Diffusion of Multiple Information: On Information Resilience and the Power of Segregation*

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Abstract

We introduce two pieces of information (memes) into a diffusion process in which memes are transmitted when agents meet and forgotten at an exogenous rate. At most one meme can be transmitted at each meeting, which one depends on preferences over memes. We find that the conditions under which a unique meme becomes endemic are sufficient for both to become endemic. Segregation according to information preferences leads to polarization, i.e., nobody is informed of both memes, and a loss of information. We show how the likelihood of segregation depends on information preferences and on parameters of the diffusion process.

Keywords: Social Networks, Information Transmission, Multiple States, Segregation

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1. Introduction

Since the seminal works of Lazarsfeld et al. (1948) and Katz and Lazarsfeld (1955), the importance of social networks in the diffusion of information has been well documented. It is less well known how different pieces of information, or *memes*,[1] interact in this diffusion process. In the production of news, such as in print or TV media, it is obvious that fixed coverage space is shared by different news stories. Arguments about politicians aiming to "bury" unfavorable news arise from this. In the social diffusion process, a similar constraint is present: Communication time is limited and has to be shared among everything an individual talks about. Making use of Twitter data, Leskovec et al. (2009) and Weng et al. (2012) have shown that the total volume of tweets is roughly constant over time, despite significant variation in the topics of tweets. In addition, their data shows that (i) at any point in time numerous hashtags diffuse simultaneously, (ii) there are significant differences in the number of times a hashtag is retweeted, and (iii) hashtags crowd each other out. Diffusion models of a unique information are not equipped to explain these patterns. Similarly, it is not clear that these patterns are necessarily due to strategic choices in the transmission of memes. The sheer volume of topics that are being transmitted online suggests that, if agents face a choice about which meme to transmit, this choice may well be between memes of entirely unrelated topics, for which strategic considerations appear unlikely.[2]

The present paper introduces a parsimonious diffusion model of multiple memes under a communication constraint, which reproduces the above patterns. To keep the analysis tractable and to isolate the effect of limited communication

[1] The Merriam-Webster dictionary defines a meme as "an idea, behavior, style, or usage that spreads from person to person within a culture". It therefore provides a meaningful way to talk about pieces or bits of information.

[2] Consider this concrete example: According to the NIFTY project, which tracks memes on Twitter, in April 2015 the two most frequently tweeted memes were "Star Wars: The Force Awakens" (relating to the trailer release of the movie) and "A Rape on Campus" (relating to the veracity of a journal article). It is unlikely that strategic considerations played a role in the decision to tweet one as opposed to the other of these. Other months similarly showcase the breadth of topics that diffuse simultaneously online.

time, we build on a standard diffusion process with epidemiological roots, the *Susceptible-Infected-Susceptible* (SIS) framework.[3] In the model, each period each agent randomly meets a subset of other agents. At any meeting, there is a chance that communication occurs, in which case an informed agent passes a meme on. Agents forget memes at an exogenous rate. The novelty of our process lies in the existence of two memes. At each meeting, if communication occurs, each agent can pass on only one meme. Within the model, the choice of what to talk about is determined by intrinsic information preferences of agents, capturing the idea that individuals are more likely to talk about things that interest them more.

Within a mean-field approximation of this process, the literature has established the conditions under which a single meme exhibits a positive steady-state, in which a constant fraction of the population is informed about it in the long run. In our first main result, we show that the conditions that guarantee existence, uniqueness, and stability of a positive steady-state for *either* meme in our model are identical to the ones previously derived. That is, we show that information is extremely resilient. This result notwithstanding, we are able to rank information steady-states according to interest: The meme that is preferred by the majority of the population will exhibit a higher steady-state. We also show that crowding out of information always occurs.

The resilience of information occurs in a society in which all agents interact randomly. Instead, it is well documented that individuals exhibit *homophily*, a tendency to interact relatively more with others that are similar to themselves. We show that segregation according to information preferences leads to a segregation of information: If groups do not interact with each other, within each group, only the preferred meme exhibits a positive steady-state. Interestingly, both memes exhibit lower steady-states in a segregated society than in a fully integrated one, i.e., segregation leads to a loss of information overall.

[3]Within economics, this process has been employed by, e.g., Jackson and Rogers (2007b), López-Pintado (2008), Jackson and Yariv (2010), and Jackson and López-Pintado (2013), among others.

While segregation implies a loss of information, we find that it increases the fraction of the population informed about their *preferred* meme. Given this result, we extend our model to endow agents with specific utilities from being informed. In particular, we assume that being informed with the preferred meme provides a utility flow that is larger than the utility flow of being informed with the alternative meme. This extension allows us to analyze the factors that influence the likelihood of segregation, as opposed to focusing only on its impact. In fact, segregation may be a Nash equilibrium in our model. We find that the likelihood of observing a segregated society is increasing in the extremism of information preferences, and that it tends to be driven by members of the smaller group. Segregation is less likely to be an equilibrium the more meetings agents have per period, *ceteris paribus*.

2. Related Literature

Our paper is part of a growing literature that studies diffusion processes on networks. The model we introduce is a direct extension of the SIS framework that has been employed by various authors, among them Jackson and Rogers (2007b), López-Pintado (2008), Jackson and Yariv (2010), Galeotti and Rogers (2013) and Galeotti and Rogers (2015), as well as Izquierdo et al. (2018). This literature itself builds on work on epidemiological and contagion models in the natural sciences, such as Bailey (1975), Pastor-Satorras and Vespignani (2001a,b), Pastor-Satorras and Vespignani (2002), Watts (2002), or Dodds and Watts (2004).[4] The simultaneous diffusion of multiple states in this framework has been addressed by Pathak et al. (2010), Karrer and Newman (2011), Beutel et al. (2012) and Prakash et al. (2012). In contrast to the present paper, in these models infection with one virus/state provides full or partial immunity

[4] More broadly, the paper is also related to network processes of learning, best response dynamics, or explicit adoption decisions. These processes however differ significantly from the SIS model we employ. See, e.g., Jackson (2008) or Goyal (2012) for an excellent introduction to the literature, as well as Acemoglu and Ozdaglar (2011) for a recent overview of models of belief and opinion dynamics.

against the other. Such immunity introduces a tendency for the more virulent state to be the only one that survives in the population. This result is in stark contrast to our findings of information resilience. Resilience seems to be corroborated by the vast array of different topics that diffuse simultaneously on online social networks. The difference in the results is interesting from a technical point of view, as it highlights the importance of the stage at which the diffusion constraint is placed (i.e., whether on the infection or on the transmission likelihood). To the best of our knowledge, our paper is the first that introduces two distinct pieces of information / memes that compete for limited communication time into the SIS framework. Outside this framwork and complementary to our paper, Campbell et al. (2019) model the simultaneous diffusion of "mainstream" and "niche" content on a network. The decision of which type of content to transmit is modeled in the same way as in our paper, but all communicating agents are informed of at least one type of content. This creates a different evolutionary dynamic, in which the focus lies on transmission rather than awareness of information.

Diffusion of competing products or innovations has been analyzed in models of influence maximization, e.g., by Dubey et al. (2006), Bharathi et al. (2007), Borodin et al. (2010) and Goyal et al. (2019). These models differ significantly from an SIS diffusion process, both with respect to the modelling characteristics, and the questions that they aim to answer. The above papers are based on threshold models, in which contagion occurs on a fixed network and nodes never recover. The central question in this strand of literature is which nodes a player with a fixed budget would choose to infect to maximize the contagion of his product. In Goyal et al. (2019), e.g., the focus is on how the efficiency of a "seeding" strategy depends on the precise diffusion process and its interaction with the network structure. A paper that does employ a random network is Jiménez-Martínez (2019). Their analysis highlights that adoption of a product may be driven simply by the number of neighbors instead of their respective adoption choice, reminiscent of how information spreads in our own model. Similarly to previous work on the diffusion of multiple states in the SIS model, in all of these papers being infected with one product precludes infection with another, differently from our model. Strategic targeting, albeit in a different framework, is also the focus of Grabisch et al. (2017). The authors investigate which nodes should be targeted by strategic agents with opposing (fixed) beliefs that wish to influence the average opinion in a DeGroot model (DeGroot, 1974).

We are also related to the literature that has investigated the impact that homophily has on information diffusion and its potential to lead to polarization. In existing models, homophily may either hinder diffusion, as in Granovetter (1973) and Golub and Jackson (2012), or be beneficial, as in the adoption of a behavior in Jackson and López-Pintado (2013). Studies such as Rosenblat and Mobius (2004), Sunstein (2009), Baccara and Yariv (2010), Gentzkow and Shapiro (2011) or Flaxman et al. (2013) have investigated the relationship between homophily and polarization. The focus of these studies has been predominantly the impact of biased news/information consumption, and its potential to lead to polarization. Given the rise in internet usage, an important question is whether this rise might increase segregation and hence polarization. Gentzkow and Shapiro (2011) and Flaxman et al. (2013) find that online news consumption is not substantially more segregated than offline consumption of news, providing an argument against a link between internet usage and polarization. On the other hand, in a more recent paper Halberstam and Knight (2016) investigate homophily among Twitter users, which is used both as an Online Social Network (OSN) and as a tool to consume news. They find higher levels of homophily for the social network aspect. Our results on the importance of biases in social interactions, as opposed to news consumption, complement those of Halberstam and Knight (2016). Indeed, in our model, a complete segregation in communication according to type will lead to a situation in which all agents are only informed of their preferred meme. This is independent of the initial seeding of information, which may proxy as consumption of news. Our results are indicative that a driving force of polarization may be biased communication patterns as opposed to biased news consumption.

We do not model links as costly. Homophily has been shown to arise endogenously if links to the same types are cheaper than links to agents of another type (Jackson and Rogers, 2005; Galeotti et al., 2006; Dev, 2018), if the meeting process is biased (Jackson and Rogers, 2007a; Bramoullé et al., 2012), or if preferences are biased towards links within the same group such as in Currarini et al. (2009) or Currarini et al. (2010). In our model, own-type links may provide greater benefits than links across types as agents are more likely to become informed of their preferred meme, thus adding a different rationale why own-type links may be preferred.

While we model communication of memes, the choice of which message to transmit is non-strategic. Strategic information transmission has been the subject of, e.g., Hagenbach and Koessler (2010), Galeotti et al. (2013), or Bloch et al. (2018).

The rest of the paper is organized as follows. Section 3 presents the model and derives the steady-states of each meme, in particular, our result on information resilience. Section 4 relates the ranking of steady-states to information preferences and to network characteristics. Section 5 investigates the impact of homophily and derives the conditions under which agents themselves wish to segregate according to information interests. Section 6 provides extensions to the main model and Section 7 concludes. Various proofs are relegated to the Appendix.

3. The Model

3.1. Propagation Mechanism

There exists an infinite number of agents, indexed by i, who represent nodes of a network. Time is continuous. Meetings between agents signify links, and the degree of agent i, k_i, denotes the number of meetings that i has at each point in time. The distribution of degrees is P, such that $P(k)$ is the probability that a randomly drawn node has degree k. A fraction $g_{\mathrm{A}} \in [0, 1]$ of the population belongs to group A and the complement $g_{\mathrm{B}} = 1 - g_{\mathrm{A}}$ belongs to group B. Group membership determines informational preferences: There exist two memes, A

and B, and members of group A prefer meme A to meme B, and *vice versa*. We assume that these preferences are common knowledge and that they encompass both the consumption and the transmission of information, such that an agent who, e.g., prefers the topic of football over political news prefers both to talk about the most recent football results and to hear about them. We assume that memes A and B are independent of each other, as with football news as opposed to political news. With the exception of informational preferences, there is no difference between members of the two groups, an assumption that will be relaxed in Section 6.

