign, IL (2021).

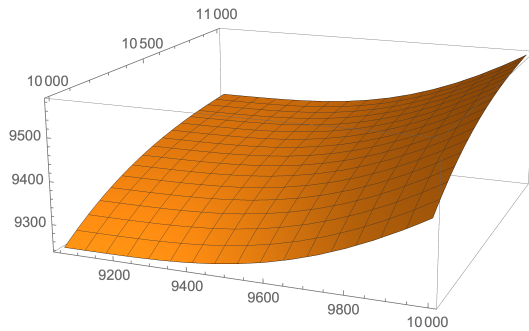


Figure 1: $\Psi(y_m, y_M)$

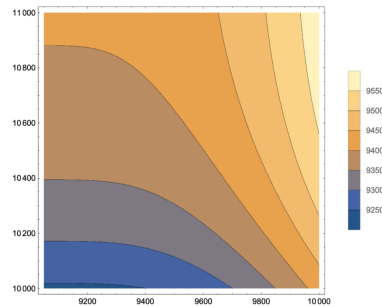


Figure 2: Level curves of $\Psi(y_m, y_M)$

5 Concluding remarks

The retrospective rating plan is a particular kind of risk-sharing agreement evolving over time, where the insurance premium is a function of the actual loss paid under the contract. Knowing the expected value of the premium is important in implementing a retrospective rating plan for both the insurer and the policyholder, so that they can plan their business.

In this paper we have analyzed some properties of the expected retrospective premium that can offer possible interpretations and practical applications. For example, by setting the initial values \bar{y}_m and \bar{y}_M in order to guarantee an initial expected premium $\Psi(\bar{y}_m, \bar{y}_M)$, upon renewal of the contract the parameters may be subject to changes. Thanks to (11), the corresponding new value of the expected premium $\Psi(y_m, y_M)$ can be approximated around the point (\bar{y}_m, \bar{y}_M) :

$$\begin{aligned} \Psi(y_m, y_M) \approx & \Psi(\bar{y}_m, \bar{y}_M) + LTF_Y(\bar{y}_m)(y_m - \bar{y}_m) \\ & + LT(1 - F_Y(\bar{y}_M))(y_M - \bar{y}_M). \end{aligned} \tag{27}$$

Note that (27) just requires the knowledge (or estimate) of the values $F_Y(\bar{y}_m)$ and $F_Y(\bar{y}_M)$.

Following [1] and [2], the topic here studied can be also further investigated under different assumptions.

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