# Management Science

## Default Ambiguity: Credit Default Swaps Create New Systemic Risks in Financial Networks

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**Abstract.** We study financial networks and reveal a new kind of systemic risk arising from what we call *default ambiguity*—that is, a situation where it is impossible to decide which banks are in default. Specifically, we study the *clearing problem*: given a network of banks interconnected by financial contracts, determine which banks are in default and what percentage of their liabilities they can pay. Prior work has shown that when banks can only enter into debt contracts with each other, this problem always has a unique maximal solution. We first prove that when banks can also enter into *credit default swaps* (CDSs), the clearing problem may have no solution or multiple conflicting solutions, thus leading to default ambiguity. We then derive sufficient conditions on the network structure to eliminate these issues. Finally, we discuss policy implications for the CDS market.

## 1. Introduction

During the 1996 East Asia crisis, "[i]n Indonesia, ... 75 percent of all businesses were put into distress, while in Thailand close to 50 percent of bank loans became nonperforming" (Stiglitz 2002, p. 112). All of these firms were interconnected, and as a result of the complexity of this network, regulators were facing a phenomenon we call *default ambiguity*. As then World Bank Chief Economist Joseph Stiglitz describes it,

> Every firm owed money to every other firm. But ... you couldn't tell whether they were bankrupt or not, because that depended on whether they got paid money that was owed to them by other firms who might or might not be in default, depending on whether the firms that owed *them* money went bankrupt. (Stiglitz 2016, at 0:51h)

In other words, *default ambiguity* is a situation where one cannot tell which banks are in default. Stiglitz (2016) points out that this led to a paralysis ("it took years to resolve it"), resulting in large welfare losses because banks' resolution could not be carried out quickly.

It may be intuitive to expect that default ambiguity can arise when the financial authority only has imperfect information about banks' contractual obligations. For instance, Haldane (2009) described a related effect on asset prices in the 2008 financial crisis. In this paper, we show that, remarkably, default ambiguity can also arise in a *perfect information setting*, where the whole financial network is known to the financial authority.

In the perfect information setting, default ambiguity can be studied in terms of the *clearing problem*: given a network of banks (or other financial institutions) interconnected by financial contracts, determine which banks are in default and for the defaulting banks what percentage of their liabilities they can still pay to their creditors (i.e., we are looking for the *recovery rate* of each bank). As in Eisenberg and Noe (2001), we assume that all payments are made simultaneously and in accordance with standard bankruptcy regulations. The banks' assets may lose part of their value when banks default (i.e., the banks incur *default costs*).[1]

An interpretation of the clearing problem is that in a financial crisis, a *clearing authority* (e.g., a central bank) observes the whole network of contracts, seeks to solve the clearing problem, and prescribes to each bank how much it has to pay to every other bank. The clearing problem is challenging because banks typically rely on payments they receive from other banks to meet their obligations, and banks can form an intricate web of contractual relations with each other. Default ambiguity arises when the clearing problem has no solution or when there are multiple conflicting solutions (i.e., none of which is simultaneously best for all banks).

Eisenberg and Noe (2001) and Rogers and Veraart (2013) showed that financial networks where banks can only enter into simple *debt contracts* (i.e., loans from one bank to another) have two very desirable properties from a clearing perspective: First, the clearing problem always has a solution (we call this property *existence*). Second, there is always a solution that maximizes the equity of each bank simultaneously (we call this property *maximality*).[2] Thus, although there may be multiple solutions, the maximal solution is the obvious choice for the clearing authority to implement (because it is simultaneously best for all banks).[3]

In this work, we study financial networks that contain debt contracts as well as *credit default swaps* (CDSs). A CDS is a financial derivative in which the writer insures the holder of the contract against the default of a third party, the *reference entity*. The holder may or may not have an exposure to the reference entity. Prior work has shown that the network structure of CDSs has a significant effect on systemic risk (Duffie and Zhu 2011, Loon and Zhong 2014). A large part of the CDS market is made up of CDSs where the reference entity is itself a financial institution.[4] An analysis of CDS transaction data by D'Errico et al. (2018) has shown that the financial institutions (including reference entities) in the CDS market are tightly connected, implying the presence of circular relationships involving holders, writers, and reference entities.

We ask, under which conditions can financial systems still be cleared when they contain such CDSs in addition to debt? We take existence and maximality as desiderata for the design of a financial system. We then derive constraints on the network structure under which the financial system is guaranteed to satisfy these two desiderata, independent of banks' external assets (i.e., assets that do not depend on other banks).[5] Similar to prior work on the clearing problem, our approach is agnostic to how networks have formed. Thus, our results apply to *any* network, including those that could arise in equilibrium from a decentralized process of network formation.

In this work, we are the first to present an analytically tractable model for the clearing problem with CDSs on financial institutions (Section 2).[6] Our first major finding is that, in networks with debt and CDSs, default ambiguity can occur (Section 3). We first show that if there are default costs, then existence is not always satisfied.[7] The intuition for this is that with CDSs, a bank A can hold a position on another bank B whereby A is better off if B is worse off, referred to as a *short position* hereafter. In a dense network of debt and CDS contracts, a bank may easily find itself *indirectly* holding a short position on itself (i.e., bank A is better off if bank A is worse off), which intuitively leads to a contradiction.[8] By contrast, in a debt-only network, banks only hold *long positions* on each other (if one bank is worse off, then the other is also worse off) so that this phenomenon does not exist. If the clearing authority was facing a situation where no solution exists in a crisis, a "paralysis" such as in the East Asia crisis may ensue because it would not be clear how to proceed. One might wonder which changes to our model might restore existence in the general case. We provide a discussion on this at the end of Section 3.1.

Second, we show that even in situations where existence is satisfied, maximality may not be satisfied. This resolves an open question by Demange (2016), who conjectured that if one extended the Eisenberg/Noe model to CDSs, "multiple and noncomparable ratios might then clear the market" (p. 967). The intuition for our result is that CDSs can give rise to a situation in which two banks happen to hold a short position on each other. In this case, exactly one of the two banks can be well off while the other one is doing poorly, but it is not possible to maximize both equities at the same time. Again, because networks of debt obligations contain only long positions, this effect can only be observed in networks with CDSs, with or without default costs. In a situation where no maximal solution exists, the clearing authority would have to choose among the different solutions, which would imply favoring the equity (and thus shareholders' profits) of one bank over that of another one. This, in turn, might lead to major lobbying activities, as banks would have an incentive to influence the clearing authority to select a solution that is favorable to them.

Note that solving the clearing problem is not only relevant in a financial crisis. Regulators such as the European Central Bank regularly conduct *stress tests* to evaluate how likely certain banks are to default given adverse economic scenarios. As regulators progressively take on a macroprudential (i.e., systemwide) perspective, stress tests increasingly take network effects into account.[9] In the future, it seems prudent to also include CDSs in network-based stress tests, given the important role they played in the 2008 financial crisis.[10] Our work shows that the inclusion of CDSs

may lead to an inconclusive outcome of a stress test due to default ambiguity. Another real-world application that illustrates the importance of our findings is the recent provision to resolve a failing bank within one weekend (Single Resolution Board 2016). If default ambiguity arose in this application, this would hinder the quick resolution of the bank.

To eliminate these issues regarding default ambiguity, we next study what constraints on the network structure are sufficient to guarantee existence and maximality. To this end, we first introduce the *colored dependency graph*, a new analysis framework to capture the dependencies among banks, in particular among the three parties (holder, writer, and reference entity) involved in a CDS (Section 4). By restricting the cycles in this dependency graph, we then derive sufficient conditions under which existence and/or maximality are satisfied (Section 5). Furthermore, we provide an algorithm to compute a solution in this case. The conditions we derive provide *ex ante* guarantees; that is, they are robust to any possible future shock on the banks' external assets. Ex ante guarantees are important for practical applications because the mere possibility that the market could not be cleared in the future could undermine trust of market participants and bring about a liquidity crisis today. Furthermore, if a bank anticipated a future incentive to influence the clearing authority's choice of a solution, then the bank would have a motivation to already start lobbying today.

We last discuss potential policy implications. We show within our model that the policy of routing all contracts through a central counterparty does not guarantee existence. By contrast, when "naked" CDSs (i.e., CDSs that are held without also holding a corresponding debt contract) are not allowed, then existence and maximality are always fulfilled. Our results thus contribute to the debate on a possible regulation of the CDS market (Section 6).

Prior work on financial networks has primarily focused on *financial contagion* (i.e., how local shocks to market participants' portfolios spread through the network and cause systemic crises). Researchers have considered two questions in particular: First, what is the impact of network topology on contagion compared with other factors such as correlation between banks' asset portfolios (Allen and Gale 2000, Elsinger et al. 2006, Acemoglu et al. 2012, Glasserman and Young 2015)? And second, how can the likelihood of an individual bank to trigger contagion be measured (Hu et al. 2012, Acemoglu et al. 2015, Battiston et al. 2016, Demange 2016)? Bardoscia et al. (2017) have shown how specific closed chains in networks of credit contracts are a sufficient condition for instability. This prior work has shown that financial contagion can amplify the effect of a small shock

leading to a large loss. By contrast, default ambiguity describes a situation in which the effect of a shock on a financial network is not even mathematically well defined. This means that neither the interbank payments nor the systemwide losses can be determined. In this sense, the risk of a financial system to experience default ambiguity is more fundamental than the risk of financial contagion. Our dependency analysis framework constitutes a new tool to study this risk and inform regulatory policy.

## 2. Formal Model and Visual Representation

Our model is based on the model by Eisenberg and Noe (2001) and its extension to default costs by Rogers and Veraart (2013). Both of these prior models were restricted to debt contracts. We define an extension to credit default swaps. Following said prior work, we assume a static model where a financial system is given exogenously and all contracts are evaluated simultaneously. We adjust the notation where necessary.

### 2.1. The Model

**Banks and External Assets.** Let $N$ denote a finite set of *banks*. Each bank $i \in N$ holds a certain amount of *external assets*, denoted by $e_i \geq 0$. Let $e = (e_i)_{i \in N}$ denote the vector of all external assets.

**Contracts.** There are two types of contracts: *debt contracts* and *CDSs*. Every contract gives rise to a conditional or unconditional obligation to pay a certain amount, called a *liability*, from its *writer* to its *holder*. Banks that cannot fulfill this obligation are said to be *in default*. The *recovery rate* $r_i$ of a bank $i$ is the share of its liabilities it is able to pay. Thus, $r_i = 1$ if $i$ is not in default and $r_i < 1$ if $i$ is in default. Let $r = (r_i)_{i \in N}$ denote the vector of all recovery rates.

A *debt contract* obliges the writer $i$ to unconditionally pay a certain amount to the holder $j$. The amount is called the *notional* of the contract and is denoted by $c_{i,j}^{\emptyset}$. A *credit default swap* obliges the writer $i$ to make a conditional payment to the holder $j$. The amount of this payment depends on the recovery rate of a third bank $k$, called the *reference entity*. Specifically, the payment amount of the CDS from $i$ to $j$ with reference entity $k$ and *notional* $c_{i,j}^{k}$ is $c_{i,j}^{k} \cdot (1 - r_k)$. The contractual relationships between all banks are represented by a three-dimensional matrix $c = (c_{i,j}^{k})_{i \in N, j \in N, k \in N \cup \{\emptyset\}}$. Zero entries indicate the absence of the respective contract.