Agents can be uninformed of both memes (*susceptible*, S), or informed of either or both. We denote being informed of meme A only by $I_{\mathrm{A\setminus B}}$, and being informed of only B by $I_{\mathrm{B\setminus A}}$, while being informed of both memes by I_{AB}. Thus, the set of states in which an agent can be is $\{S, I_{\mathrm{A\setminus B}}, I_{\mathrm{B\setminus A}}, I_{\mathrm{AB}}\}$. Transition between states occurs in the following way. Agents in S transition into $I_{\mathrm{A\setminus B}}$ if they become informed of meme A and into $I_{\mathrm{B\setminus A}}$ if they become informed of meme B. Agents in $I_{\mathrm{A\setminus B}}$ ($I_{\mathrm{B\setminus A}}$) transition into I_{AB} if they become informed of meme B (A). Similarly, they transition into S if they forget A (B). Agents in I_{AB} transition into $I_{\mathrm{A\setminus B}}$ ($I_{\mathrm{B\setminus A}}$) if they forget meme B (A). We denote by ν the rate at which information is transmitted at a meeting and by δ the rate at which it is forgotten. In line with the previous literature and the epidemiological roots of the model, we refer to ν as the (per contact) *infection rate* and δ as the *recovery rate*.[5]

A central assumption is that at each meeting agents can communicate at most one meme, as communication time is limited. We assume that in this case, an agent transmits his preferred meme, conditional on communication taking place at all.[6] Note that if agents are either in state $I_{\mathrm{A\setminus B}}$ or $I_{\mathrm{B\setminus A}}$, their

[5] If agents never forgot, all information would eventually be known by everybody. Apart from being uninteresting, this does not seem to be a relevant case for many of the memes that diffuse through social interactions. Much of the information that is transmitted as chit-chat is not immediately payoff relevant. Such information may be a prime target to be forgotten under memory limitations.

[6] This implies that a fraction g_l of agents in state I_{AB} will transmit meme l. Our results

information preferences will not matter for the rate at which they pass on meme l. In particular, agents are non-strategic in the way they pass on information. They neither distort the meme they possess, nor do they strategically choose to not transmit a meme. Aside from strategic considerations, other reasonable frameworks to model information transmission clearly exist. It is worthwhile highlighting that any such framework in which the existence of a second meme does not impose any restriction on the transmission of the first and *vice versa* will lead to a parallel diffusion of the memes in which each meme's diffusion can be studied individually. The standard one meme SIS model then applies.[7] The present model is kept deliberately simple with respect to further intricacies of the transmission process of memes to clearly identify the role that competition between memes in communication plays. Section 6 discusses the relaxation of some of the simplifying assumptions employed here.

Following the literature, we model the diffusion of the memes under the assumption that the network of meetings is realized every period and we solve for the *mean-field approximation* of the system. Formally, we define $\rho_{\mathrm{A\setminus B}}(k)$, $\rho_{\mathrm{B\setminus A}}(k)$ and $\rho_{\mathrm{AB}}(k)$ as the proportion of degree-k agents in the three infection states, $I_{\mathrm{A\setminus B}}$, $I_{\mathrm{B\setminus A}}$, and I_{AB}, respectively. We denote the corresponding *prevalences* in the population overall as $\rho_{\mathrm{A\setminus B}} = \sum_k P(k)\rho_{\mathrm{A\setminus B}}(k)$, $\rho_{\mathrm{B\setminus A}} = \sum_k P(k)\rho_{\mathrm{B\setminus A}}(k)$ and $\rho_{\mathrm{AB}} = \sum_k P(k)\rho_{\mathrm{AB}}(k)$. By definition, $\rho_{\mathrm{A}}(k) = \rho_{\mathrm{A\setminus B}}(k)+\rho_{\mathrm{AB}}(k)$, $\rho_{\mathrm{B}}(k) = \rho_{\mathrm{B\setminus A}}(k)+\rho_{\mathrm{AB}}(k)$, and $\rho(k) = \rho_{\mathrm{A\setminus B}}(k) + \rho_{\mathrm{B\setminus A}}(k) + \rho_{\mathrm{AB}}(k)$, with equivalent relationships for overall prevalences.

Let $\langle\cdot\rangle$ denote the expectation operator. Given the distribution of degrees, $P(k)$, the probability that a randomly encountered node has degree k is $\tilde{P}(k) = \frac{P(k)k}{\langle k\rangle}$. Our assumption that both groups are identical except for informational

will not change if we instead assume that g_l is the probability that a single agent in state I_{AB} passes on information l. This assumption would not allow us to investigate questions of the effect of segregation according to information preferences.

[7] An intuitive example is a model in which the rate at which an agent communicates a meme depends on his type, such that an agent of type l communicates meme l at rate ν_l and meme $-l$ at rate $\nu_{\text{-}l}$ with $\nu_l > \nu_{\text{-}l}$. Given group sizes g_{A} and g_{B}, such a transmission protocol would be identical to a model in which meme A is being transmitted at rate $\nu_A = g_A\nu_l + g_B\nu_{\text{-}l}$ and meme B at rate $\nu_B = g_A\nu_{\text{-}l} + g_B\nu_l$ in a parallel fashion.

preferences extends to the distribution of degrees, i.e., $P(k) = P_{\mathrm{A}}(k) = P_{\mathrm{B}}(k)$. It also extends to interaction patterns across groups being random, i.e., agents of either group meet members of the same / the other group at a rate that is determined by the relative sizes of the groups. Currarini et al. (2009) term such interactions *baseline homophily*. While many social interactions exhibit a tendency towards homophily beyond this level, we believe that this assumption captures well the fact that social networks are likely to have formed long before any particular pair of memes diffuses on them. In this case, we do expect that the degree of homophily of the network is unrelated to the types of currently diffusing memes. Denote by θ_l the probability that a randomly encountered agent will transmit meme l for $l \in \{A, B\}$, if communication occurs. Then,

$$\theta_{\mathrm{A}} = \sum_k \tilde{P}(k)\left[\rho_{\mathrm{A}}(k) - g_{\mathrm{B}}\rho_{\mathrm{AB}}(k)\right], \tag{1}$$

$$\theta_{\mathrm{B}} = \sum_k \tilde{P}(k)\left[\rho_{\mathrm{B}}(k) - g_{\mathrm{A}}\rho_{\mathrm{AB}}(k)\right]. \tag{2}$$

In the SIS model with only one meme, θ_l is equal to the probability that a randomly encountered node is informed of meme l. Equations (1) and (2), instead, show that in the present model, for $g_l \in (0,1)$ and $\rho_{\mathrm{AB}}(k) > 0$, θ_l is strictly lower than this probability. Thus, in our model the probability of becoming informed of either meme is lower than in the standard SIS model. Note that $\rho_{\mathrm{A}}(k)$, $\rho_{\mathrm{B}}(k)$, and $\rho_{\mathrm{AB}}(k)$ are the same in either group. This is an outcome of our assumption that interactions are random across groups and it leads to the result that a fraction g_l of informed agents (whether in $\rho_{\mathrm{A}}(k)$, $\rho_{\mathrm{B}}(k)$, or $\rho_{\mathrm{AB}}(k)$) belongs to group l, which increases the tractability of the model tremendously.[8]

Following the standard SIS model with a unique meme, we assume that the infection rate ν is sufficiently small that it approximates the chance that an agent becomes informed through his k independent interactions at t. The rate

[8] We return to this point in more detail in Section 5, where we look at the implications of different levels of homophily.

at which a susceptible agent becomes infected with either meme l is then $k\nu\theta_l$. Similarly, we assume that the recovery rate δ is sufficiently small such that δ approximates the probability that an agent forgets a particular meme at time t.[9]

As A and B are about independent topics, we assume that the two memes diffuse through the population independently of each other. Knowledge of one does not make knowledge of the other any more or less likely. The propagation process exhibits a steady-state if the following three differential equations are satisfied,

$$\frac{\partial \rho_{\mathrm{A}}(k)}{\partial t} = (1-\rho_{\mathrm{A}}(k))k\nu\theta_{\mathrm{A}} - \rho_{\mathrm{A}}(k)\delta = 0, \tag{3}$$

$$\frac{\partial \rho_{\mathrm{B}}(k)}{\partial t} = (1-\rho_{\mathrm{B}}(k))k\nu\theta_{\mathrm{B}} - \rho_{\mathrm{B}}(k)\delta = 0, \tag{4}$$

$$\begin{aligned}\frac{\partial \rho_{\mathrm{AB}}(k)}{\partial t} &= (\rho_{\mathrm{A}}(k) - \rho_{\mathrm{AB}}(k))k\nu\theta_{\mathrm{B}} + \\ &\quad + (\rho_{\mathrm{B}}(k) - \rho_{\mathrm{AB}}(k))k\nu\theta_{\mathrm{A}} - 2\rho_{\mathrm{AB}}(k)\delta = 0,\end{aligned} \tag{5}$$

i.e., the proportion of agents who become aware of a meme at t equals the proportion of agents who forget it.[10]

[9]In essence, this assumption implies that at most one information is forgotten at any t. This seems reasonable for short time intervals. Importantly, as at most one meme can be transmitted per meeting, it ensures that the setup is not exogenously biased against the survival of a meme.

[10]We assume that δ is the unique rate at which both A and B are forgotten. There are numerous alternative ways to model forgetting, e.g., the preferred meme might be forgotten at a lower rate, or being aware of multiple memes increases the rate at which all of them are forgotten. On the other hand, it might also be the complexity of a meme that is the determining factor in forgetting, something that is entirely exogenous to the model. In this case, the steady-state prevalences of the two memes would differ by construction. Instead, the unique value of δ allows us to derive very cleanly the impact that the existence of a second meme has on the diffusion process, without additional complications.

3.2. Steady-States

Define $\lambda = \frac{\nu}{\delta}$ as the *diffusion rate* of information. The steady-state conditions of $\rho_{\rm A}(k)$, $\rho_{\rm B}(k)$, and $\rho_{\rm AB}(k)$ can be written as

$$\rho_{\rm A}(k) = \frac{k\lambda\theta_{\rm A}}{1+k\lambda\theta_{\rm A}}, \quad (6)$$

$$\rho_{\rm B}(k) = \frac{k\lambda\theta_{\rm B}}{1+k\lambda\theta_{\rm B}}, \quad (7)$$

$$\rho_{\rm AB}(k) = \frac{k^2\lambda^2\theta_{\rm A}\theta_{\rm B}}{(1+k\lambda\theta_{\rm A})(1+k\lambda\theta_{\rm B})} = \rho_{\rm A}(k)\rho_{\rm B}(k), \quad (8)$$

and substitution of these conditions into equations (1) and (2) yields:

$$H^A(\theta_{\rm A},\theta_{\rm B}) = \sum_k \tilde{P}(k)\frac{k\lambda\theta_{\rm A}}{1+k\lambda\theta_{\rm A}}\left[1-g_{\rm B}\frac{k\lambda\theta_{\rm B}}{1+k\lambda\theta_{\rm B}}\right], \quad (9)$$

$$H^B(\theta_{\rm A},\theta_{\rm B}) = \sum_k \tilde{P}(k)\frac{k\lambda\theta_{\rm B}}{1+k\lambda\theta_{\rm B}}\left[1-g_{\rm A}\frac{k\lambda\theta_{\rm A}}{1+k\lambda\theta_{\rm A}}\right]. \quad (10)$$

Denote the steady-state values of $\theta_{\rm A}$ and $\theta_{\rm B}$ as $\bar{\theta}_{\rm A}$ and $\bar{\theta}_{\rm B}$, respectively. These are determined as the fixed points such that $\bar{\theta}_{\rm A} = H^A(\bar{\theta}_{\rm A},\bar{\theta}_{\rm B})$ and $\bar{\theta}_{\rm B} = H^B(\bar{\theta}_{\rm A},\bar{\theta}_{\rm B})$, which by equations (6)-(8) determine the steady-states of $\rho_l(k)$ (and hence ρ_l) for $l \in \{A, B\}$, which we denote by $\bar{\rho}_l(k)$ and $\bar{\rho}$. Due to the inherent symmetry of the model, in the remainder of the paper we focus, without loss of generality, on the case in which $g_{\rm A} \geq g_{\rm B}$.