Note that when banks enter contracts, there typically is an initial payment. For example, debt contracts arise because the holder lends an amount of money to the writer, and the holder of a CDS pays a premium to obtain the CDS. In our model, we assume that any such initial payments have been made at an

earlier time and are implicitly reflected in the external assets.

We make two *sanity assumptions* to rule out pathological cases. First, we require that no bank enters into a contract with itself or on itself (i.e., $c_{i,i}^{\emptyset} = c_{i,i}^{j} = c_{i,j}^{j} = c_{i,j}^{i} = 0$ for all $i, j \in N$). Second, as CDSs are defined as insurance on debt, we require that any bank that is a reference entity in a CDS must also be the writer of a debt contract (i.e., if $\sum_{k,l \in N} c_{k,l}^{i} > 0$, then $\sum_{j \in N} c_{i,j}^{\emptyset} > 0$ for all $i \in N$).

For any bank $i$, the *creditors of $i$* are those banks that are holders of contracts for which $i$ is the writer (i.e., the banks to which $i$ owes money). Conversely, the *debtors of $i$* are the writers of contracts of which $i$ is the holder (i.e., the banks that owe money to $i$). Note that the two sets can overlap: for example, a bank could hold a CDS on one reference entity while writing a CDS on another reference entity, both with the same counterparty.

**Default Costs.** We model default costs following Rogers and Veraart (2013): there are two *default cost parameters* $\alpha, \beta \in [0, 1]$. Defaulting banks are only able to pay to their creditors a share of $\alpha$ of their external assets and a share of $\beta$ of their incoming payments. Thus, $\alpha = \beta = 1$ means that there are no default costs, and $\alpha = \beta = 0$ means that assets held by defaulting banks are worthless. The values $1 - \alpha$ and $1 - \beta$ are the *default costs*.[11]

**Financial System.** A *financial system* is a tuple $(N, e, c, \alpha, \beta)$ where $N$ is a set of banks, $e$ is a vector of external assets, $c$ is a three-dimensional matrix of contracts, and $\alpha$ and $\beta$ are default cost parameters.

**Liabilities, Payments, and Assets.** For two banks $i, j$ and a vector of recovery rates $r$, the *liability of $i$ to $j$ at $r$* is the amount of money that $i$ has to pay to $j$ if recovery rates in the financial system are given by $r$, denoted by $l_{i,j}(r)$. It arises from the aggregate of any debt contract and all CDSs from $i$ to $j$:

$$l_{i,j}(r) := c_{i,j}^{\emptyset} + \sum_{k \in N} (1 - r_k) \cdot c_{i,j}^{k}.$$

The *total liabilities of $i$ at $r$* are the aggregate liabilities that $i$ has toward other banks given the recovery rates $r$, denoted by $l_i(r)$:

$$l_i(r) := \sum_{j \in N} l_{i,j}(r).$$

The actual *payment $p_{i,j}(r)$ from $i$ to $j$ at $r$* can be lower than $l_{i,j}(r)$ if $i$ is in default. By the *principle of proportionality* (discussed below), a bank that is in default makes payments for its contracts in proportion to the respective liability:

$$p_{i,j}(r) := r_i \cdot l_{i,j}(r).$$

The *total assets $a_i(r)$ of a bank $i$ at $r$* consist of its external assets $e_i$ and the incoming payments:

$$a_i(r) := e_i + \sum_{j \in N} p_{j,i}(r).$$

In case bank $i$ is in default, its *assets after default costs $a_i'(r)$* are the assets reduced according to the factors $\alpha$ and $\beta$. This is the amount that will be paid out to creditors:

$$a_i'(r) := \alpha e_i + \beta \sum_{j \in N} p_{j,i}(r).$$

**Clearing Recovery Rate Vector.** Following Eisenberg and Noe (2001), we call a recovery rate vector $r$ *clearing* if it is in accordance with the following three principles of bankruptcy law:

1. *Absolute priority*: Banks with sufficient assets pay their liabilities in full. Thus, these banks have recovery rate 1.

2. *Limited liability*: Banks with insufficient assets to pay their liabilities are in default and pay all of their assets to creditors after default costs have been subtracted. Thus, these banks have recovery rate $a_i'(r)/l_i(r) < 1$.

3. *Proportionality*: In case of default, payments to creditors are made in proportion to the respective liability.

The principle of proportionality is automatically fulfilled in our model by the definition of the payments $p_{i,j}(r)$. The other two principles lead to the following definition.

**Definition 1** (Clearing Recovery Rate Vector). Let $X = (N, e, c, \alpha, \beta)$ be a financial system. A *recovery rate vector* is a vector of values $r_i \in [0, 1]$ for each $i \in N$. We denote by $[0, 1]^N$ the space of all possible recovery rate vectors. Define the *update function*

$$F : [0, 1]^N \to [0, 1]^N,$$

$$F_i(r) := \begin{cases} 1 & \text{if } a_i(r) \geq l_i(r) \\ \frac{a_i'(r)}{l_i(r)}, & \text{if } a_i(r) < l_i(r). \end{cases} \quad (1)$$

A recovery rate vector $r$ is called *clearing* for $X$ if it is a fixed point of the update function (i.e., if $F_i(r) = r_i$ for all $i$). We also call a clearing recovery rate vector a *solution* to the clearing problem.

**Equity.** For any bank $i$, its *equity $E_i(r)$* is the positive difference between assets and liabilities. This is the profit that the owners of bank $i$ get to keep after clearing:

$$E_i(r) := \max(0, a_i(r) - l_i(r)).$$

## 2.2. Example and Visual Representation

Figure 1 shows a visual representation of an example financial system. There are three banks N = {A, B, C},

**Figure 1.** (Color online) Example Financial System with $\alpha = \beta = 0.5$
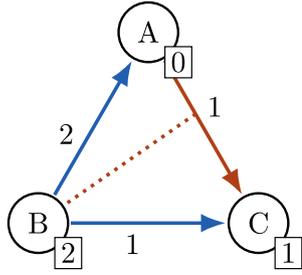


drawn as circles, with external assets of $e_A = 0$, $e_B = 2$, and $e_C = 1$, drawn as rectangles on top of the banks. Debt contracts are drawn as blue arrows from the writer to the holder, and they are annotated with the notionals $c_{B,A}^{\emptyset} = 2$ and $c_{B,C}^{\emptyset} = 1$. CDSs are drawn as orange arrows with a dashed line connecting to the reference entity, and they are also annotated with the notionals: $c_{A,C}^{B} = 1$. Default cost parameters $\alpha = \beta = 0.5$ are given in addition to the picture. A solution for this example is $r_A = 1$, $r_B = 1/3$, and $r_C = 1$. The liabilities at this recovery rate vector are $l_{B,A}(r) = 2$, $l_{B,C}(r) = 1$, and $l_{A,C}(r) = 2/3$. Payments are $p_{B,A}(r) = 2/3$, $p_{B,C}(r) = 1/3$, and $p_{A,C}(r) = 2/3$, and equities are $E_A(r) = 0$, $E_B(r) = 0$, and $E_C(r) = 1$. This is the only solution.

## 2.3. Discussion of Our Formal Model

Note that our addition of CDSs to the Rogers and Veraart (2013) model substantially changes its mathematical properties. The liabilities $l_{i,j}(r)$ now depend on the recovery rate vector $r$, and the assets $a_i(r)$ contain terms of the form $c_{j,i}^k \cdot r_j \cdot (1 - r_k)$. Thus, the update function $F_i(r)$ depends on $r$ in a way that is both *nonlinear* and *nonmonotonic*: an increase in some recovery rate $r_l$ could lead to a higher or lower value of $F_i(r)$ for another bank $i$. Because $F_i(r)$ is nonmonotonic, we cannot in general find a solution to the clearing problem by simply iterating the function $F$. The iteration sequence may cycle among different recovery rate vectors without even getting near a solution (see Appendix A for an example). For the same reason, Eisenberg and Noe's (2001) algorithm for computing clearing payments in debt-only systems cannot be applied to systems with CDSs.

Prior work has modeled financial networks almost exclusively as weighted binary graphs where edges reflect binary long relationships such as debt (Eisenberg and Noe 2001, Cifuentes et al. 2005, Rogers and Veraart 2013) and cross-ownership (Vitali et al. 2011, Elliott et al. 2014). Barucca et al. (2016) presented a unified framework for such models. However, CDSs give rise to *ternary* relationships because the holder is affected by the financial health of both the writer and the reference entity, and they imply both long and short positions. Weighted-graph models cannot accurately represent these features, whereas our model captures them well.[12]

## 3. Existence and Maximality in General Financial Systems

In this section, we explore the possible shapes of the set of solutions for a financial system with debt and CDSs. We construct financial systems that have no solution or multiple conflicting solutions. Consequently, neither existence nor maximality is guaranteed in general.

At the heart of our constructions lies the following lemma, which may be of independent interest to some readers. The lemma demonstrates a gap in the space of possible solutions: the recovery rate of any bank is either 1 or below $\alpha$ or $\beta$.

**Lemma 1.** *Let $X = (N, e, c, \alpha, \beta)$ be a financial system, $r$ clearing for $X$, and let $i \in N$ be a bank. If $r_i < 1$, then the following hold:*

(1) *If $i$ has only external assets (i.e., $\sum_j p_{j,i}(r) = 0$), then $r_i \leq \alpha$. If, in addition, $\alpha > 0$, then $r_i < \alpha$.*

(2) *If $i$ has only interbank assets (i.e., $e_i = 0$), then $r_i \leq \beta$. If, in addition, $\beta > 0$, then $r_i < \beta$.*

(3) *In any case, $r_i \leq \max(\alpha, \beta)$. If $\alpha > 0$ or $\beta > 0$, then $r_i < \max(\alpha, \beta)$.*

**Proof.** We prove part (3). From the definition, it follows that $a_i'(r) \leq \max(\alpha, \beta) \cdot a_i(r)$. Because $r_i < 1$, we must have $a_i(r) < l_i(r)$ (in particular, $l_i(r) > 0$) and $r_i = F_i(r) = a_i'(r)/l_i(r) \leq \max(\alpha, \beta) \cdot a_i(r)/l_i(r) \leq \max(\alpha, \beta)$. If $\alpha > 0$ or $\beta > 0$, then the last inequality is strict.

The proofs of parts (1) and (2) are analogous. □

### 3.1. Existence of a Solution

What is perhaps most surprising about financial networks with CDSs is that as soon as there are any default costs, the existence of a solution can no longer be guaranteed.

**Theorem 1** (No Solution with Default Costs). *For any pair $(\alpha, \beta)$ with $\alpha < 1$ or $\beta < 1$ there exists a financial system $(N, e, c, \alpha, \beta)$ that has no clearing recovery rate vector.*

**Proof.** If $\beta < 1$, consider the system in Figure 2. Let $\delta = 3 \cdot 1/(1 - \beta)$. Assume, toward a contradiction, that there is a clearing recovery rate vector $r$.

• If $r_A = 1$, then $p_{C,B}(r) = l_{C,B}(r) = \delta(1 - r_A) = 0$; hence $a_B(r) = 0$ and $p_{B,A}(r) = 0$. This implies $a_A(r) = 0 < 1 = l_A(r)$, and thus $r_A = 0$— a contradiction.

• If $r_A < 1$, then $r_A \leq \beta$ by Lemma 1. Thus, $p_{C,B}(r) = l_{C,B}(r) = \delta(1 - r_A) \geq \delta(1 - \beta) = 3$. Now, $a_B(r) = 3 \geq 2 = l_B(r)$, so $p_{B,A}(r) = l_{B,A}(r) = 2$. Hence $a_A(r) \geq l_A(r)$, and so $r_A = 1$—a contradiction.

The proof for the case $\alpha < \beta = 1$ is provided in Appendix B. It uses a similar construction but where A has positive external assets. □

**Figure 2.** (Color online) Financial System with No Solution for $\beta < 1$
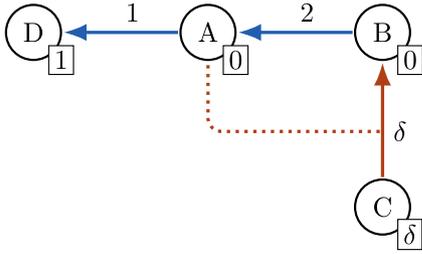


The system in Figure 2 is paradoxical because A is implicitly holding a CDS (and is thus short) on itself: if A is in default, it receives a payment because of the CDS written on it, so it is not in default, and vice versa. Although A *actually* holding a CDS on itself would be absurd, having B in between makes the paradox much less obvious. Supervisory authorities could only notice that the two scenarios are, in fact, equivalent once they are aware of network effects and have detailed knowledge about the contract structure, including the ternary relationships introduced by CDSs.

In addition, although it is hard to imagine why a bank would ever buy a CDS on itself, Figure 2 could have formed in an entirely sensible way. For example, B could have borrowed money from A and later placed a speculative bet on A's default before both banks were hit by a shock that wiped out their external assets. With only knowledge of their own assets and liabilities, none of the banks would have noticed any problem.

Figure 2 is a particularly simple example to show nonexistence because of its small size and zero external assets for all relevant banks. Note that these features are not essential for nonexistence. We present a larger example where nonexistence arises in a much more indirect way in the electronic companion of this paper.

**Remark 1.** We know from Rogers and Veraart (2013) that no example such as in Theorem 1 can be constructed using only debt contracts. Note that it can also not be constructed using only CDSs because in a financial system consisting of *only* CDSs, the recovery rate vector $(1, \ldots, 1)$ (nobody defaults) is always clearing: under this recovery rate vector, no liabilities arise, and thus, indeed, no bank defaults. Therefore, nonexistence can only arise in systems with debt *and* CDSs.

It turns out that the nonexistence of a solution hinges on the presence of default costs.

**Theorem 2** (Existence of a Solution Without Default Costs). *Any financial system $(N, e, c, \alpha = 1, \beta = 1)$ has a clearing recovery rate vector.*

**Proof.** Because $\alpha = \beta = 1$, we can simplify the update function $F$ from Definition 1 to $F_i(r) = \min(1, a_i(r)/l_i(r))$ on the set $L_i := \{r \mid l_i(r) > 0\}$. Note that $L_i$ is an open set because $l_i$ is continuous, and note that $F_i$ is continuous

on $L_i$. We use this fact and apply a fixed-point theorem. Care must be taken because $F_i$ is not, in general, continuous on $[0,1]^N \setminus L_i$. Consider the set-valued function $\rho$ defined by

$$\rho : [0,1]^N \to \mathscr{P}([0,1]^N), \quad \text{where } \mathscr{P}(S) \text{ denotes the}$$
$$\text{power set of } S;$$

$$\rho(r) := \underset{i \in N}{\times} \rho_i(r), \quad \text{where } \rho_i(r) := \begin{cases} F_i(r) & \text{if } r \in L_i, \\ [0,1] & \text{if } r \notin L_i. \end{cases}$$

If there is an $r$ such that $r \in \rho(r)$, then $r$ can be made clearing by setting the recovery rates of banks with zero liabilities to 1. This is because for all $i$, if $r \in L_i$, then $F_i(r) = r_i$ by choice of $r$, and if $r \notin L_i$, then $F_i(r) = 1$, and no other bank depends on $i$ because of our sanity assumptions.

It remains to show that an $r$ with $r \in \rho(r)$ exists. By the Kakutani (1941) fixed-point theorem, this is the case if (1) the domain of $\rho$ is compact and convex, (2) the set $\rho(r)$ is convex for each $r$, and (3) the graph of $\rho$, $G_\rho := \{(r,s) \mid s \in \rho(r)\}$, is a closed set. Properties (1) and (2) are obvious.

To prove (3), it suffices to show that for each $i$, the graph of $\rho_i$, $G_{\rho_i} := \{(r, s_i) \mid s_i \in \rho_i(r)\}$, is closed. To this end, let $(r^k, s_i^k)_{k \in \mathbb{N}}$ be a sequence in $[0,1]^N \times [0,1]$ converging to some point $(r, s_i)$ such that $s_i^k \in \rho_i(r^k)$ for each $k$. We need to show that $s_i \in \rho_i(r)$. If $r \notin L_i$, then trivially, $s_i \in \rho_i(r) = [0,1]$. If $r \in L_i$, then $s_i \in \rho_i(r) = \{F_i(r)\}$ because $F_i$ is continuous on the open set $L_i$. □

In financial systems without default costs, money is never lost, just redistributed. Theorem 2 shows that these systems always have a solution. It does not apply once default costs are present because the update function $F$ then has a discontinuity where the assets of a bank are equal to its liabilities (i.e., when a bank is just on the verge of defaulting). This discontinuity creates a gap in the space of possible recovery rates (see Lemma 1) and can give rise to nonexistence.

One might wonder what changes to our model might restore existence in the general case. Our model differs from Eisenberg and Noe (2001) only in that we allow for default costs and CDSs. Thus, if one seeks to represent these two features and aims to guarantee existence, the only option would be to use a clearing model different from simultaneous clearing. Perhaps the first alternative that comes to mind is *sequential clearing*, where the contracts are not evaluated at the same time but in some order. The result of this procedure, however, heavily depends on the order of evaluation, as the following example shows.

**Example 1** (Sequential Clearing). We define a natural sequential clearing procedure: The debt contracts are evaluated in a predetermined order, and banks pay their liabilities based on their external assets and

payments received so far (i.e., their "cash" holdings). If a bank cannot pay a liability, then it enters bankruptcy. Default costs are subtracted from the bank's cash holdings, and the recovery rate is computed based on the remaining cash. Then all CDSs written on the bank are triggered and are evaluated next. The process ends when all debt contracts have been evaluated.

Now assume that this procedure is applied to Figure 2 for $\beta = 0.5$, so that $\delta = 6$.
- If the debt contract from A to D is evaluated first, then A defaults with $r_A = 0$, B receives six in the CDS, and A receives two from B. The resulting equities are $E_A(r) = 2$ and $E_B(r) = 4$.
- If the debt contract from B to A is evaluated first, then B defaults with $r_B = 0$, A receives nothing and defaults with $r_A = 0$, and B receives six in the CDS. We have $E_A(r) = 0$ and $E_B(r) = 6$.

The order of evaluation in sequential clearing could be chosen at random or based on some objective criterion, such as the maturity of the contracts. In any case, it would introduce an element of arbitrariness and an opportunity for strategic manipulation. Although other sequential clearing procedures could be defined, it seems unlikely that this problem could be fully avoided.

Alternative clearing models that go beyond sequential clearing have been proposed in the literature; each has its own limitations. Csóka and Herings (2018) studied a *decentral clearing* procedure where payments are made incrementally in an arbitrary order. In a setting with CDSs, the result of this procedure depends on the order in which payments are made, similar to sequential clearing.[13] Banerjee and Feinstein (2018) defined a *dynamic clearing* procedure where multiple rounds of simultaneous payments are performed while a CDS is triggered with a delay of one round after its reference entity has defaulted. A solution always exists and can be chosen in a natural way. However, CDSs in this model cannot represent a complete insurance on a debt exposure. This is because, if the writer of a debt contract fails, the holder still incurs a loss *in that round*, irrespective of any CDS they might hold. This loss may be enough to send the holder into permanent bankruptcy. Acemoglu et al. (2015) studied a simultaneous clearing model where default costs arise exclusively from the partial liquidation of illiquid projects. This assumption would ensure existence even with CDSs via continuity in a similar way to Theorem 2, but it also precludes modeling any kind of *discontinuous* default costs such as time delays or operational losses.

### 3.2. Multiplicity of Solutions
We now show that even when the clearing problem has a solution, there can be multiple ones, and the structure of the set of solutions may not be economically desirable. We discuss this structure in terms

of the banks' aggregate preferences. Recall that we denote the equity of a bank $i$ by $E_i(r)$. We assume that, when there are multiple solutions, banks prefer those that maximize their equity.

**Definition 2** (Preferred and Maximal Solution). Fix a financial system $X$. A bank $i$ is said to *weakly prefer* a solution $r$ over another solution $r'$ if $E_i(r) \geq E_i(r')$. A solution $r$ is called *maximal* if it is weakly preferred to all other solutions by all banks.[14]

Our second desideratum, *maximality*, requires that a maximal solution exists. Otherwise, any solution the clearing authority could select would be opposed by at least one bank because this bank could achieve strictly higher equity in a different solution.[15] Such a situation is illustrated in the following theorem.

**Theorem 3** (No Maximal Solution). *For any $\alpha$ and $\beta$, there exists a financial system $(N, e, c, \alpha, \beta)$ that has a clearing recovery rate vector but no maximal one.*

**Proof.** We use the financial system in Figure 3 with $\delta = 1/(1 - \beta)$ if $\beta < 1$ and $\delta > 1$ arbitrary if $\beta = 1$. It is easy to verify that $r^0 := (0, 1, 1, 1)$ and $r^1 := (1, 0, 1, 1)$ (where entries are in alphabetical order) are clearing. In any potential other solution, we must have $r_C = r_D = 1$ and $0 < r_A, r_B < 1$.

For $\beta < 1$, no other solution exists: if $r$ *was* another one, then because $r_A < 1$, by Lemma 1, we have $r_A \leq \beta$, so $a_B(r) = \delta(1 - r_A) \geq \delta(1 - \beta) = 1$. Thus, $r_B = 1$—a contradiction.