Remark 1. *For any given diffusion rate* $\lambda \geq 0$*, there exists a steady-state in which* $\bar{\theta}_l = \bar{\rho}_l(k) = \bar{\rho}_l = 0$ *for either or both* $l \in \{A, B\}$.

The existence of a steady-state in which nobody is informed is trivial. If the initial conditions are such that no agent is informed of a meme, nobody ever will be. Questions of interest concern the existence of a steady-state in which $\bar{\rho}_l > 0$ for at least one $l \in \{A, B\}$ and its characteristics. Henceforth, we will denote, with slight abuse of notation, by $\bar{\rho}_l(k)$ and $\bar{\rho}_l$ the positive steady-state values of meme prevalence, and by $\bar{\theta}_l$ the positive steady-state of θ_l for either $l \in \{A, B\}$.

3.3. Existence of Non-zero Steady-States

To analyze the existence of steady-states in which $\bar{\rho}_l > 0$ for either or both $l \in \{A, B\}$, we adapt the following definition from López-Pintado (2008).

Definition 1. *For each $l \in \{A, B\}$, let λ_l^d be such that the following two conditions are satisfied:*

(i) For all $\lambda > \lambda_l^d$, there exists a positive steady-state for meme l, i.e., a steady-state in which a strictly positive fraction of the population is informed about it, while it does not exist for $\lambda \leq \lambda_l^d$.

(ii) For all $\lambda > \lambda_l^d$, starting from an infinitesimally small fraction of agents informed about l, the dynamics converge to a positive steady-state, while for $\lambda \leq \lambda_l^d$, they converge to a steady-state in which no agent is informed about l.

We call λ_l^d the diffusion threshold of meme l.[11]

Furthermore, we are interested in how the diffusion threshold and the prevalence of either meme compare to the case in which meme l is the unique meme diffusing on the network. We therefore define the following concepts.

Definition 2. *Let λ^d be the diffusion threshold in case a unique meme diffuses through the network.*

Definition 3. *Let $\tilde{\rho}$ denote the positive steady-state prevalence of a meme if it is the unique meme that diffuses through the network, with corresponding $\tilde{\theta}$ and $\tilde{\rho}(k)$.*

For the present diffusion process, it has been established[12] that $\lambda^d = \frac{\langle k \rangle}{\langle k^2 \rangle}$. We are now in a position to state our first set of results regarding the existence and stability of positive steady-states for either meme $l \in \{A, B\}$.

[11] López-Pintado (2008) in fact defines a *critical threshold* above which a positive steady-state exists, and a *diffusion threshold* above which a positive steady-state is stable. In the present setting, these two thresholds always coincide.

[12] See, e.g., Jackson (2008) or López-Pintado (2008) and the references therein.

Theorem 1. *The existence and magnitude of a diffusion threshold* λ_l^d *are as follows:*

(i) If $g_l \in (0,1)$ *for both* $l \in \{A, B\}$*, a diffusion threshold exists and is given by* $\lambda_{\text{A}}^d = \lambda_{\text{B}}^d = \lambda^d = \frac{\langle k \rangle}{\langle k^2 \rangle}$*.*

(ii) If $g_l = 1$ *for either* $l \in \{A, B\}$*, a diffusion threshold exists and is given by* $\lambda_l^d = \lambda^d = \frac{\langle k \rangle}{\langle k^2 \rangle}$*. For* $\lambda > \lambda_l^d$*,* $\bar{\theta}_l = \tilde{\theta}$*,* $\bar{\rho}(k) = \tilde{\rho}(k)$*, and* $\bar{\rho}_l = \tilde{\rho}$*. There does not exist a diffusion threshold for the composite meme,* $-l$*.*

Independent of the value of g_l*, for either* $l \in \{A, B\}$ *there exists at most one steady-state in which* $\bar{\theta}_l > 0$*, and therefore* $\bar{\rho}_l > 0$*.*

Proof. See Appendix A. □

Our result that λ_l^d is identical to λ^d for all interior values of g_l highlights an enormous resilience of information. Any combination of network structure $P(k)$ and λ that is sufficient to make a unique meme endemic is also sufficient to make multiple memes endemic, provided that any interest exists in the population. This is despite the fact that, as evidenced by equations (1) and (2), each individual meme is transmitted at a strictly lower rate than ν, the rate in the standard SIS model. This suggests that λ_l^d should be larger than λ^d for at least one l. Briefly, the intuition as to why this is not the case is that the negative externality the two memes impose on each other is determined through $\rho_{\text{AB}}(k)$, which approaches zero faster as λ approaches λ^d from above than either $\rho_{\text{A}}(k)$ or $\rho_{\text{B}}(k)$.

It is noteworthy that the results of Theorem 1 are the exact opposite of the results in Prakash et al. (2012). They focus on the case of competing viruses which endow the host with immunity against each other. In such a framework, only the more infectious virus survives in the population. While such immunity seems a realistic feature for some viruses or product adoption, it appears less intuitive for the diffusion of information. In comparison with previously established results, Theorem 1 highlights that the exact nature of competition between the diffusing states is crucial in determining the likelihood of their survival.

Theorem 1 provides a possible argument for why such vast amounts of different topics are being discussed both offline and on OSNs, such as Twitter. Assume that $\lambda > \lambda^d$. Then any meme that is deemed the most interesting by a positive fraction of the population will survive on this network, no matter how much of a niche topic it might be. Note that survival of a meme does not necessarily imply that a sizeable fraction of the population will be informed of it. We now turn to the magnitude of meme prevalences.

4. Information Prevalence and Network Structure

4.1. Relative Information Prevalence

Theorem 1 shows that the predictions of our model are in line with the observation that many memes survive simultaneously in a population. The second aspect of communication that has been highlighted is that memes' prevalences differ. Ultimately, the prevalence of information l is determined by $\bar{\theta}_l$, which is the steady-state rate at which l is talked about. In general, it is not possible to explicitly solve for $\bar{\theta}_l$. Nevertheless, there are a number of positive results that can be derived regarding relative meme prevalence. In the remainder of the paper we focus on the case in which $\lambda > \lambda^d$ and $\bar{\theta}_l > 0$ for both $l \in \{A, B\}$.[13] We also restrict ourselves to consider only finite values of λ. We refer to meme l as the "majority" meme if $g_l > 1/2$.

Proposition 1. *Consider a given degree distribution P, $\lambda > \lambda^d$ and $g_l \in (0, 1)$. Then, $\bar{\theta}_{\mathrm{A}} = \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}}\bar{\theta}_{\mathrm{B}}$. That is, $\bar{\theta}_{\mathrm{A}} \geq \bar{\theta}_{\mathrm{B}}$, $\bar{\rho}_{\mathrm{A}}(k) \geq \bar{\rho}_{\mathrm{B}}(k)$, and $\bar{\rho}_{\mathrm{A}} \geq \bar{\rho}_{\mathrm{B}}$ if and only if $g_{\mathrm{A}} \geq g_{\mathrm{B}}$. Inequalities are strict if $g_{\mathrm{A}} > g_{\mathrm{B}}$, and in the case of $p_l(k)$, if k is finite.*

Proof. The relation between $\bar{\theta}_{\mathrm{A}}$ and $\bar{\theta}_{\mathrm{B}}$ is proven in Lemma 3 in Appendix A. The ranking of steady-states follows directly from this relation and the fact that

[13] If we set $\bar{\theta}_l = 0$ for either $l \in \{A, B\}$, the composite meme $-l$ will uniquely diffuse on the network and the results of López-Pintado (2008) apply.

$\bar{\rho}_l(k)$ is strictly increasing in $\bar{\theta}_l$, while $\bar{\rho}_l$ is increasing in $\bar{\rho}_l(k)$. $\square$

Proposition 1 is independent of the type of degree distribution P, and independent of the diffusion rate λ, or degree k.[14] It shows that relative interest uniquely determines which meme exhibits a higher prevalence in the long run, independent of any parameters of the diffusion process, including the network structure. A driving force behind this result is the fact that random interactions imply that *all* agents become informed about meme l at a higher rate than meme $-l$ if and only if $g_l > g_{\text{-l}}$. We show in Section 6 that the importance of relative exposure in prevalence rankings remains intact also in environments where groups are less symmetric. Proposition 1 also shows that relative communication rates, which is what is observed in data such as Twitter, are determined entirely through relative interest. This is not true for relative meme prevalences.

Proposition 2. *For each $l \in \{A, B\}$ and $g_l \in (0, 1)$, $\bar{\theta}_l$, $\bar{\rho}_l(k)$, and $\bar{\rho}_l$ are strictly increasing in λ. Furthermore, for $g_{\text{A}} > g_{\text{B}}$:*

(i) $\frac{\bar{\rho}_{\text{A}}(k)}{\bar{\rho}_{\text{B}}(k)}$ *and* $\frac{\bar{\rho}_{\text{A}}}{\bar{\rho}_{\text{B}}}$ *are strictly decreasing in λ.*

(ii) $\frac{\bar{\rho}_{\text{A}}(k)}{\bar{\rho}_{\text{B}}(k)}$ *is strictly decreasing in k.*

Proof. See Appendix B. $\square$

The fact that all steady-state measures ($\bar{\theta}_l$, $\bar{\rho}_l(k)$, and $\bar{\rho}_l$), are increasing in λ is unsurprising, and in line with the one-meme model. The decrease in prevalence ratios in both k and λ is driven by an increased importance of $\bar{\rho}_{\text{AB}}(k)$ in both memes' prevalences. Through it, any increase in the diffusion rate λ will increase the prevalence of the minority meme relatively more. We now turn to investigate how changes in the degree distribution P affect relative meme prevalence.

[14] Where we specify finite k for strict inequalities, it is due to the fact that an agent who meets every other agent at t will become informed of both memes for sure.

4.2. Stochastic Dominance and Relative Prevalence

To analyze the impact of the network structure, we focus on the effect of a change in the degree distribution in the sense of first order stochastic dominance.[15] In particular, the degree distribution P' first order stochastically dominates the distribution P if $\sum_{k=0}^{Y} P'(k) \leq \sum_{k=0}^{Y} P(k)$ for all Y with strict inequality for some Y.