For $\beta = 1$, there is exactly one other solution $r^2 = (\zeta, \zeta, 1, 1)$, where $\zeta = (\delta^2 - \delta)/(\delta^2 - 1)$. This is because $r$ is a solution with $r_A, r_B < 1$ iff $r_A = \delta(1 - r_B)$ and $r_B = \delta(1 - r_A)$. It is easy to verify that $r_B = r_A = \zeta$ is the unique solution of this linear equation system.

For any value of $\beta$, bank A has a positive equity of $\delta - 1$ in $r^1$ and equity 0 (because it is in default) in the other solution(s). Thus, A strictly prefers $r^1$. Analogously, B strictly prefers $r^0$. This implies that none of the solutions of this system is maximal.[16] □

To see why the solution structure in the previous theorem is economically undesirable, consider the $\beta < 1$ case in the above proof and imagine a clearing authority faced with the problem of actually clearing

**Figure 3.** (Color online) Financial System with No Maximal Solution
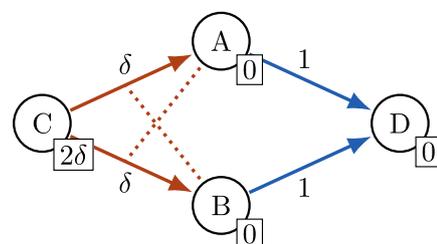
the market: there are two solutions, one where A defaults and one where B defaults. Choosing among the solutions means giving preference to one of the banks. It is not clear how this decision should be made, and the clearing authority may even be legally prohibited from making such a trade-off. If a choice among nonmaximal solutions were legally allowed, then a bank may have a large incentive to lobby for the implementation of a solution that it prefers most. Note that, in contrast to nonexistence, nonmaximality can even occur in systems without default costs.[17]

If clearing were done sequentially in the scenario from Theorem 3, one of the two solutions, $r^0$ or $r^1$, would be chosen based on which of the two debt contracts is evaluated first. In practice, such a scenario could lead to severe incentive problems. In today's financial practice, whether a CDS is triggered is decided by so-called *determinations committees*, which consist of the most active dealers in addition to nondealer members (International Swaps and Derivatives Association 2012). In Figure 3, A, B, and C would be members of the determinations committee. Taking the perspective of A, it would be rational to try to convince the other members that B's financial situation qualifies as a default. This triggers the CDS, A receives the payment and does not default, and B receives nothing. Thus, in hindsight, it appears as though A made a correct objective assessment about B. Of course, B would argue exactly the opposite of A. By contrast, when a maximal solution exists, it can be implemented without having to make any choices that could be manipulated.

**Remark 2.** Note that Rogers and Veraart (2013) have previously observed multiple solutions in debt-only networks stemming from default costs. However, the form of multiplicity they observed is much less problematic because in debt-only systems, there always exists a maximal solution.

# 4. Dependency Analysis Framework: The Colored Dependency Graph

In Section 3, we have shown that introducing CDSs into the well-established clearing model by Eisenberg and Noe (2001) has the effect that existence and maximality are no longer guaranteed. In this section, we develop an analysis framework, which we call the "colored dependency graph," to better understand how and when this effect arises. In Section 5, we then show how to use the colored dependency graph to derive sufficient conditions under which the two desiderata are satisfied.

## 4.1. Covered and Naked CDS Positions
At the level of an individual bank, we need to distinguish between two fundamentally different uses of

CDSs. For the purposes of illustration, consider a financial system with a single CDS where the CDS writer cannot default. If the holder of the CDS also holds at least an equal amount of debt written by the reference entity, then the CDS holder is long on the reference entity: a worse situation of the reference entity would, at most, be offset by the CDS payment, but it could never be beneficial for the holder. This use of a CDS is called *covered*. By contrast, if the holder holds no or not enough debt written by the reference entity, then it is short on the reference entity: a worse financial situation of the reference entity would benefit the holder. This use of a CDS is called *naked*. See the top row of Figure 4 for a depiction of a prototypical (a) debt contract, (b) naked CDS, and (c) covered CDS. For the formal definition in general financial systems, we must consider the notional of *all* CDSs that a bank holds on a reference entity to classify a CDS position as covered or naked.

**Definition 3** (Covered and Naked CDS Position). Let $X = (N, e, c, \alpha, \beta)$ be a financial system. A bank $j$ has a *covered CDS position* toward another bank $k$ if

$$\sum_{i \in N} c_{i,j}^k \leq c_{k,j}^{\emptyset}.$$

Otherwise, $j$ has a *naked CDS position* toward $k$.

## 4.2. The Colored Dependency Graph
We can now define the colored dependency graph (or just the "dependency graph"), in which long and short positions among banks are represented by green edges (with filled arrow tips) and red edges (with hollow arrow tips), respectively.

**Definition 4** (Colored Dependency Graph). Let $X = (N, e, c, \alpha, \beta)$ be a financial system. The *colored dependency*

**Figure 4.** (Color online) Prototypical Financial Systems (Top) and Their Colored Dependency Graphs (Bottom)
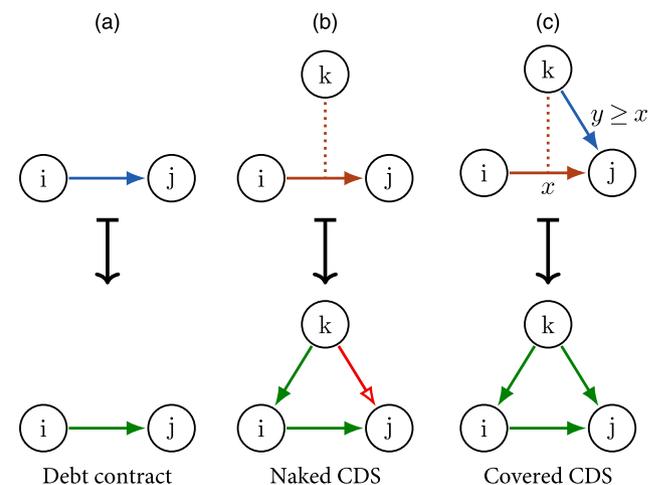


*Note.* Green edges with filled arrow tips indicate long positions, and red edges with hollow arrow tips indicate short positions.

*graph* $CD(X)$ is the graph with nodes $N$ and edges of colors red and green constructed as follows:

(1) For each $i, j \in N$, if $c_{i,j}^{\emptyset} > 0$ or $c_{i,j}^{k} > 0$ for any $k \in N$, then add a green edge $i \rightarrow j$.

(2) For each $i, k \in N$, if $c_{i,j}^{k} > 0$ for any $j \in N$, then add a green edge $k \rightarrow i$.

(3) For each $j, k \in N$, if $j$ has a naked CDS position toward $k$, then add a red edge $k \rightarrow j$.

The definition of the colored dependency graph can be understood in terms of the three primitive contract patterns illustrated in Figure 4: debt contracts, naked CDSs, and covered CDSs. In each case, the holder of any contract is long on the writer because, in case the writer defaults, the lower the recovery rate of the writer, the lower the payment that the holder receives. This is expressed by rule (1) in Definition 4. In case of a debt contract, this is the only dependency that is induced, whereas a CDS gives rise to two additional dependencies. The writer of a CDS is always long on the reference entity because, the lower the recovery rate of the reference entity, the higher the liability for the writer. This is expressed by rule (2) in Definition 4. The position of the holder of a CDS toward the reference entity depends on whether it is a naked or a covered CDS: only the holder of a naked CDS is short on the reference entity, expressed by rule (3) in Definition 4. A covered CDS, on the other hand, only gives rise to a long position together with the debt contract.

The following proposition shows the usefulness of the framework in capturing the directional behavior of the update function $F$. We will repeatedly use it in Section 5 when deriving sufficient conditions. The proof is straightforward and thus omitted.

**Proposition 1** (The Colored Dependency Graph and the Update Function). *For any two banks $i$ and $j$, we let $r_{-ij}$ denote a vector of recovery rates of all banks excluding $i$ and $j$. Then the following holds:*

(1) *If there exists an $r_{-ij}$ such that, holding $r_{-ij}$ fixed, the function $F_j$ is increasing[18] in $r_i$, then there is a green edge from $i$ to $j$ in $CD(X)$.*

(2) *If there exists an $r_{-ij}$ such that, holding $r_{-ij}$ fixed, the function $F_j$ is decreasing in $r_i$, then there is a red edge from $i$ to $j$ in $CD(X)$.*

(3) *If there is no edge from $i$ to $j$ of any color, then $F_j$ is independent of $r_i$. The converse is not necessarily the case.*

**Remark 3** (Parallel Edges). Both a red and a green edge can be present in the dependency graph in the same direction between the same two banks. In this case, whether a long or a short effect is present depends on the recovery rates of the other banks. The two edges do not cancel out.

If a financial system contains only debt contracts, then the colored dependency graph only has green edges; specifically, it has a green edge $i \rightarrow j$ whenever

$c_{i,j}^{\emptyset} > 0$. Notice that this graph coincides with the "financial structure graph" introduced by Eisenberg and Noe (2001). For systems with debt and CDSs, our colored dependency graph provides an elegant conversion from the ternary relations introduced by CDSs to binary relations, making them amenable to graph-theoretic analysis.[19]

Figure 5 depicts the colored dependency graphs of two financial systems that exhibit very different behavior: Figure 5, panel (a) corresponds to the example financial system from Figure 1, which has a unique solution. Figure 5, panel (b) corresponds to the financial system from Figure 2, which has no solution. We immediately see some similarities and differences: both graphs have a red edge; panel (a) has no *directed cycle*, whereas (b) has two of them, A–B–A and A–C–B–A; and the former cycle contains a red edge. All of these features will be of importance in the analysis in Section 5. Note that, although the cycles in Figure 5, panel (b) happen to be very short, this is not a necessary condition for the nonexistence of a solution. The electronic companion provides a more involved example with longer cycles.

## 5. Analysis of Restricted Network Structures

With our analysis framework in place, we now use it to describe sufficient conditions under which our desiderata are fulfilled. We show that one can guarantee our desiderata by restricting the ways in which the edges in the dependency graph may form cycles. We present three domain restrictions where we successively allow more cycles and receive successively fewer guarantees.

### 5.1. Acyclic Financial Systems

If there are no cycles in the colored dependency graph, then the clearing problem has a *unique* solution. As this solution is trivially maximal, both desiderata are fulfilled.

**Theorem 4** (Existence and Uniqueness in Acyclic Financial Systems). *Let $X$ be a financial system such that $CD(X)$ has no cycles. Then $X$ has a unique clearing recovery rate vector.*

**Figure 5.** (Color online) The Colored Dependency Graphs of the Financial Systems from (a) Figure 1 and (b) Figure 2



unique solution                no solution
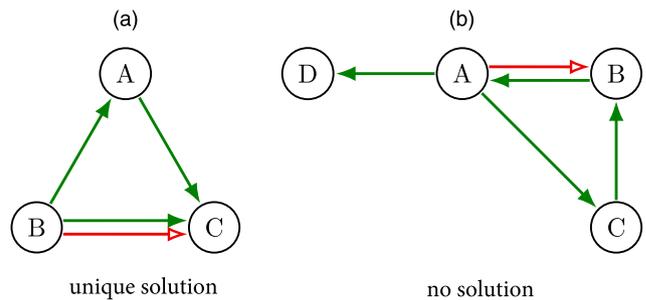
**Proof.** Without loss of generality, assume that $N = \{1 \ldots n\}$ and banks are sorted in *topological order*; that is, whenever there is an edge $i \to j$ in $CD(X)$, we have $i \leq j$. This is possible because $CD(X)$ has no cycles by assumption. To find a solution $r$, iterate over banks $i$ in order. In each step, set $r_i := F_i(r_1, \ldots r_{i-1})$, where $r_1, \ldots, r_{i-1}$ have already been computed. This is well defined by Proposition 1. In the end, $r$ is clearing by construction.

Toward uniqueness, if $r$ and $r'$ are both clearing, it follows by induction on $i$ that $r_i = F_i(r_1, \ldots, r_{i-1}) = F_i(r'_1, \ldots, r'_{i-1}) = r'_i$ for all $i$, where the middle equality is by induction hypothesis. Note that $F_1$ is a constant function. □

Theorem 4 shows formally that default ambiguity in financial systems with CDSs is due to cycles in the dependency graph. Note that we must consider *all* dependency edges here, including those originating at reference entities of CDSs. It is not sufficient to consider the graph of *liabilities*, where an edge exists from the writer of each contract to the holder, corresponding to only rule (1) in Definition 4. This graph would be acyclic for all our counterexamples in Section 3, although they clearly did not fulfill our desiderata. Thus, the more sophisticated colored dependency graph is necessary to capture the behavior of a financial system with CDSs.

## 5.2. Green Core Systems

The previous theorem required a very strong assumption; in reality, financial systems *do* contain cycles in the dependency graph, but not all of them pose a problem. In fact, we know from Rogers and Veraart (2013) that debt-only financial systems, even if they contain cycles, always satisfy existence and maximality. At the same time, debt-only systems always have a completely green dependency graph; that is, banks are only long on each other. In this section, we show that *all* financial systems with a completely green dependency graph satisfy existence and maximality, thus generalizing Rogers and Veraart's result. We consider a slightly more general class of financial systems that we call *green core systems*.

**Definition 5** (Green Core System). A financial system $X$ is called a *green core system* if in $CD(X)$, banks with an incoming red edge (i.e., the holders of naked CDS positions) have no outgoing edges. We call the set of these banks the *leaf set* and the other banks the *core*.

An example of a green core system is shown in Figure 6. Banks in the leaf set have no liabilities (otherwise, they would have an outgoing green edge) and hence always have recovery rate 1. This does not render the leaf set obsolete: allowing a leaf set keeps the definition of green core systems general enough so that

**Figure 6.** (Color online) Colored Dependency Graph of a Green Core System with Core $C$ and Leaf Set $L$
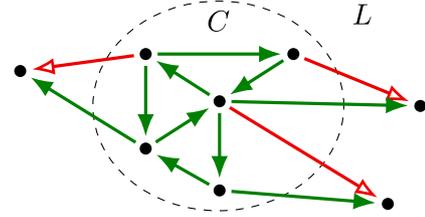


banks in the core can be writers of naked CDSs. This feature will also be essential in Section 5.3, where we consider even more general network structures that are composed of multiple green core systems that can be connected by red edges.

Green core systems always have a solution that is best for all banks *in the core*. We call such a solution *core maximal*. Our proof is constructive.

**Theorem 5** (Existence and Core Maximality in Green Core Systems). *In any green core system, the following holds:*

(1) *There exists a recovery rate vector that maximizes both the recovery rate and the equity of all banks in the core.*

(2) *The iteration sequence $(r^n)$ defined by $r^0 = (1, \ldots, 1)$ and $r^{n+1} = F(r^n)$ converges to this recovery rate vector.*

**Proof.** The main technical challenge lies in proving the following lemma.

**Lemma 2.** *Consider a green core system with core $C$ and leaf set $L$.*

(1) *The update function $F$ is monotonic and continuous from above, where the order relation is a pointwise comparison of recovery rate vectors.*

(2) *If $i \in C$, then the equity $E_i$ is monotonic, also with respect to a pointwise comparison.*

The proof of Lemma 2 is given in Appendix C. The lemma formalizes the fact that because all relevant dependency edges are green, a decrease in any bank's recovery rate can only affect the other banks in the core in a negative way. In addition, this happens in a continuous fashion.

From part (1) of the lemma, it follows via a standard technique from lattice theory (see Lemma 3 in Appendix C) that the sequence $(r^n)$ converges to a solution that maximizes the recovery rate of each bank. By part (2) of the lemma, this solution also maximizes all equities in the core. □

Theorem 5 shows that green core systems always satisfy existence and core maximality. Furthermore, the proof of the theorem tells us that green core systems are structurally very similar to debt-only systems. They share the following properties, which have previously been observed for debt-only systems by Rogers and Veraart (2013). First, the update function is monotonic and continuous from above. Second, the set of

solutions even forms a *complete lattice* (which follows from monotonicity of $F$ via the Knaster–Tarski fixed-point theorem; see, e.g., Granas and Dugundji 2003). Third, a core-maximal solution can be found via the iteration sequence provided in part (2) of Theorem 5. A subtle difference to the debt-only case is that a core-maximal solution of a green core system can contain irrational numbers, whereas the maximal solution of a debt-only system is always rational.[20]

A special case of Theorem 5 is a situation in which naked CDSs are not present.

**Corollary 1** (Existence and Maximality Without Naked CDSs). *If no bank in a financial system has a naked CDS position toward another bank, then there exists a maximal clearing recovery rate vector.*

**Proof.** In this case, the colored dependency graph contains only green edges. The financial system is hence trivially a green core system where the core consists of all banks.  □

Corollary 1 has important implications for regulatory policy regarding naked CDSs, which we discuss in detail in Section 6.

### 5.3. Systems Without Red-Containing Cycles

We know from Theorem 5 and Corollary 1 that default ambiguity can be attributed to the presence of red edges in the dependency graph. Green core systems restrict these edges in an extreme way, only allowing them to leaf banks. But we know from Theorem 4 (acyclic systems) that red edges to nonleaf banks do not *always* pose a problem. In this section, we study in which situations they do. Our main result is that, regarding existence, only red edges that are part of a cycle of dependencies can pose a problem.

**Theorem 6** (Existence Without Red-Containing Cycles). *Assume that in the colored dependency graph of a financial system, there is no cycle that contains a red edge. Then a clearing recovery rate vector exists.*

Our proof of the theorem is constructive by using an algorithm. Unfortunately, simply iterating the function $F$ such as in the green core case does not work anymore in the more general no-red-containing-cycle case (we provide an example in Appendix D). Instead, our algorithm exploits the structure of the dependency graph. Recall from above that the solutions of a financial system with debt and CDSs may be irrational. Given this, it is impossible to design an algorithm that can compute an exact solution in finite time. Instead, we devise an *approximation algorithm* that computes an arbitrarily accurate *approximate* solution. We now first describe our approximate solution concept and our algorithm. We then prove correctness of the algorithm and Theorem 6.

**Definition 6** (Approximately Clearing Recovery Rate Vector). Let $X$ be a financial system, and let $\varepsilon \geq 0$. A recovery rate vector $r$ is called $\varepsilon$-*approximately clearing* or an $\varepsilon$-*solution* for $X$ if $\|F(r) - r\| \leq \varepsilon$, where $\|r\| := \max_i |r_i|$ is the maximum norm.

We now describe our *core iteration algorithm* to compute an $\varepsilon$-solution in a financial system $X$ when no cycle in $CD(X)$ contains a red edge. Given are $\varepsilon$ and $X$. We begin by partitioning the dependency graph into strongly connected components; or *cores*. Each of these corresponds to the core of a green core system. A core is a minimal set of banks such that all banks with which these banks are in cycles are also part of the core. By partitioning the graph in this way, the connections between different cores form an acyclic graph, so we can sort them in topological order (i.e., edges only go from earlier to later cores in the order, but never in the other direction).[21] Figure 7 provides an example for such a dependency graph. We now iterate over cores. By assumption, all edges within a core are green, so we can use the iteration sequence from Section 5.2 to compute an $\varepsilon$-solution for each of them.

More in detail, let $C_1, \ldots, C_m$ be the cores in topological order. We store recovery rates in a vector $r$. Initially, $r$ is the empty vector. In step $k \in \{1, \ldots, m\}$, we define a function

$$F^k : [0,1]^{C_k} \rightarrow [0,1]^{C_k},$$
$$F_i^k(s) := F_i(r \frown s),$$

where the symbol "$\frown$" denotes a concatenation of vectors. This corresponds to the update function $F$ restricted to $C_k$ with previously computed recovery rates of the previous cores $C_1, \ldots, C_{k-1}$ given by $r$. Function $F^k$ is well defined because, by the topological ordering, each bank $i \in C_k$ depends only on the banks in $C_1, \ldots, C_k$. We iterate the function $F^k$ starting at $s = (1, \ldots, 1)$ until $\|F^k(s) - s\| \leq \varepsilon$. We then add the recovery rates computed in $s$ to $r$ and continue with the next core. The algorithm stops when all cores have been visited.

**Figure 7.** (Color online) Dependency Graph Where No Cycle Contains a Red Edge
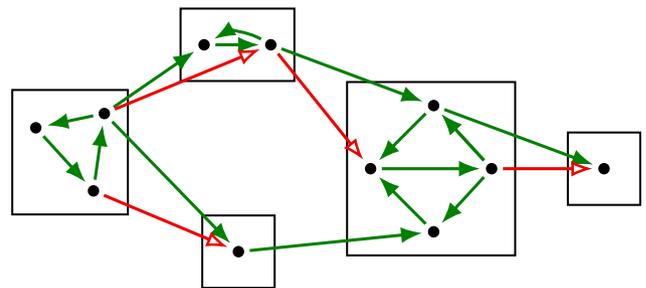


*Notes.* Cores are marked by black rectangles. The topological ordering of cores is from left to right; the two cores second from the left can be visited in any order.

**Proposition 2** (Correctness of the Core Iteration Algorithm).
*Assume that in the colored dependency graph of a financial system, no cycle contains a red edge. Then for any $\varepsilon > 0$, the core iteration algorithm computes an $\varepsilon$-approximately clearing recovery rate vector.*

**Proof.** To see that $r$ is an $\varepsilon$-solution when the algorithm terminates, let $i$ be a bank, let $k$ be such that $i \in C_k$, and let $s$ be $r$ restricted to the indices in $C_k$. By the topological ordering, $F_i(r)$ depends only on the $r_j$ with $j \in \bigcup_{l \leq k} C_l$. Hence, $F_i(r) = F_i^k(s)$, and therefore $|F_i(r) - r_i| = |F_i^k(s) - s_i| \leq \|F^k(s) - s\| \leq \varepsilon$ as required, where the last inequality holds by the algorithm's stopping criterion.