Proposition 3. *Let P' and $\tilde{P}'$ first order stochastically dominate P and $\tilde{P}$ respectively. Let $g_l \in (0,1)$ for both $l \in \{A, B\}$ and $\lambda > \lambda^d$. Then,*

(i) $\bar{\theta}'_l > \bar{\theta}_l$, $\bar{\rho}'_l(k) > \bar{\rho}_l(k)$, and $\bar{\rho}'_l > \bar{\rho}_l$ for each $l \in \{A, B\}$.

(ii) $\frac{\bar{\rho}'_{\mathrm{A}}(k)}{\bar{\rho}'_{\mathrm{B}}(k)} < \frac{\bar{\rho}_{\mathrm{A}}(k)}{\bar{\rho}_{\mathrm{B}}(k)}$ and $\frac{\bar{\rho}'_{\mathrm{A}}}{\bar{\rho}'_{\mathrm{B}}} < \frac{\bar{\rho}_{\mathrm{A}}}{\bar{\rho}_{\mathrm{B}}}$ if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$.

Proof. See Appendix C. □

Both Proposition 2 and 3 show that any form of improvements in the transmission of information are relatively more important for memes that are, *ex ante*, less likely to be transmitted. Compared to communication offline, OSNs might be characterized by an increased λ or indeed a first order stochastic dominant shift in P (allow agents to have more meetings). If so, our results predict that increased online communication disproportionally benefits the prevalence of minority memes.

4.3. Crowding Out of Information

The third aspect of information diffusion that is highlighted in the Twitter data of Leskovec et al. (2009) is the fact that hashtags crowd each other out, a fact that lies at the heart of the prospect of "burying" news. This is a direct

[15] We focus on first order stochastic dominance as it appears a change that might naturally occur with an increasing importance of online communication. Another change in the degree distribution investigated by, e.g., Jackson and Rogers (2007b) would be a mean-preserving spread. We omit this comparison here for two reasons. First, it seems a case less related to the rise in OSN's. Second, Jackson and Rogers (2007b) show that its impact depends on the value of λ. As the value of λ above which a mean-preserving spread increases prevalence cannot be explicitly solved for, our analysis cannot go beyond re-iterating the results of Jackson and Rogers (2007b) for both memes individually.

consequence of the fact that overall communication stays roughly constant, while hashtags are retweeted at different rates.

In the context of our model, an intuitive measure of crowding out is the difference between a meme's prevalence without competition from another meme and the one with it, i.e., $\tilde{\rho} - \bar{\rho}$. Without solving explicitly for $\bar{\theta}_l$, the exact level of crowding out cannot be determined. We can, however, establish its existence, and find boundaries on the relation between $\bar{\theta}_l$ and $\tilde{\theta}$.

Proposition 4. *For any P, $\lambda > \lambda^d$ and $g_l \in (0,1)$, crowding out is positive:*

$$\bar{\theta}_l \in \left(g_l \tilde{\theta} \ , \ \frac{g_l}{1 - g_l g_{\text{-}l}} \tilde{\theta} \right).$$

As $\tilde{\theta} > \bar{\theta}_l$, it follows that $\tilde{\rho}(k) > \bar{\rho}_l(k)$ and $\tilde{\rho} > \bar{\rho}_l$.

Proof. See Appendix D. □

While it is clear from the bounds on $\bar{\theta}_l$ that g_l is a significant determinant in crowding out, its exact value depends on the degree distribution and λ in non-obvious ways. Its extent can neither be ranked according to FOSD of degree distributions, nor according to λ for a given degree distribution. Nevertheless, the derived bounds show that it can be extensive.

5. Segregation and Integration

5.1. Information Prevalence under Segregation

In the preceding analysis, agents of groups A and B interact randomly with each other, irrespective of group membership. Currarini et al. (2009) show that homophily patterns in the data appear to go beyond random interactions, a concept they term *inbreeding homophily.*[16]

To incorporate homophily into our model, we denote by $\beta_l \in [0,1]$ the probability that an agent belonging to group l will meet another agent from the

[16] One of the earliest work on homophily in general is Lazarsfeld et al. (1954). See also the survey by McPherson et al. (2001).

same group. This parameter captures the level of homophily of group l. When $\beta_l = 1$, all meetings are within the same group, and at $\beta_l = 0$ all meetings are with members of the opposite group. We recover random interactions by setting $\beta_l = g_l$. Interactions are homophilous if $\beta_l > g_l$. To ensure that meetings across groups are well defined, we require that

$$g_{\mathrm{A}}(1-\beta_{\mathrm{A}}) = g_{\mathrm{B}}(1-\beta_{\mathrm{B}}),$$

i.e., the fraction of the population that is of type A and meets type B agents is the same as the fraction who is type B and meets type A.

By allowing $\beta_l \neq g_l$, we have to keep track of how the prevalence of meme l evolves in group l. We denote the prevalence of meme l among degree-k agents in group l by $\rho_l^l(k)$, where subscript l denotes *meme* l and superscript denotes *group* l. Overall prevalences ρ_l^l and the probability of a type l agent transmitting meme l when communication takes place (θ_l^l) are defined analogously. Any steady-state of the system now has to be a solution to the following equations:

$$\rho_{\mathrm{A}}^{\mathrm{A}}(k) = \frac{\lambda k \theta_{\mathrm{A}}^{\mathrm{A}}}{1+\lambda k \theta_{\mathrm{A}}^{\mathrm{A}}}, \tag{11}$$

$$\rho_{\mathrm{A}}^{\mathrm{B}}(k) = \frac{\lambda k \theta_{\mathrm{A}}^{\mathrm{B}}}{1+\lambda k \theta_{\mathrm{A}}^{\mathrm{B}}}, \tag{12}$$

$$\rho_{\mathrm{B}}^{\mathrm{A}}(k) = \frac{\lambda k \theta_{\mathrm{B}}^{\mathrm{A}}}{1+\lambda k \theta_{\mathrm{B}}^{\mathrm{A}}}, \tag{13}$$

$$\rho_{\mathrm{B}}^{\mathrm{B}}(k) = \frac{\lambda k \theta_{\mathrm{B}}^{\mathrm{B}}}{1+\lambda k \theta_{\mathrm{B}}^{\mathrm{B}}}, \tag{14}$$

with

$$\theta_{\mathrm{A}}^{\mathrm{A}} = \sum_k \tilde{P}(k)\left[\beta_{\mathrm{A}}\rho_{\mathrm{A}}^{\mathrm{A}}(k) + (1-\beta_{\mathrm{A}})\rho_{\mathrm{A}}^{\mathrm{B}}(k)[1-\rho_{\mathrm{B}}^{\mathrm{B}}(k)]\right], \tag{15}$$

$$\theta_{\mathrm{A}}^{\mathrm{B}} = \sum_k \tilde{P}(k)\left[\beta_{\mathrm{B}}\rho_{\mathrm{A}}^{\mathrm{B}}(k)[1-\rho_{\mathrm{B}}^{\mathrm{B}}(k)] + (1-\beta_{\mathrm{B}})\rho_{\mathrm{A}}^{\mathrm{A}}(k)\right], \tag{16}$$

$$\theta_{\mathrm{B}}^{\mathrm{A}} = \sum_k \tilde{P}(k)\left[\beta_{\mathrm{A}}\rho_{\mathrm{B}}^{\mathrm{A}}(k)[1-\rho_{\mathrm{A}}^{\mathrm{A}}(k)] + (1-\beta_{\mathrm{A}})\rho_{\mathrm{B}}^{\mathrm{B}}(k)\right], \tag{17}$$

$$\theta_{\mathrm{B}}^{\mathrm{B}} = \sum_k \tilde{P}(k)\left[\beta_{\mathrm{B}}\rho_{\mathrm{B}}^{\mathrm{B}}(k) + (1-\beta_{\mathrm{B}})\rho_{\mathrm{B}}^{\mathrm{A}}(k)[1-\rho_{\mathrm{A}}^{\mathrm{A}}(k)]\right]. \tag{18}$$

Note that by setting $\beta_l = g_l$, we recover our original model. Furthermore, it is apparent that β_l only matters for prevalence as long as both memes diffuse simultaneously. By setting, e.g., $\theta^{\mathrm{A}}_{\mathrm{B}} = \theta^{\mathrm{B}}_{\mathrm{B}} = 0$, we see that $\theta^{\mathrm{A}}_{\mathrm{A}} = \theta^{\mathrm{B}}_{\mathrm{A}} = \tilde{\theta}$ is a steady-state of the system for any value of β_l. Similarly, there always exists a steady-state in which the prevalence of both memes is zero. Without restricting either β_l further, we are able to derive the following result.

Lemma 1. *Let $g_{\mathrm{A}} \geq g_{\mathrm{B}}$ and the degree of homophily be given by $\beta_l \in [0,1]$. For all $\lambda \leq \frac{\langle k \rangle}{\langle k^2 \rangle}$ the steady-state in which no agent is informed about either meme l is asymptotically stable. It is unstable for all $\lambda > \frac{\langle k \rangle}{\langle k^2 \rangle}$.*

Proof. See Appendix E. □

While Lemma 1 stops short of proving the existence or the stability of one (or more) positive steady-state(s) for the system, it does establish that information survival itself is independent of the level of homophily whenever some degree of interaction takes place between groups. If $\beta = 1$ the population is split into two separate groups, each of which homogeneously prefers one meme. Theorem 1 has established the diffusion threshold in this scenario, and it follows in particular that under full segregation, in each group only the preferred meme will survive. The implications of segregation are thus stark: Independent of the amount of initial media coverage (i.e., the "seed" of meme l), the degree distribution P, or the diffusion rate λ, meme B will never exhibit a positive steady-state in group A and *vice versa*. Our next Theorem establishes that this result is specific to the case of full segregation.

Theorem 2. *Let $\lambda > \lambda^d$ and $g_{\mathrm{A}} \geq g_{\mathrm{B}}$. For any degree of homophily such that $\beta_l \in [0,1)$ for both $l \in \{0,1\}$, it is the case that if meme l exhibits a positive prevalence in group l, it also exhibits a positive prevalence in group $-l$.*

Proof. See Appendix F. □

Theorem 2 mirrors qualitatively the main result of Theorem 1. As long as some interaction between groups occurs, information will survive either in both groups, or in neither.

The corner case of full segregation can be employed for further analysis. It serves as a useful benchmark against which to compare the case of random interacions. In what follows, we refer to a *segregated* society as one in which $\beta_{\text{A}} = \beta_{\text{B}} = 1$, and to an *integrated* society as one in which $\beta_l = g_l$. Our next result establishes that the prevalence of *either* meme is lower in a segregated society than in an integrated one.

Theorem 3. *For $g_l \in (0,1)$ and $\lambda > \lambda^d$, the prevalence of meme l is $\bar{\rho}_l$ in an integrated society and it is $g_l\tilde{\rho}$ in a segregated society. The following holds:*

(i) *$\bar{\rho}_l > g_l\tilde{\rho}_l$; information prevalence is higher in an integrated society. The information loss due to segregation is larger for meme A than meme B if and only if $g_{\text{A}} > g_{\text{B}}$.*

(ii) *$g_l\bar{\rho}_l < g_l\tilde{\rho}$; the proportion of the population informed about their preferred meme is higher in a segregated society.*

Proof. The second point is immediate as $\tilde{\rho} > \bar{\rho}_l$. The inequality and ranking of information loss established in the first point are derived in Appendix G. □

To the best of our knowledge, the result that segregation can lead to a decrease in total prevalence is novel in the literature. It goes beyond the polarizing impact of having no agent informed about both A and B in the long run. Indeed, if memes A and B are entirely unrelated, there might not be perceivable benefits of being informed about both simultaneously. Nevertheless, even if segregation does not lead to polarization, it has an impact on information. This impact falls disproportionally on the prevalence of the majority meme, thus segregation reduces particularly the steady-state prevalence of information that might be considered mainstream.