It remains to show that the algorithm terminates—that is, that the iteration sequence for $F^k$ reaches the stopping criterion $\|F^k(s) - s\| \leq \varepsilon$ after finitely many steps for each $k$. First note that $F^k$ is monotonic and continuous from above. This follows just like in Lemma 2, where we also need to account for the effects of earlier cores on the financial subsystem $C_k$: CDSs written by banks in $C_k$ on banks in earlier cores give rise to additional fixed liabilities, and incoming payments from earlier cores to $C_k$ give rise to additional assets. These manifest as constants that do not affect the argument in the proof. Now the iteration sequence converges to a maximal fixed point of $F^k$ like in Theorem 5. In particular, we reach the stopping criterion after finitely many steps.[22] □

Given the core iteration algorithm, it is now straightforward to prove existence in systems without red-containing cycles.

**Proof of Theorem 6.** We "run" the algorithm with $\varepsilon = 0$ to receive a constructive proof of existence. The stopping criterion $\|F^k(s) - s\| = 0$ is not attained after finitely may steps but in the limit of the iteration sequence. All other steps of the proof of Proposition 2 remain the same. □

Theorem 6 generalizes and unifies the existence statements of Theorems 4 and 5: individually, neither *cycles* nor *red edges going to nonleaf nodes* are a problem; only *red-containing cycles* can cause nonexistence. Thus, the no-red-containing-cycle condition is the most general (weakest) condition we have derived for existence (as it also covers acyclic and green core systems). Regarding maximality, the weakest condition we have derived is "acyclic or no naked CDSs" (Theorem 4 and Corollary 1). The absence of red-containing cycles does not guarantee maximality because cores with an incoming red edge may be made worse off when the recovery rates of earlier cores are maximized. It is an open question whether a condition exists that is weaker than acyclic or no naked CDSs and guarantees maximality. However, because our analysis has shown that cycles and red edges in the dependency graph are essential

factors for nonmaximality, it seems unlikely that a simple condition that fulfills this requirement can be found.

In this section, we have shown that our dependency analysis framework can be used to derive sufficient conditions for existence and maximality. However, they are not *necessary conditions*. Although it may be possible to derive stronger guarantees by taking even more information about the contract structure into account, we should not expect to obtain *equivalence conditions*: Our computational complexity results in Schuldenzucker et al. (2019) imply that any condition that is equivalent to existence or maximality would be NP-hard to check (informally, this would take exponential runtime) and would therefore be of limited use. By contrast, our framework has yielded sufficient conditions that are simple and easy to check. Thus, we argue that our colored dependency graph hits a "sweet spot" by capturing the most important interactions among contracts, enabling us to distinguish between long and short positions as well as between covered and naked CDSs.

## 6. Discussion: Policy Relevance
We evaluate two recent policies regarding their effectiveness for protecting against default ambiguity under the assumptions of our model: *central counterparty clearing* and *banning naked CDSs*.

The regulatory frameworks EMIR (in Europe) and Dodd–Frank (in the United States) mandate the use of a *central clearing counterparty* (CCP) for a large part of the over-the-counter derivatives market.[23] In its most extreme form, this means that all contracts are routed via a central node: a bank A would not write a contract to a bank B directly, but rather bank A would write a contract to a highly capitalized central entity $S$, and $S$ would write a contract to bank B. One of the desired effects is that the CCP would absorb a shock on the banks, prevent it from spreading through the network, and thus prevent financial contagion. Although using a CCP simplifies the network of liabilities, surprisingly, it is not effective for protecting against default ambiguity in our model. Figure 8 provides an example: there are three banks that hold CDSs and write debt together with a CCP $S$.[24] Note that $S$ has very high external assets such that it cannot default. This system does not have a solution (the proof is given in Appendix E). Indeed, when we look at the colored dependency graph (Figure 8, bottom panel), we see that there is still a red-containing cycle A–B–C–A. At a higher level, we see that although a CCP can help reduce *counterparty risk* (i.e., the risk to a bank that a debtor cannot pay its liability), the *flow of fundamental risk* (i.e., the risk that the reference entity in a CDS has a higher or lower recovery rate than expected; see D'Errico et al. 2018) still takes place

**Figure 8.** (Color online) Financial System with a CCP and No Solution and Its Colored Dependency Graph
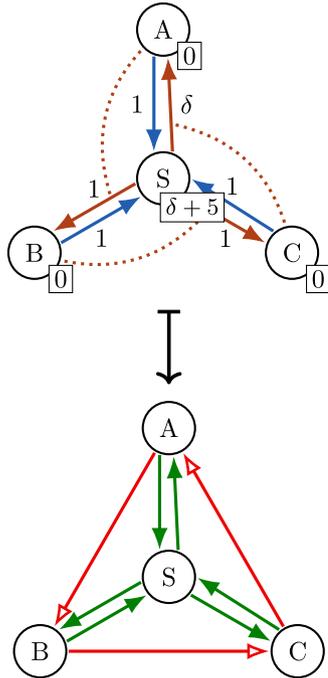


*Notes.* Top Panel: Financial system with a CCP $S$ where $\beta < 1$ is arbitrary and $\delta := 1/(1 - \beta + \beta^2 - \beta^3)$. There is no clearing recovery rate vector. Bottom Panel: The corresponding colored dependency graph (note that we omit offsetting positions that the CCP $S$ has with other banks as they do not affect our results).

directly between the banks, essentially "around" the CCP. This is enough to lead to nonexistence of a solution. Overall, our example shows that requiring banks to trade all CDSs via a CCP, even if the CCP is very well capitalized, does not guarantee existence of a solution to the clearing problem. This result is formally proven in Appendix E.

Another policy that has seen adoption in Europe since the European sovereign debt crisis in 2011 is *banning naked CDSs*. A CDS on a European sovereign state can only be bought if a corresponding (debt) exposure is present as well (European Commission 2011; also see European Securities and Markets Authority 2017b). Corollary 1 shows that if *all* naked CDSs are banned, not only those on sovereigns, then the clearing problem is guaranteed to have a maximal solution. Thus, under the assumptions of our model, this policy is effective against default ambiguity. Note that in this paper, we refrain from recommending the adoption of any particular policy. Instead, our findings illustrate how our framework can be used to help inform regulatory policy.

## 7. Conclusion

In this paper, we have shown that financial networks that contain debt contracts and CDSs are prone to a phenomenon we call *default ambiguity*—that is, a situation where it is impossible to decide which banks are in default. For many years, the total notional of CDSs written on financial institutions has exceeded USD1 trillion worldwide. Although the European Commission has previously acknowledged that CDSs can give rise to new kinds of systemic risk, they have not yet considered the risk of default ambiguity. Our new dependency analysis framework reveals that default ambiguity hinges on the presence of cycles in the colored dependency graph. Table 1 summarizes our findings. As we have shown, the more we relax the restrictions on the type of these cycles, the weaker the guarantees we obtain for our desiderata. To find a solution for the restricted network structures we have studied, one can use the *core iteration algorithm* we have provided.

Our results illustrate that, to understand the behavior of financial systems with CDSs, it is essential to consider the *ternary* relations they introduce, including the reference entities. If we had instead only considered the writer–holder relationships, all of our counterexamples in Section 3 would have looked like simple acyclic graphs, and we would only have captured one of the three dependencies arising from a CDS. Our insights may help bring about a paradigm

**Table 1.** Summary of Our Results

| Contracts | Network structure | Existence | | Maximality | Reference |
|---|---|---|---|---|---|
| Debt only | Any network structure | ✓ | | ✓ | Eisenberg and Noe (2001), Rogers and Veraart (2013) |
| Debt + CDSs | External reference entities only | ✓ | | ✓ | See "Debt only" above[a] |
| | Acyclic dependency graph | ✓ (unique) | | ✓ | This paper (Section 5.1) |
| | No naked CDSs | ✓ | | ✓ | This paper (Section 5.2) |
| | Green core systems | ✓ | | (✓) (in the core) | This paper (Section 5.2) |
| | No red-containing cycles | ✓ | | ✗ | This paper (Section 5.3) |
| | Red-containing cycles | ✓ (no default costs)[b]/✗ (with default costs) | | ✗ | This paper (Section 3) |

[a] When reference entities are external to the financial system, then CDS liabilities can be considered constant for the purpose of clearing, and the results for financial systems without CDSs carry over to this case.

[b] Remember that our proof of this result is nonconstructive. Indeed, we have shown in a separate piece of work (Schuldenzucker et al. 2019) that, in general financial systems without default costs, finding a solution is PPAD-hard (informally, any algorithm would need exponential runtime in the worst case). This implies that any *practical* algorithm would have to use heuristics.

shift in the literature on systemic risk in CDS markets where, so far, either the reference entities were aggregated or the interactions across different reference entities were not taken into account.

From a conceptual perspective, the reason why CDSs can give rise to default ambiguity is that the holder of a naked CDS may profit from financial distress of another market participant. Note that this phenomenon is not exclusive to CDSs. For example, the holder of a bond put option and the writer of a bond call option both benefit if the issuer of the underlying bond is in financial distress, and therefore the price of the bond declines. Stock options exhibit similar behavior. Thus, we expect that these markets would also be susceptible to default ambiguity. Our framework can be extended to these other derivative markets in a straightforward way (in particular, to options).

Our dependency analysis framework enables ex ante guarantees that hold irrespective of a shock to banks' external assets. If the external assets are known or can be bounded, future work may be able to derive stronger sufficient conditions for existence and maximality. To do this, one could extend the colored dependency graph with weights that represent the "strength" of the dependency. This will be a challenging task, however, because in contrast to standard weighted-graph models, this "weighted dependency graph" would have to represent highly nonlinear effects.

An important problem that is closely related to clearing is *network valuation* of contracts. Studying this problem requires a model with uncertainty about the future value of banks' external assets. Barucca et al. (2016) designed such a model for debt-only networks by extending the Eisenberg and Noe (2001) model. Researchers interested in network valuation with CDSs could similarly extend our new model. This would raise new questions regarding whether a consistent vector of CDS valuations exists and what is needed for market prices to reflect these true values.

## Acknowledgments

## Appendix A. Example That Iterating the Update Function Does Not Generally Converge to a Solution

Consider Figure A.1.[25] The unique solution of this system is $r_A = 6/7$, $r_B = 3/7$, and $r_C = r_D = 1$. However, the iteration sequence defined by $r^0 = (1, 1, 1, 1)$ and $r^{n+1} = F(r^n)$ does not converge to this solution but rather exhibits cycling behavior: we have $r^1 = (1, 0, 1, 1)$, $r^2 = (0, 0, 1, 1)$, $r^3 = (0, 1, 1, 1)$, $r^4 = (1, 1, 1, 1) = r^0$, etc. One may think that the cycling behavior is due to an unfortunate choice of the starting point $r^0$, but this is not the case: the iteration sequence does not converge for *any* starting point other than the solution itself. To see this, let $\Delta \neq 0$ and $r_B = 3/7 + \Delta$. It is easy to see from the definition of $F$ that

$$F_B(F(r)) = \min(1, \max(0, 3(1 - 2r_B)))$$
$$= \min\left(1, \max\left(0, \frac{3}{7} - 6\Delta\right)\right).$$

Thus, after two iterations, the distance to the solution has increased sixfold until the sequence again enters the infinite loop above.