The distinction between overall meme prevalence and meme prevalence within each group is also noteworthy. If, e.g., A is a piece of celebrity gossip and B a piece of political news, the value that agents in group A put on being informed about B (and *vice versa*) might be limited. That is, while overall information is lost due to segregation, it increases prevalence of memes among those that attach a higher value to it. This leads us to question under which conditions agents themselves prefer a segregated society to an integrated one, which we address now.

5.2. *Welfare under Segregation and Integration*

While segregation reduces information prevalence and leads to polarization, the model so far is too general to allow for a concrete welfare analysis. To this end, we impose additional structure on the utility agents gain from being informed. To keep the analysis as tractable as possible, we assume that agents derive utility directly from being informed about memes A and/or B. We assume that an agent in group l receives a flow utility of h while he is informed about meme l and a flow utility of s while he is informed about meme $-l$, where $h \geq s \geq 0$. Such utility flows could arise if agents truly value information in itself, but also if they value it because there is the possibility that it will be useful at an uncertain, future, date.[17] Agents then care about $\bar{\rho}_l(k)$, which is the time that an agent of degree k spends being informed about l in steady-state. We also assume that agents care only about the steady-state values of $\bar{\rho}_l(k)$ and $\bar{\rho}_{\text{-}l}(k)$. As the case of zero prevalences is uninformative, we focus again on the case where $\lambda > \lambda^d$. The utility of an agent with degree k in group l in an integrated and a segregated society is then

$$
\begin{aligned}
U(k)_{l|int} &= h\bar{\rho}_l(k) + s\bar{\rho}_{\text{-}l}(k), \quad \text{and} && (19)\\
U(k)_{l|seg} &= h\tilde{\rho}(k). && (20)
\end{aligned}
$$

Following these utilities and the results of Theorem 3, it is immediate that in a society in which all agents place positive utility only on one meme, i.e.,

[17]It is, e.g., possible that agents might value to be informed not so much because it provides them with any benefit in itself, but because there is a chance that these topics might be discussed in their presence, and not being informed would brand them as ignorant. Alternatively, the information might pertain to the state of the world and an agent knows that at an uncertain point in the (distant) future he will have to take an action whose payoff depends on the state. In either case, the expected utility of an agent would be increasing in the amount of time he is informed, which is captured with our parsimonious utility function.

$h > 0$ and $s = 0$, all agents are better off in a segregated society. Conversely, if both memes are valued equally by all agents, $h = s > 0$, all agents are strictly better off in an integrated society. For all other values of h and s, an agent of group l and degree k is strictly better off in a segregated society if and only if

$$\frac{s}{h} < \frac{\tilde{\rho}(k) - \bar{\rho}_l(k)}{\bar{\rho}_{\text{-}1}(k)}. \tag{21}$$

This formalizes the intuitive idea that segregation is more likely to be beneficial to agents who have more extreme preferences for the two memes. The smaller the ratio $\frac{s}{h}$, the more likely it is that an agent with degree k and of group l will prefer a segregated society to an integrated one. The exact value of $\frac{s}{h}$ at which agents are indifferent between segregation and integration depends on the values of k, g_l, λ, and P. Let $m_l(k) \equiv \frac{\tilde{\rho}(k) - \bar{\rho}_l(k)}{\bar{\rho}_{\text{-}1}(k)}$. The larger $m_l(k)$, the broader is the range of $\frac{s}{h}$ for which an agent prefers a segregated society. The effects of λ and P on $m_l(k)$ are interrelated and no general statements can be made about them.[18] However, we are able to state the following positive results.

Theorem 4. *For all $\lambda > \lambda^d$,*

- *$m_l(k)$ is decreasing in k for each $l \in \{A, B\}$. The higher an agents' degree, the broader is the range of $\frac{s}{h}$ for which he prefers an integrated society.*
- *$m_{\text{A}}(k) < m_{\text{B}}(k)$ if and only if $g_{\text{A}} > g_{\text{B}}$. Conditional on degree, an agent that belongs to the minority group prefers an integrated society for a smaller range of $\frac{s}{h}$.*

Proof. See Appendix H. □

Theorem 4 shows that for a fixed value of $\frac{s}{h}$, segregation is always more likely to be preferred by members of the minority group, and/or by agents that

[18] Although it is noteworthy that there is a general tendency for λ to decrease $m_l(k)$, i.e., a tendency for increases in information transmission to increase agents' welfare of integration, relative to segregation.

have fewer meetings per period. With the exception of a regular network, in which all agents have the same degree, segregation must not be unanimously preferred by all members of a group. Furthermore, unless $g_{\text{A}} = g_{\text{B}}$, it is possible for agents of one group (the minority group) to prefer segregation while similar agents of the majority group prefer an integrated society.

5.3. Endogenous Segregation

We now turn to the question of the emergence of a segregated or integrated society. In particular, we ask under which conditions either type of society is a (strict) Nash equilibrium, if agents can choose whether to interact randomly across types, or to restrict their interactions to members of their own group.[19]

In a society in which all interactions are governed by random interactions, an agent of group l and degree k obtains a utility given by equation (19). If he deviates and interacts only with members of his own group, this deviation will have no impact on overall meme prevalence. By deviating, the agent becomes informed of meme l at rate $\hat{\theta}_l = \sum_k \tilde{P}(k)\bar{\rho}_l(k)$ at a meeting, and of meme $-l$ at rate $\hat{\theta}_{\text{-l}} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{-l}}(k)[1 - \bar{\rho}_l(k)]$ (both conditional on communicating).[20] Therefore, his prevalence of meme l will be $\hat{\rho}_l(k) = \frac{k\lambda\hat{\theta}_l}{1+k\lambda\hat{\theta}_l}$ and that of meme $-l$ will be $\hat{\rho}_{\text{-l}}(k) = \frac{k\lambda\hat{\theta}_{\text{-l}}}{1+k\lambda\hat{\theta}_{\text{-l}}}$. This implies that random interactions are a strict (and stable) Nash equilibrium if for both l and all k

$$\begin{aligned} h\hat{\rho}_l(k) + s\hat{\rho}_{\text{-l}}(k) &< h\rho_l(k) + s\rho_{\text{-l}}(k) \quad \Rightarrow \\ \frac{s}{h} &> \frac{\hat{\rho}_l(k) - \rho_l(k)}{\rho_{\text{-l}}(k) - \hat{\rho}_{\text{-l}}(k)} \equiv n_l(k). \end{aligned} \tag{22}$$

If, on the other hand, $\frac{s}{h} < n_l(k)$ for at least some k, random interactions are not an equilibrium.

[19]Given the lack of positive results for more general levels of homophily, we restrict the action set of agents to a choice between full integration and full segregation. While this is obviously restrictive, our results provide useful insights into whenever either interaction pattern is *not* an equilibrium, which translate to richer action sets.

[20]As the agent only meets members of group l, everybody who is informed of l will pass this on, while only those who are only aware of $-l$ will pass meme $-l$ on.

In contrast, the utility an agent receives in a fully segregated society is given by (20). If he deviates from this pattern and interacts randomly, he will meet another member of group l at rate g_l, and a member of group $-l$ at rate $g_{\text{-}1}$. In the first case, conditional on communication, he will become informed of meme l at rate $\tilde{\theta}$ and will never hear meme $-l$. In the second, he will only ever be told of meme $-l$, again at rate $\tilde{\theta}$. Which implies that his prevalences when interacting randomly with others will be $\check{\rho}_l(k) = \frac{g_l k\lambda\tilde{\theta}}{1+g_l k\lambda\tilde{\theta}}$ and $\check{\rho}_{\text{-}1}(k) = \frac{g_{\text{-}1} k\lambda\tilde{\theta}}{1+g_{\text{-}1} k\lambda\tilde{\theta}}$ respectively. Segregation is a strict (and stable) Nash equilibrium if for all k in at least one group

$$\begin{aligned} h\tilde{\rho}(k) &> h\check{\rho}_l(k) + s\check{\rho}_{\text{-}1}(k) \quad \Rightarrow \\ \frac{s}{h} &< \frac{1 + g_{\text{-}1}k\lambda\tilde{\theta}}{(1+k\lambda\tilde{\theta})(1+g_l k\lambda\tilde{\theta})} \equiv \eta_l(k). \end{aligned} \tag{23}$$

These results allow us various insights into the occurence of either an integrated or a segregated society.

Proposition 5. *For all $\lambda > \lambda^d$,*

- *$n_{\text{A}}(k) < n_{\text{B}}(k)$ and $\eta_{\text{A}} < \eta_{\text{B}}$ if $g_{\text{A}} > g_{\text{B}}$, i.e., ceteris paribus, segregation is more likely to be Nach equilibrium if group sizes are heterogenous.*
- *for both l, $\eta_l(k)$ is strictly decreasing in k, λ, and a FOSD shift of the degree distribution.*
- *if segregation is preferred to integration, i.e., $\frac{s}{h} < m_l(k)$, for at least some k of either l, integration is not a Nash equilibrium. If $\frac{s}{h} < m_l(k)$ holds for all members of either l, a segregated society is a stable Nash equilibrium.*

Proof. See Appendix I. The ranking of $\eta_l(k)$ is immediate from equation (23). □

Lemmas 6 and 7, which are employed in the proof, establish that for any (finite) k, l, and λ, both $n_l(k) > m_l(k)$ and $\eta_l(k) > m_l(k)$. Thus, whenever a segregated society is preferred by all members of either group, segregation is the unique Nash equilibrium. The converse does not hold. If $m_l(k) < \frac{s}{h} < \min\{n_l(k), \eta_l(k)\}$ for all agents, an integrated society provides higher welfare to all, yet segregation remains the only Nash equilibrium.[21] It is similarly interesting that segregation is more likely to be the unique Nash equilibrium if group sizes are heterogeneous. In this sense, we expect segregation to be driven by the minority group.

Insofar as OSNs might create a possibility to segregate where previously none existed, our results of Section 5.1 indicate that increased popularity of these networks has the potential to polarize a society and reduce information prevalence. Section 5.2 establishes that segregation is more likely to be beneficial to agents that are (i) particularly interested in niche or very specialized pieces of information (small g_l), (ii) are extreme in their valuation of information (small $\frac{s}{h}$), and/or (iii) are comparatively "anti-social", in the sense that they have few meetings per period (small k). The results of the present Section show that preferences for segregation can be linked to the endogenous emergence of a segregated society. In particular, a segregated society is also more likely to emerge if $\frac{s}{h}$ is small, and it is the unique Nash equilibrium if it is preferred by all members of the minority group.

On the other hand, OSNs might do more than simply offer a possibility to segregate. They might facilitate communication or interactions, i.e., increase either λ, k, or lead to a FOSD shift in the distribution of meetings. Either increase makes segregation less likely as an equilibrium as it decreases $\eta_l(k)$. Note that our results pertaining to either integration or segregation *not* being an equilibrium are valid even if we allow for a richer action set of agents, including more general levels of homophily.[22]

[21] A conflict between stability and efficiency in network formation has been shown to arise in a variety of settings, see e.g., Jackson (2005) for an excellent overview. Masson et al. (2018) is another recent example, albeit in a different setting. Izquierdo et al. (2018) have found that homophily might be inefficiently high, reminiscent of our result regarding inefficient segregation.

[22] In fact, for the case of equal group sizes and considering all potential levels of homophily, we can show that random interactions are a Nash equilibrium if and only if $s = h$ for all agents. Thus, it seems likely that we should observe some form of biased interaction patterns.

6. Extensions

6.1. Degree Distributions Depending on Type

The benchmark model assumes that types are identical except for their information preferences. In the present Section, we relax this assumption and allow the two types to differ with respect to their degree distributions, such that $P_{\text{A}}(k) \neq P_{\text{B}}(k)$. We keep the assumptions that each meme is being communicated at rate ν and that only the preferred meme will be communicated by an agent aware of both memes.

With different degree distributions across types, we define $\hat{g}_l = \frac{g_l \langle k \rangle_l}{\langle k \rangle}$ where $\langle k \rangle$ is the average degree in the whole network and $\langle k \rangle_l$ the average degree in group l. Thus, $\hat{g}_l$ describes the likelihood that an agent of type l is being met at a meeting. A steady-state is determined by a fixed point of the system:

$$H^A(\theta_{\text{A}}, \theta_{\text{B}}) = \hat{g}_{\text{A}} \sum_k \tilde{P}_{\text{A}}(k) \frac{\lambda k \theta_{\text{A}}}{(1 + \lambda k \theta_{\text{A}})} + \hat{g}_{\text{B}} \sum_k \tilde{P}_{\text{B}}(k) \frac{\lambda k \theta_{\text{A}}}{(1 + \lambda k \theta_{\text{A}})(1 + \lambda k \theta_{\text{B}})}, \tag{24}$$

$$H^B(\theta_{\text{A}}, \theta_{\text{B}}) = \hat{g}_{\text{A}} \sum_k \tilde{P}_{\text{A}}(k) \frac{\lambda k \theta_{\text{A}}}{(1 + \lambda k \theta_{\text{A}})(1 + \lambda k \theta_{\text{B}})} + \hat{g}_{\text{B}} \sum_k \tilde{P}_{\text{B}}(k) \frac{\lambda k \theta_{\text{B}}}{(1 + \lambda k \theta_{\text{B}})}. \tag{25}$$

This allows us the following result.

Proposition 6. *The steady-state in which $\bar{\theta}_l = 0$ for both $l \in \{A, B\}$ is asymptotically stable if and only if $\lambda \leq \frac{\langle k \rangle}{g_{\text{A}} \langle k^2 \rangle_{\text{A}} + g_{\text{B}} \langle k^2 \rangle_{\text{B}}}$. For a steady-state to exist in which $\bar{\theta}_{\text{A}} > 0$ and $\bar{\theta}_{\text{B}} > 0$, it is necessary that $g_l \in (0, 1)$.*

Proof. The proof of Proposition 6 follows the steps outlined in Appendix A. □

Proposition 6 highlights that our earlier results on the severe impact of segregation on information survival are robust to asymmetries in degree distributions. Similarly, the condition for asymptotic stability of the zero steady-state appears an intuitive generalization of our earlier results. Finally, we are able to show

The relevant derivations are available on request.

that it is no longer true that the meme preferred by the majority group will automatically exhibit a larger prevalence. Instead, in the present generalization the ratio $\bar{\theta}_{\mathrm{A}}/\bar{\theta}_{\mathrm{B}}$ depends on the three factors that influence the likelihood that an agent will be told meme l, namely group size, average degree per group, and the relative degree distributions.[23]

6.2. Information Transmission to Please Neighbors

Within our model, strategic information transmission to influence other agents' beliefs or actions is not likely. Each agent knows that they are too small to affect the overall prevalences of memes, and he is not affected by any decision his neighbors take based on the information he passes on. It is more likely that information transmission might be tailored to increase the utility of neighbors, i.e., it is conceivable that an agent who is aware of both memes passes on the preferred meme of his neighbor rather than his own. Under the assumption that types are common knowledge and an agent can send a different message to each of his neighbors, this change in assumption renders the model intractable. If types are unobserved, however, the present analysis continues to hold. In that case, an agent puts probability g_l (or β_l, in the case of homophily) that his neighbor is of type l, and the present model remains unchanged. Unobserved types might describe the reality of chat rooms where agents can form reasonable estimates of the likelihood that they will meet their own types, but not necessarily observe these directly. Similarly, the analysis would remain unchanged if an agent had to communicate the same meme to all his neighbors (like a tweet) and would choose the frequency with which to send meme l according to the composition of types in the neighborhood.

[23]The exact relationship is

$$\bar{\theta}_{\mathrm{A}} = \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}}\bar{\theta}_{\mathrm{B}} \cdot \frac{\langle k\rangle_{\mathrm{A}} \sum_k \tilde{P}_{\mathrm{A}}(k)\frac{(\lambda k)^2}{(1+\lambda k\bar{\theta}_{\mathrm{A}})(1+\lambda k\bar{\theta}_{\mathrm{B}})}}{\langle k\rangle_{\mathrm{B}} \sum_k \tilde{P}_{\mathrm{B}}(k)\frac{(\lambda k)^2}{(1+\lambda k\bar{\theta}_{\mathrm{A}})(1+\lambda k\bar{\theta}_{\mathrm{B}})}}.$$

7. Conclusion

The present paper introduces communication constraints into a standard SIS diffusion model: While two memes diffuse simultaneously on the network, at each meeting an agent can pass on at most one of these. The choice of which meme to pass on is driven by intrinsic preferences, and agents can be grouped according to which meme they prefer. In essence, the existence of communication constraints introduces opportunity costs in the diffusion process. To the best of our knowledge, communication costs of any type have not before been analyzed in a SIS framework.

We find that our parsimonious model is in line with stylized communication patterns found in Twitter data, such as differences in prevalences and crowding out of memes. Most importantly, our model predicts that information is resilient, in the sense that the conditions under which a unique meme exhibits a positive steady-state are identical to the conditions under which both memes exhibit positive steady-states. Thus, it is able to provide one possible rationalization for why so many different topics are discussed simultaneously online.

When we allow for segregated interactions among agents, we find that segregation leads to polarization, a loss of information overall, but an increase in the fraction of agents informed of their preferred meme. We extend our model by introducing explicit utility flows from being informed, which allows us to investigate the factors that drive segregation. We find that extremism of information preferences and low number of meetings increase the likelihood of segregation. The larger the size of the group that prefers a meme, the smaller are the incentives for agents of this group to segregate. In fact, we find that segregation is more likely (and integration less likely) to be a Nash equilibrium the more dispersed the group sizes are in the population.

We believe that our results relating to the impact (and the causes) of segregation are of particular interest when applied to the rise of Online Social Networks. Much information that diffuses on these is casual chit-chat, which we think is well captured by our model. Our results imply that for otherwise identical memes (with respect to, e.g., ν and δ) only the preferred meme will survive in a segregated group and there is an overall loss of information, both potential costs of segregation.

Our model is kept deliberately simple to highlight the impact of opportunity costs in the diffusion of information. There are a number of extensions that we believe would be promising areas of future research. One of these would be to consider the incentives of players exogenous to the network that provide the initial information "seed" of each meme. The uniqueness and stability of meme prevalences in the present model imply that there is no scope for strategic information seeding. Related to this is the question of how agents choose which information to communicate. In our model, memes are unrelated to each other and each agent is too insignificant to affect the steady-state prevalence of either meme. As such, linking the communication choice to intrinsic preferences appears a valid approximation of the choice process. However, if either of these conditions did not hold, strategic considerations will play a role in the transmission process. Alternatively, forgetting is a complex matter, and might depend on preferences, or the number of memes an agent has been exposed to. We believe that these are interesting aspects of the diffusion process that deserve closer attention.

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Appendix A. Proof of Theorem 1

Generically, the potential steady-states of memes A and B can be of three types: (i) a steady-state in which neither meme exhibits a positive prevalence, (ii) a steady-state in which $\bar{\theta}_l > 0$ for either $l = A$ or $l = B$, and zero for the other meme, and (iii) a steady-state in which $\bar{\theta}_l > 0$ for both $l \in \{A, B\}$. As highlighted in Remark 1, the case in which neither meme exhibits a positive prevalence trivially always exists. Based on this, the second type of steady-state also always exists. It is trivial to show that when setting $\theta_l = 0$ for either l, the model collapses to the one-meme model. Existence and magnitude of λ_l^d for the composite meme then follow immediately from the arguments in López-Pintado (2008). So does the fact that at most one steady-state exists in which $\bar{\rho}_l(k) > 0$.

We therefore focus on proving the existence and magnitude of λ_l^d for the case where $\theta_l > 0$ for both l. To do so, we employ four lemmas. Due to the symmetry of memes A and B, we can change the labels of the information to apply any arguments that we make about A also for B. For expositional simplicity, we will therefore focus on meme A in these lemmas, without loss of generality.

In steady-state, both equation (1) and (2) must be satisfied. Under the condition that $\theta_l > 0$ for both l, these can be re-arranged to show that at any positive steady-state, if one exists, the following equations must be satisfied,

$$1 = \sum_k \tilde{P}(k) \frac{k\lambda}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)} (1 + g_{\mathrm{A}} k\lambda\bar{\theta}_{\mathrm{B}}), \tag{A.1}$$

$$1 = \sum_k \tilde{P}(k) \frac{k\lambda}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)} (1 + g_{\mathrm{B}} k\lambda\bar{\theta}_{\mathrm{A}}). \tag{A.2}$$

First, Lemma 2 proves the the existence and magnitude of λ_l^d for $g_l = 1$, and that at most one steady-state exists in which $\bar{\theta} > 0$.

Lemma 2. *The arguments in López-Pintado (2008) are sufficient to show that for $g_l = 1$, $\bar{\theta}_l = \tilde{\theta} > 0$ exists if and only if $\lambda > \frac{\langle k \rangle}{\langle k^2 \rangle}$, that there exists only one steady-state in which $\bar{\theta}_l > 0$, and that this state is stable. For $g_l = 0$ there exists no diffusion threshold.*

Proof. It is immediate that if $g_{\text{A}} = 1$, the steady-state condition for $\bar{\theta}_{\text{A}}$ in equation (A.1) is identical to the condition when A is the only information on the network. López-Pintado (2008) has proven existence and stability of a positive steady-state in this case, as well as the fact that at most one positive steady-state exists.

It is easy to see that if $g_{\text{A}} = 1$ and $g_{\text{B}} = 0$, there is no $\bar{\theta}_{\text{B}} > 0$ that solves equation (A.2) and (A.1) simultaneously, independently of the value of λ. Consequently, for $g_l = 0$ there exists no diffusion threshold. $\square$

The following Lemma is a useful step in the derivation of steady-states when $g_l \in (0, 1)$.