**Figure A.1.** (Color online) Financial System Where Iterating $F$ Does Not Converge to a Solution; Let $\alpha = \beta = 1$ (No Default Costs)
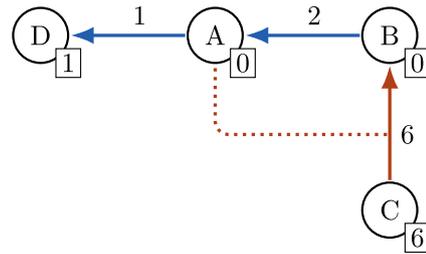


## Appendix B. Omitted Proofs from Section 3

We describe a financial system that does not have a solution and where $\alpha < \beta = 1$, thus completing the proof of Theorem 1.

**Proof of Theorem 1** ($\alpha < \beta = 1$). Consider Figure B.1, a variant of Figure 2, with values for $e_A$, $\gamma$, and $\delta$ chosen as follows: let $e_A \in (0, 1)$ arbitrary, set $\gamma = 1 - (1 + \alpha)/2 \cdot e_A$, and let $\delta \geq \gamma/(1 - \alpha e_A - \gamma)$. It is easy to see that (1) $e_A < 1$, (2) $e_A + \gamma > 1$, and (3) $\alpha e_A + \gamma < 1$. We have $\gamma > 0$ by definition and $\delta > 0$ by (3), so this is a well-defined financial system.

We perform a case distinction such as in the proof for $\beta < 1$. Assume toward a contradiction that $r$ is clearing.

**Figure B.1.** (Color online) Financial System with No Solution for $\alpha < \beta = 1$
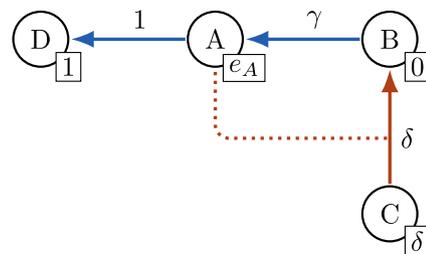
- If $r_A = 1$, then $p_{C,B}(r) = 0$, so $a_B(r) = 0$ and $p_{B,A}(r) = 0$. Thus, $a_A(r) = e_A < 1$, which implies that $r_A < 1$—a contradiction.
- If $r_A < 1$, then $A$ is in default, so $r_A = (\alpha e_A + p_{B,A}(r))/1 \le \alpha e_A + \gamma$. Thus, $p_{C,B}(r) = \delta(1 - r_A) \ge \delta(1 - \alpha e_A - \gamma) \ge \gamma$, so $B$ is not in default and $p_{B,A}(r) = \gamma$. Now $a_A(r) = e_A + \gamma > 1$ by (2), so $A$ is not in default, and $r_A = 1$—a contradiction. □

## Appendix C. Omitted Proofs from Section 5

We show that in a green core system, the update function is monotonic and continuous from above and the equities of core banks are monotonic, which is the main ingredient to the proof of Theorem 5.

**Proof of Lemma 2.** As the main step of the proof, we show that for all $i \in C$, the assets $a_i(r)$ and the assets after default costs $a_i'(r)$ are monotonically increasing in $r$.

- *The assets $a_i$ and $a_i'$ are monotonic*: It suffices to show that the total incoming payments of bank $i$, $\sum_j p_{j,i}(r)$, are monotonically increasing in $r$. To this end, let

$$q_{k,i}(r) := r_k c_{k,i}^{\emptyset} + (1 - r_k) \sum_j r_j c_{j,i}^k.$$

Observe that $\sum_j p_{j,i}(r) = \sum_k q_{k,i}(r)$. Each individual summand $q_{k,i}(r)$ is monotonically increasing in $r$ by the green core property, which can be seen as follows. Let $r \le r'$ pointwise. Then

$$
\begin{aligned}
q_{k,i}(r') - q_{k,i}(r) &= r_k' c_{k,i}^{\emptyset} - r_k c_{k,i}^{\emptyset} + (1 - r_k') \sum_j r_j' c_{j,i}^k \\
&\quad - (1 - r_k) \sum_j r_j c_{j,i}^k \\
&\ge r_k' c_{k,i}^{\emptyset} - r_k c_{k,i}^{\emptyset} + (1 - r_k') \sum_j r_j c_{j,i}^k \\
&\quad - (1 - r_k) \sum_j r_j c_{j,i}^k \\
&= (r_k' - r_k) \cdot \left( c_{k,i}^{\emptyset} - \sum_j r_j c_{j,i}^k \right) \\
&\ge (r_k' - r_k) \cdot \left( c_{k,i}^{\emptyset} - \sum_j c_{j,i}^k \right) \ge 0,
\end{aligned}
$$

where the last inequality holds because $r_k' - r_k \ge 0$ by assumption and $c_{k,i}^{\emptyset} - \sum_j c_{j,i}^k \ge 0$ because we are in a green core system, so $i$ must have a covered CDS position toward $k$.

- *The equity $E_i$ is monotonic for $i \in C$*: First note that the liabilities $l_i(r)$ are monotonically decreasing in $r$ in any financial system, as can be seen directly from the definition. As $a_i(r)$ is monotonically increasing by the above argument, $E_i(r) = \max(0, a_i(r)) - l_i(r)$ is monotonically increasing.
- *The function $F$ is monotonic and continuous from above*: Because $F_i$ is constant 1 for $i \in L$, it suffices to show the statement for each $F_i$ with $i \in C$. To this end, note that $F_i$ is of form

$$
F_i(r) = \begin{cases} f(r) & \text{if } h(r) \ge 0, \\ g(r) & \text{if } h(r) < 0, \end{cases}
$$

where $f(r) := 1$, $g(r) := a_i'(r)/l_i(r)$, and $h(r) := a_i(r) - l_i(r)$ are all monotonic and continuous. It is easy to see that this implies that $F_i$ is monotonic and continuous from above. □

The following lemma has become a standard proof technique in financial network theory, for example in Rogers and Veraart (2013). It can be viewed as a special case of the Kleene

or Tarski–Kantorovitch fixed-point theorems (see Granas and Dugundji 2003). We restate and prove it here because there is no standard reference for it.

**Lemma 3.** *Let N be any finite set, and let $F : [0,1]^N \to [0,1]^N$ be any function that is monotonic and continuous from above, where the order relation is given by pointwise ordering. Then F has a pointwise maximal fixed point, and the iteration sequence $(r^n)$ defined by $r^0 = (1, \ldots, 1)$ and $r^{n+1} = F(r^n)$ converges to this maximal fixed point.*

**Proof.** We proceed in three steps.

(i) *The sequence $(r^n)$ is descending and convergent*: We show by induction that $(r^n)$ is a descending sequence; that is, $r^n \ge r^{n+1}$ pointwise. For $n = 0$, this is trivial because $r^0$ is the maximal element of $[0,1]^N$. For $n > 0$, and assuming $r^{n-1} \ge r^n$, we have $r^n = F(r^{n-1}) \ge F(r^n) = r^{n+1}$ by monotonicity of $F$. Because $(r^n)$ is also bounded from below by $(0, \ldots, 0)$, it must be convergent. Call the limit of the sequence $r$.

(ii) *The point r is greater or equal to any fixed point of F*: It suffices to show that any $r^n$ is greater or equal to any fixed point $r^*$ of $F$. We proceed by induction: for $n = 0$, the statement is obvious; for $n > 0$, and assuming $r^{n-1} \ge r^*$, we receive by monotonicity of $F$ that $r^n = F(r^{n-1}) \ge F(r^*) = r^*$.

(iii) *The point r is a fixed point of F*: Because $F$ is continuous from above and $(r^n)$ is descending, we have $F(r) = F(\lim_n r^n) = \lim_n F(r^n) = \lim_n r^{n+1} = \lim_n r^n = r$. □

## Appendix D. Example That Iterating the Update Function Is Not Effective in the No-Red-Containing-Cycle Case

Consider Figure D.1 and let $\alpha = \beta = 0.5$. The cores of the dependency graph in topological order are {A, B}, {C}, {D}, and {E}. The unique solution of this system is given by $r_A = r_B = 0$ and $r_C = r_D = r_E = 1$. This is because $C$ and $E$ cannot default, and $A$ and $B$ must default with recovery rate 0 as they together have no assets but an outgoing liability; from this it follows that $D$ has assets exactly equal to its liabilities.

**Figure D.1.** (Color online) Financial System Where Iterating the Update Function Does Not Converge
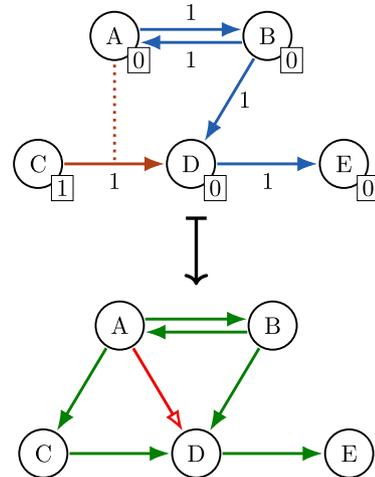


*Notes.* Top Panel: Financial system with no red-containing cycle where iterating $F$ does not converge to a solution (let $\alpha = \beta = 0.5$). Bottom Panel: Its colored dependency graph.

Simply iterating the update function $F$ does not converge in this system. To see this, let $r^0 = 1, \ldots, 1$ and $r^{n+1} = F(r^n)$ for each $n$. We first consider the recovery rates of bank $A$ and $B$ as the iteration sequence proceeds. Note that $B$ defaults in step 1, and then, following the default of $B$, $A$ defaults in step 2. We thus have from the definition of $F$ for $r_A^n$ and $r_B^n$ (recall that default costs are 0.5):

$$r_A^n = \frac{1}{2} r_B^{n-1} \text{ for } n \geq 2 \text{ and } r_A^0 = r_A^1 = 1,$$

$$r_B^n = \frac{1}{4} r_A^{n-1} \text{ for } n \geq 1 \text{ and } r_B^0 = 1.$$

Solving these recursive equations yields $r_A^n = 2^{-3\lfloor n/2 \rfloor}$ for all $n$ and $r_B^n = 2^{-3\lceil n/2 \rceil + 1}$ for $n \geq 1$ and $r_B^0 = 1$. We observe that for $n \geq 2$, $r_A^n < r_B^n$ if $n$ is even and $r_A^n > r_B^n$ if $n$ is odd. This is because $\log(r_A^n) - \log(r_B^n) = -3\lfloor n/2 \rfloor + 3\lceil n/2 \rceil - 1 = 3\lceil n/2 \rceil - \lfloor n/2 \rfloor - 1$ and $\lceil n/2 \rceil - \lfloor n/2 \rfloor$ is 0 if $n$ is even and 1 if $n$ is odd.