Lemma 3. *For $g_l \in (0, 1)$, any steady-state such that $\bar{\theta}_l > 0$ for both l has the property that*

$$\bar{\theta}_{\text{A}} = \frac{g_{\text{A}}}{g_{\text{B}}}\bar{\theta}_{\text{B}}. \tag{A.3}$$

Proof. Equations (A.1) and (A.2) show that for any $g_{\text{A}} \in (0, 1)$ any steady-state has the property stated in equation (A.3), as this is the only condition under which both (A.1)and (A.2) hold simultaneously. $\square$

Lemma 4. *For $g_l \in (0, 1)$, a steady-state in which $\bar{\theta}_l > 0$ exists if and only if $\lambda > \frac{\langle k \rangle}{\langle k^2 \rangle}$. There exists at most one steady-state in which $\bar{\theta}_l > 0$.*

Proof. To prove Lemma 4, note that by Lemma 3, we can state that any potential pair of steady-states for θ_{A} and θ_{B}, which we denote $\acute{\theta}_{\text{A}}$ and $\acute{\theta}_{\text{B}}$ respectively, has the property that $\acute{\theta}_{\text{A}} = g_{\text{A}}/g_{\text{B}}\acute{\theta}_{\text{B}}$. We can employ this relationship to write the steady-state condition for θ_{A} as a function of $\acute{\theta}_{\text{A}}$.

$$H^A(\acute{\theta}_{\text{A}}) = \sum_k \tilde{P}(k)\frac{k\lambda\acute{\theta}_{\text{A}}}{1 + k\lambda\acute{\theta}_{\text{A}}}\frac{1 + g_{\text{B}}k\lambda\acute{\theta}_{\text{A}}}{1 + \frac{g_{\text{B}}}{g_{\text{A}}}k\lambda\acute{\theta}_{\text{A}}}. \tag{A.4}$$

Fixed points such that $H^A(\acute{\theta}_{\text{A}}) = \acute{\theta}_{\text{A}}$ correspond to the steady-state $\bar{\theta}_{\text{A}}$ that is consistent with $\bar{\theta}_{\text{B}}$. We follow the arguments put forward in Jackson and Rogers (2007b) and López-Pintado (2008) to show the existence of a fixed point in which $\bar{\theta}_{\text{A}} > 0$, and that at most one such point exists. First, note that

$$H^A(0) = 0, \tag{A.5}$$

$$H^A(1) = \sum_k \tilde{P}(k) \frac{k\lambda}{1+k\lambda} \frac{1+g_{\text{B}}k\lambda}{1+\frac{g_{\text{B}}}{g_{\text{A}}}k\lambda} < 1. \tag{A.6}$$

The second result is immediate since $\sum_k \tilde{P}(k) = 1$, and both factors that multiply $\tilde{P}(k)$ in $H^A(1)$ are less than 1 (strictly so if $g_{\text{A}} \in (0,1)$). Furthermore, taking first and second order derivatives of $H^A(\acute{\theta}_{\text{A}})$ with respect to $\acute{\theta}_{\text{A}}$ yields

$$H^{A'}(\acute{\theta}_{\text{A}}) = \sum_k \tilde{P}(k) \frac{k\lambda\left[1+2g_{\text{B}}k\lambda\acute{\theta}_{\text{A}}\right]}{(1+k\lambda\acute{\theta}_{\text{A}})^2(1+\frac{g_{\text{B}}}{g_{\text{A}}}k\lambda\acute{\theta}_{\text{A}})^2} > 0, \tag{A.7}$$

$$\begin{aligned} H^{A''}(\acute{\theta}_{\text{A}}) = \sum_k \tilde{P}(k) \Bigg\{ & -\frac{2k^2\lambda^2(1+2g_{\text{B}}k\lambda\acute{\theta}_{\text{A}})}{(1+k\lambda\acute{\theta}_{\text{A}})^3(1+\frac{g_{\text{B}}}{g_{\text{A}}}k\lambda\acute{\theta}_{\text{A}})^2} + \\ & + \frac{2k^2\lambda^2\left[g_{\text{B}} - \frac{g_{\text{B}}}{g_{\text{A}}} - \frac{g_{\text{B}}^2}{g_{\text{A}}}k\lambda\acute{\theta}_{\text{A}}\right]}{(1+k\lambda\acute{\theta}_{\text{A}})^2(1+\frac{g_{\text{B}}}{g_{\text{A}}}k\lambda\acute{\theta}_{\text{A}})^3} \Bigg\} < 0, \end{aligned} \tag{A.8}$$

i.e., $H^A(\acute{\theta}_{\text{A}})$ is strictly increasing and concave in $\acute{\theta}_{\text{A}}$. This implies that $\bar{\theta}_{\text{A}} > 0$ exists if $H^{A'}(0) > 1$, and that in this case, only one such positive point exists. In fact,

$$H^{A'}(0) = \sum_k \tilde{P}(k)k\lambda = \sum_k \frac{P(k)k^2\lambda}{\langle k\rangle} = \lambda\frac{\langle k^2\rangle}{\langle k\rangle} \tag{A.9}$$

which is larger than 1 if and only if $\lambda > \frac{\langle k\rangle}{\langle k^2\rangle}$, identical to the one-meme case. This completes the proof of existence of a possible positive steady-state $\bar{\theta}_l > 0$ for $l \in \{A, B\}$, and that at most one $\bar{\theta}_l > 0$ exists. □

In the one-meme case, concavity of $H(\theta)$ implies stability of the single positive steady-state as well as its existence. But since $H^A(\acute{\theta}_{\text{A}})$ is derived under the condition that $\acute{\theta}_{\text{A}} = \frac{g_{\text{A}}}{g_{\text{B}}}\acute{\theta}_{\text{B}}$, convergence to the steady-state does not follow from the above arguments. Instead, Lemma 5 is employed.

Lemma 5. *Whenever $\bar{\theta}_l > 0$ exists, it is asymptotically stable.*

Proof. We conduct the stability analysis through the eigenvalues of the Jacobian of the system

$$H^A(\theta_{\text{A}}, \theta_{\text{B}}) - \theta_{\text{A}} = 0, \tag{A.10}$$

$$H^B(\theta_{\text{A}}, \theta_{\text{B}}) - \theta_{\text{B}} = 0. \tag{A.11}$$

The entries of the Jacobian are,

$$\frac{\partial H^A}{\partial \theta_{\text{A}}} - 1 = \sum_k \tilde{P}(k) \frac{k\lambda}{(1 + k\lambda\theta_{\text{A}})^2} \left[1 - g_{\text{B}} \frac{k\lambda\theta_{\text{B}}}{1 + k\lambda\theta_{\text{B}}} \right] - 1, \tag{A.12}$$

$$\frac{\partial H^A}{\partial \theta_{\text{B}}} = -g_{\text{B}} \sum_k \tilde{P}(k) \frac{k^2\lambda^2\theta_{\text{A}}}{(1 + k\lambda\theta_{\text{A}})(1 + k\lambda\theta_{\text{B}})^2}, \tag{A.13}$$

$$\frac{\partial H^B}{\partial \theta_{\text{A}}} = -g_{\text{A}} \sum_k \tilde{P}(k) \frac{k^2\lambda^2\theta_{\text{B}}}{(1 + k\lambda\theta_{\text{A}})^2(1 + k\lambda\theta_{\text{B}})}, \tag{A.14}$$

$$\frac{\partial H^B}{\partial \theta_{\text{B}}} - 1 = \sum_k \tilde{P}(k) \frac{k\lambda}{(1 + k\lambda\theta_{\text{B}})^2} \left[1 - g_{\text{A}} \frac{k\lambda\theta_{\text{A}}}{1 + k\lambda\theta_{\text{A}}} \right] - 1. \tag{A.15}$$

At $\bar{\theta}_{\text{A}} = \bar{\theta}_{\text{B}} = 0$, the eigenvalues of the Jacobian are $\frac{\partial H^A}{\partial \theta_{\text{A}}} - 1$ and $\frac{\partial H^B}{\partial \theta_{\text{B}}} - 1$, both of which are equal to $\sum_k \tilde{P}(k) k\lambda - 1$. I.e., the zero steady-state is stable if $\lambda < \frac{\langle k \rangle}{\langle k^2 \rangle}$ and unstable if $\lambda > \frac{\langle k \rangle}{\langle k^2 \rangle}$.

At $\bar{\theta}_{\text{A}} > 0, \bar{\theta}_{\text{B}} = 0$, again the eigenvalues are $\frac{\partial H^A}{\partial \theta_{\text{A}}} - 1$ and $\frac{\partial H^B}{\partial \theta_{\text{B}}} - 1$. In this case,

$$\frac{\partial H^A}{\partial \theta_{\text{A}}} - 1 = \sum_k \tilde{P}(k) \frac{k\lambda}{(1 + k\lambda\bar{\theta}_{\text{A}})^2} - 1 < 0, \tag{A.16}$$

$$\frac{\partial H^B}{\partial \theta_{\text{B}}} - 1 = \sum_k \tilde{P}(k) \left[\frac{k\lambda}{1 + k\lambda\bar{\theta}_{\text{A}}} + g_{\text{B}} \frac{k^2\lambda^2\bar{\theta}_{\text{A}}}{1 + k\lambda\bar{\theta}_{\text{A}}} \right] - 1 > 0. \tag{A.17}$$

Since at $\bar{\theta}_{\text{A}} > 0, \bar{\theta}_{\text{B}} = 0$, it is the case that $\sum_k \tilde{P}(k) \frac{k\lambda}{1 + k\lambda\bar{\theta}_{\text{A}}} = 1$. I.e., this steady-state is unstable, too. Symmetry implies that the same argument applies for $\bar{\theta}_{\text{A}} = 0, \bar{\theta}_{\text{B}} > 0$.

Finally, for $\bar{\theta}_{\mathrm{A}} > 0, \bar{\theta}_{\mathrm{B}} > 0$, the two eigenvalues of the Jacobian are

$$\begin{aligned} r_{1,2} &= \frac{1}{2}\left\{\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} + \frac{\partial H^B}{\partial\theta_{\mathrm{B}}} - 2\pm\right. \\ &\left.\left[\left(\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} + \frac{\partial H^B}{\partial\theta_{\mathrm{B}}} - 2\right)^2 - 4\left((\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} - 1)(\frac{\partial H^B}{\partial\theta_{\mathrm{B}}} - 1) - \frac{\partial H^A}{\partial\theta_{\mathrm{B}}}\frac{\partial H^B}{\partial\theta_{\mathrm{A}}}\right)\right]^{1/2}\right\}. \end{aligned}$$

Note that $\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} < \frac{H^A}{\theta_{\mathrm{A}}}$ and $\frac{\partial H^B}{\partial\theta_{\mathrm{B}}} < \frac{H^B}{\theta_{\mathrm{B}}}$. Since at the steady-state, $\frac{H^l}{\theta_l} = 1$, this automatically implies that $\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} - 1 < 0$ and $\frac{\partial H^B}{\partial\theta_{\mathrm{B}}} - 1 < 0$ at the steady-state. Thus, for both eigenvalues to be negative, it is sufficient that $(\frac{\partial H^A}{\partial\theta_{\mathrm{A}}} - 1)(\frac{\partial H^B}{\partial\theta_{\mathrm{B}}} - 1) - \frac{\partial H^A}{\partial\theta_{\mathrm{B}}}\frac{\partial H^B}{\partial\theta_{\mathrm{A}}} > 0$. For this to hold, in turn, it is sufficient that $1 - \frac{\partial H^A}{\partial\theta_{\mathrm{A}}} > -\frac{\partial H^B}{\partial\theta_{\mathrm{A}}}$ and $1 - \frac{\partial H^B}{\partial\theta_{\mathrm{B}}} > -\frac{\partial H^A}{\partial\theta_{\mathrm{B}}}$. Given the partial derivatives, the condition that $1 - \frac{\partial H^A}{\partial\theta_{\mathrm{A}}} > -\frac{\partial H^B}{\partial\theta_{\mathrm{A}}}$ is equal to

$$1 - \sum_k \tilde{P}(k)\left[\frac{k\lambda}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)^2} + (g_{\mathrm{B}} - g_{\mathrm{A}})\frac{k^2\lambda^2\bar{\theta}_{\mathrm{B}}}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)^2\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)}\right] > 0. \quad \text{(A.18)}$$