Now consider bank $D$. The assets of $D$ consist of a CDS on $A$ and debt from $B$, so $a_D(r) = 1 - r_A + r_B$, and $l_D(r) = 1$. Thus, $D$ is in default iff $r_A > r_B$. Over the course of the iteration, whenever $n$ is even, we have $r_A^n < r_B^n$, so $D$ is not in default and $r_D^{n+1} = F_D(r^n) = 1$. Whenever $n$ is odd, we have $r_A^n > r_B^n$, so $D$ is in default and $r_D^{n+1} = F_D(r^n) < \max(\alpha, \beta) = 0.5$. Hence, $r_D^n$ changes by at least 0.5 from each iteration to the next. In particular, the sequence does not converge.

Our example may appear artificial because bank $D$ is just on the verge of defaulting in the solution. We, indeed, expect that the iteration sequence converges if this is not the case for any bank. However, it is not clear how one would detect this property if the exact solution is not yet known.

## Appendix E. Nonexistence with a CCP
We formalize and prove the fact that the presence of a highly capitalized central clearing counterparty (CCP) does not guarantee existence of a clearing recovery rate vector (see Section 6).

**Proposition 3.** *There exists a financial system $(N, e, c, \alpha, \beta)$ with a distinguished bank $S \in N$ (the CCP) such that the following holds:*

*(1) The CCP is a counterparty to each contract: For any $k \in N \cup \{\emptyset\}$ and $i, j \in N$, if $c_{i,j}^k > 0$, then $S \in \{i, j\}$.*

*(2) The CCP is running a balanced book: For any $k \in N \cup \{\emptyset\}$, $\sum_i c_{i,S}^k = \sum_i c_{S,i}^k$.*

*(3) The CCP is so highly capitalized that it cannot default: $e_S \geq \sum_i c_{S,i}^\emptyset + \sum_{i,k} c_{S,i}^k$.*

*(4) The financial system has no clearing recovery rate vector.*

**Proof.** Let $\beta < 1$. Consider a financial system with banks A, B, C, and S and contracts such as in Figure 8 together with an additional bank D (for offsetting positions). Choose $e_D$ arbitrarily, and for any $k \in N \cup \{\emptyset\}$, let $c_{D,S}^k = \sum_{i \in N \setminus \{D\}} c_{S,i}^k$ and $c_{S,D}^k = \sum_{i \in N \setminus \{D\}} c_{i,S}^k$. The system is well defined because $1 - \beta + \beta^2 - \beta^3 = (1-\beta) + \beta^2(1-\beta) > 0$, so $\delta > 0$. It is easy to see that it fulfills conditions (1)–(3). To see that it also fulfills condition (4), assume toward a contradiction that $r \in [0,1]^N$ is clearing. As $S$ is highly capitalized, $r_S = 1$.

If $r_A = 1$, then $r_B = 0$; thus $r_C = 1$, and thus $r_A = 0$, as is easily seen from the contracts—a contradiction.

If $r_A < 1$, then $r_A \leq \beta$ by Lemma 1. Then $a_B(r) \geq 1 - \beta$, so $r_B \geq \beta(1-\beta)$. Then $a_C(r) \leq 1 - \beta(1-\beta)$, so $r_C \leq \beta(1 - \beta(1-\beta))$.

Thus, $a_A(r) \geq \delta(1 - \beta(1 - \beta(1-\beta))) = \delta(1 - \beta + \beta^2 - \beta^3) = 1 = l_A(r)$: a contradiction. $\square$

## Endnotes
[1] As payments are made simultaneously and there is no timing, *default* and *technical insolvency* are equivalent conditions in our model. We use the term "default" throughout this paper.

[2] Note that the models in Eisenberg and Noe (2001) and Rogers and Veraart (2013) are based on the payments between banks instead of recovery rates. It is easy to see that the two points of view are equivalent. In debt-only financial systems, maximizing payments, recovery rates, and equities are equivalent objectives.

[3] One could also include the interests of "society" (i.e., the real economy) in our analysis by introducing it as an additional node in the network. In Section 5.2 (Corollary 1), we derive sufficient conditions for maximality that guarantee that a solution is simultaneously best for all banks and society, assuming that banks' defaults can only have a negative effect on society.

[4] The total notional of these CDSs was USD1 trillion in the second half of 2017. See Bank for International Settlements (2018, section "Single-name instruments," subsection "Financial firms").

[5] An orthogonal question is whether we can efficiently compute a solution to the clearing problem or determine algorithmically whether a solution exists. We have found in a separate stream of work (Schuldenzucker et al. 2019) where, in contrast to debt-only networks, both problems are computationally infeasible in general financial networks with CDSs. It is an open question to which extent algorithms exist that are guaranteed to be efficient in restricted cases.

[6] See our discussion at the end of Section 2.3 for a comparison with prior approaches toward modeling these networks.

[7] Following Rogers and Veraart (2013), we model default costs as multiplicative discounts $\alpha \in [0, 1]$ on the external assets and $\beta \in [0, 1]$ on the interbank assets that apply only in case of default. If $\alpha = \beta = 1$, no default costs are present, and we show using a fixed-point theorem that a solution always exists by continuity. By contrast, we show that if either $\alpha < 1$ or $\beta < 1$ (or both), default costs introduce a discontinuity, and existence is not guaranteed anymore.

[8] A similar kind of contradiction was observed by Sundaresan and Wang (2015), who considered a setting with a single bank that issues *contingent capital*—that is, debt that is converted into equity as soon as the bank's stock price falls below a threshold. For certain contract parameters, this led to nonexistence or multiplicity of an equilibrium stock price. Their situation bears some resemblance to a bank that has gone "short on itself" by buying a pathological CDS where it itself is the reference entity or "long on itself" by selling such a CDS. Sundaresan and Wang's scenario thus concerns the balance sheet of an *individual* bank. By contrast, the ambiguity we illustrate is due to the *interactions* among different contracts in a network, although each individual bank may look innocuous.

[9] The ECB's recent STAMP\euro framework, which was developed based on "top-down models used to support EU-wide stress-testing exercises" (Constâncio 2017), includes network effects as one of its central elements. Specifically, a variant of the clearing problem very close to Eisenberg and Noe (2001) is solved 20,000 times in the context of a Monte Carlo simulation to obtain a probability distribution of contagion losses (Dees et al. 2017, chapter 12).

[10] For example, Fender et al. (2008) described how the default of Lehman Brothers, which was both a major counterparty and reference entity in CDSs, had significant repercussions in money markets. Further distress in these markets could only be averted by the government rescue of AIG, another major CDS trader.

[11] Default costs could result from legal and administrative costs, from a delay in payments, or from fire sales when defaulting banks need to

sell off their assets quickly. Details can be found in Rogers and Veraart (2013).

[12] Some prior work *has* employed graph-based models for CDS networks when simplifying assumptions reduce a CDS to a binary relationship. It was implicitly assumed that either the default of a reference entity is an event exogenous to the network (Duffie and Zhu 2011, Markose et al. 2012, Brunnermeier et al. 2013) or that CDSs do not carry counterparty risk (Puliga et al. 2014, Leduc et al. 2017). Heise and Kühn (2012) studied a model of CDS networks with ternary relationships and short positions. However, they only considered recovery rates 0 and 1 and a fixed number of update steps. We are the first to study simultaneous clearing in networks where CDSs are modeled as ternary relationships.

[13] This is the case, for example, in any variant of Figure 2 where B has positive external assets. Here, B might first receive the full amount in the CDS and then pay A, or B might pay A first, which reduces the payment in the CDS.

[14] Note that a maximal solution is not necessarily unique, but all maximal solutions lead to the same equities $E_i(r)$.

[15] One could define a notion of *utility* for banks equal to their equity. Then a solution to the clearing problem would be maximal if and only if it Pareto dominates any other solution, and the situation in Theorem 3 would be one of multiple Pareto optima. Note, however, that the multiplicity we reveal here is different from a multiplicity of equilibria that is often observed in strategic games. Recovery rates are not actions chosen by banks. Rather, they are mathematically implied by the network of obligations and the rules of bankruptcy. Thus, instead of banks "choosing an equilibrium," the clearing authority chooses a solution to be implemented, which implies a requirement to treat all market participants fairly (in particular, not to advantage one bank over another one). This is only possible if maximality is fulfilled.

[16] The solution $r^2$ that exists only in the $\beta = 1$ case is strictly disfavored by A and B over all other solutions (but strictly preferred by C and D).

[17] If there exists a solution, there also exists one that maximizes total equity (except in pathological cases). One might be tempted to simply select such a solution. However, without maximality, there would still be banks that prefer another solution. For *complete markets*, we know from the second welfare theorem that one can impose lump sum transfers to move from one (less desirable) Pareto-optimal outcome to any other (more desirable) one. By contrast, without maximality, all solutions of the clearing problem that are Pareto optimal pose the issue that the clearing authority would have to favor one bank over another.

[18] It follows from the structure of $F$ that in this situation, holding $r_{-ij}$ fixed, $F_j$ is increasing at *some* point $r_i$ if and only if it is increasing at *all* points $r_i$.

[19] Leduc et al. (2017) presented a mapping from a financial system with CDSs to a weighted graph where they distinguished between naked and covered CDSs in a similar way as we do. However, they made simplifying assumptions regarding the regulatory environment such that only a subset of the possible dependencies need to be considered. For example, their model does not represent default by CDS writers or short dependencies so that naked CDSs cannot be captured. This makes their model unsuitable to study general financial systems with naked and covered CDSs.

[20] See Schuldenzucker et al. (2019, appendix B) for a financial system with CDSs and a unique and irrational solution. Inspection shows that it is a green core system. The maximal solution of a debt-only system is rational because it can be computed exactly in finite time using Eisenberg and Noe's (2001) *fictitious default algorithm*.

[21] Both computing strongly connected components and sorting in topological order can be done easily using well-known algorithms (see, e.g., Korte and Vygen 2012). Note that the topological order may not be unique.

[22] It is possible to show that the number of steps of the algorithm is bounded by a function that is polynomial with respect to the size of the financial system and $1/\varepsilon$. This makes the core iteration algorithm a computationally efficient approximation scheme. See Schuldenzucker et al. (2019) for details.

[23] Both frameworks mandate use of a CCP for certain types of derivatives (interest rate swaps and index CDSs) but not for the kind of CDSs we discuss in this paper (single-name CDSs). See European Securities and Markets Authority (2017a, c) for EMIR and the documents linked at U.S. Commodity Futures Trading Commission (2017) for Dodd–Frank.

[24] For simplicity, we assume that all CDSs and debt contracts are cleared via the same CCP. This is not necessary for our result. We further assume that S has offsetting positions with other banks (i.e., it is running a balanced book). These positions do not affect our result and are therefore omitted from the figure.

[25] Figure A.1 corresponds to Figure 2 for $\beta = 0.5$; however, in Figure A.1, we set $\alpha = \beta = 1$. That is why, in contrast to Figure 2, this system has a solution.

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