We know that at $(\bar{\theta}_{\mathrm{A}}, \bar{\theta}_{\mathrm{B}})$,

$$1 = \sum_k \tilde{P}(k)\left[\frac{k\lambda}{1 + k\lambda\bar{\theta}_{\mathrm{A}}} - g_{\mathrm{B}}\frac{k^2\lambda^2\bar{\theta}_{\mathrm{B}}}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)}\right]. \quad \text{(A.19)}$$

By substituting this expression into equation (A.18), all terms are sums over k. For equation (A.18) to be satisfied, it is then sufficient that it is satisfied for all individual terms of the sums, i.e.,

$$\begin{aligned} \frac{k\lambda}{1 + k\lambda\bar{\theta}_{\mathrm{A}}} &- g_{\mathrm{B}}\frac{k^2\lambda^2\bar{\theta}_{\mathrm{B}}}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)} - \frac{k\lambda}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)^2} + \\ &+(g_{\mathrm{B}} - g_{\mathrm{A}})\frac{k^2\lambda^2\bar{\theta}_{\mathrm{B}}}{\left(1 + k\lambda\bar{\theta}_{\mathrm{A}}\right)^2\left(1 + k\lambda\bar{\theta}_{\mathrm{B}}\right)} > 0. \end{aligned} \quad \text{(A.20)}$$

Simplifying equation (A.20), we find that it is equivalent to the condition that

$$1 + k\lambda\bar{\theta}_{\mathrm{B}} > 0, \quad \text{(A.21)}$$

which is always satisfied. Yet again due to symmetry, this also shows that

$1 - \frac{\partial H^B}{\partial \theta_B} > -\frac{\partial H^A}{\partial \theta_B}$, too. This completes the proof that for $\lambda > \frac{\langle k \rangle}{\langle k^2 \rangle}$, the uniquely stable steady-state is the one in which $\bar{\theta}_l > 0$ for both $l \in \{A, B\}$. $\square$

Appendix B. Proof of Proposition 2

To prove the first part of Proposition 2, note that $\bar{\rho}_l$ is increasing in λ if and only if $\bar{\rho}_l(k)$ is increasing in λ. For $\bar{\rho}_l(k)$ to be increasing in λ in turn it is sufficient that $\bar{\theta}_l$ is increasing in λ. We prove this now for $l = A$. As

$$H^A(\acute{\theta}_A) = \sum_k \tilde{P}(k) \frac{k\lambda\acute{\theta}_A}{1 + k\lambda\acute{\theta}_A} \frac{1 + g_B k\lambda\acute{\theta}_A}{1 + \frac{g_B}{g_A} k\lambda\acute{\theta}_A},$$

it follows that for given $\acute{\theta}_A$,

$$\frac{\partial H^A(\acute{\theta}_A)}{\partial \lambda} = \frac{k\acute{\theta}_A}{(1 + k\lambda\acute{\theta}_A)^2 (1 + \frac{g_B}{g_A} k\lambda\acute{\theta}_A)^2} \left[1 + 2g_B k\lambda\acute{\theta}_A\right] > 0. \quad \text{(B.1)}$$

Fix λ and λ' and let $\bar{\theta}_A = H^A(\bar{\theta}_A)$ for λ and $\bar{\theta}_{A'} = H^{A'}(\bar{\theta}_{A'})$ for λ'. Proposition 2 states that for any $\lambda' > \lambda$, $\bar{\theta}_{A'} > \bar{\theta}_A$.

Suppose to the contrary that $\bar{\theta}_{A'} \leq \bar{\theta}_A$. Then, as $H^A(\acute{\theta}_A)$ is concave in $\acute{\theta}_A$, it is the case that $\bar{\theta}_{A'} \leq H^A(\bar{\theta}_{A'})$. However, from equation (B.1) we know that

$$H^A(\bar{\theta}_{A'}) < H^{A'}(\bar{\theta}_{A'}) \quad \text{(B.2)}$$

which contradicts the fact that $\bar{\theta}_{A'} = H^{A'}(\bar{\theta}_{A'})$. Thus, for each $\lambda' > \lambda$, $\bar{\theta}_{A'} > \bar{\theta}_A$. Hence, $\bar{\theta}_A$ is increasing in λ. The same argument holds for $l = B$.

To show that $\frac{\bar{\rho}_A}{\bar{\rho}_B}$ is decreasing in λ if and only if $g_A > g_B$, it suffices to show that $\frac{\bar{\rho}_A(k)}{\bar{\rho}_B(k)}$ is decreasing in λ if and only if $g_A > g_B$. If this is true, $\bar{\rho}_B(k)$ is increasing in λ faster than $\bar{\rho}_A(k)$, which implies that also $\bar{\rho}_B$ is increasing in λ faster than $\bar{\rho}_A$. To this end, it is useful to employ the result that $\bar{\theta}_A = \frac{\nu_A}{\nu_B}\bar{\theta}_B$, which allows us to write $\frac{\bar{\rho}_A(k)}{\bar{\rho}_B(k)}$ as

$$\frac{\bar{\rho}_A(k)}{\bar{\rho}_B(k)} = \frac{\frac{g_A}{g_B} + k\lambda\bar{\theta}_A}{1 + k\lambda\bar{\theta}_A}. \quad \text{(B.3)}$$

This implies that

$$\frac{d\frac{\bar{\rho}_{\mathrm{A}}(k)}{\bar{\rho}_{\mathrm{B}}(k)}}{d\lambda} = \frac{k[\bar{\theta}_{\mathrm{A}} + \lambda\frac{d\bar{\theta}_{\mathrm{A}}}{d\lambda}]}{(1+k\lambda\bar{\theta}_{\mathrm{A}})^2}\left(1 - \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}}\right). \tag{B.4}$$

Which, as $\bar{\theta}_{\mathrm{A}}$ is strictly increasing in λ, is negative if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$. Finally, it is straightforward to show that

$$\frac{d\frac{\bar{\rho}_{\mathrm{A}}(k)}{\bar{\rho}_{\mathrm{B}}(k)}}{dk} = \frac{\lambda\bar{\theta}_{\mathrm{A}}}{(1+k\lambda\bar{\theta}_{\mathrm{A}})^2}\left(1 - \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}}\right) \tag{B.5}$$

which again is negative if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$, which completes the proof.

Appendix C. Proof of Proposition 3

For a single meme, Theorem 1 in Jackson and Rogers (2007b) proves that the steady-states of θ, $\rho(k)$, and ρ are increasing in a first order stochastic shift in P and $\tilde{P}$ as $H(\theta)$ is concave and $H(1) < 1$. In the present model, Lemma 4 proves concavity for $H^l(\bar{\theta}_l)$. The proof of the first point hence follows from Theorem 1 in Jackson and Rogers (2007b).

To prove the second point, it is useful to again write

$$\frac{\bar{\rho}_{\mathrm{A}}(k)}{\bar{\rho}_{\mathrm{B}}(k)} = \frac{\frac{g_{\mathrm{A}}}{g_{\mathrm{B}}} + k\lambda\bar{\theta}_{\mathrm{A}}}{1+k\lambda\bar{\theta}_{\mathrm{A}}}.$$

Taking the derivative of this expression with respect to $\bar{\theta}_{\mathrm{A}}$ yields

$$\frac{d\frac{\bar{\rho}_{\mathrm{A}}(k)}{\bar{\rho}_{\mathrm{B}}(k)}}{d\bar{\theta}_{\mathrm{A}}} = \frac{k[\bar{\theta}_{\mathrm{A}} + \lambda\frac{d\bar{\theta}_{\mathrm{A}}}{d\lambda}]}{(1+k\lambda\bar{\theta}_{\mathrm{A}})^2}\left(1 - \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}}\right) \tag{C.1}$$

which is decreasing in $\bar{\theta}_{\mathrm{A}}$ if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$. As this implies that $\bar{\rho}_{\mathrm{B}}(k)$ is increasing in $\bar{\theta}_{\mathrm{A}}$ faster than $\bar{\rho}_{\mathrm{A}}(k)$ if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$, it also implies that $\bar{\rho}_{\mathrm{B}}$ is increasing in $\bar{\theta}_{\mathrm{A}}$ faster than $\bar{\rho}_{\mathrm{A}}$ if and only if $g_{\mathrm{A}} > g_{\mathrm{B}}$. As a first order stochastic dominant change in the degree distribution implies an increase in $\bar{\theta}_{\mathrm{A}}$, the second point follows.

Appendix D. Proof of Proposition 4

We focus on $l = A$ with $g_{\rm A} > g_{\rm B} > 0$. Information prevalence in the one-meme case is given by

$$\begin{aligned}
\tilde{\rho} &= \sum_k P(k)\tilde{\rho}(k), && \text{(D.1)}\\
\tilde{\rho}(k) &= \frac{k\lambda\tilde{\theta}}{1+k\lambda\tilde{\theta}} && \text{(D.2)}\\
\tilde{\theta} &= H(\tilde{\theta}) = \sum_k \tilde{P}(k)\frac{k\lambda\tilde{\theta}}{1+k\lambda\tilde{\theta}} && \text{(D.3)}
\end{aligned}$$

Therefore, $\tilde{\rho}$ is strictly increasing in $\tilde{\rho}(k)$. Also,

$$\bar{\rho}_{\rm A}(k) = \frac{k\lambda\bar{\theta}_{\rm A}}{1+k\lambda\bar{\theta}_{\rm A}},$$

which implies that $\tilde{\rho} > \bar{\rho}_{\rm A}$ if and only if $\tilde{\theta} > \bar{\theta}_{\rm A}$.

To establish the bounds on $\bar{\theta}_{\rm A}$, we make use of the fact that at $\tilde{\theta} > 0$ and $\bar{\theta}_{\rm A} > 0$, the following conditions are satisfied,

$$\begin{aligned}
1 &= \sum_k \tilde{P}(k)\frac{k\lambda}{1+k\lambda\tilde{\theta}}, && \text{(D.4)}\\
1 &= \sum_k \tilde{P}(k)\frac{k\lambda}{1+k\lambda\bar{\theta}_{\rm A}}\left[1-g_{\rm B}\frac{k\lambda\bar{\theta}_{\rm B}}{1+k\lambda\bar{\theta}_{\rm B}}\right]. && \text{(D.5)}
\end{aligned}$$

Which means that the two sums are equal to each other, and we can write them as

$$\sum_k \tilde{P}(k)\left\{\frac{k\lambda}{1+k\lambda\tilde{\theta}} - \frac{k\lambda}{1+k\lambda\bar{\theta}_{\rm A}}\left[1-g_{\rm B}\frac{k\lambda\bar{\theta}_{\rm B}}{1+k\lambda\bar{\theta}_{\rm B}}\right]\right\} = 0. \qquad \text{(D.6)}$$

Some re-arranging shows that this implies

$$\begin{aligned}
\sum_k \tilde{P}(k) \quad & \frac{k^2\lambda^2}{(1+k\lambda\tilde{\theta})(1+k\lambda\bar{\theta}_{\rm A})(1+k\lambda\bar{\theta}_{\rm B})}\cdot && \text{(D.7)}\\
& \cdot\left\{\frac{1}{g_{\rm A}}\left[\bar{\theta}_{\rm A}(g_{\rm A}+g_{\rm B}^2) - g_{\rm A}\tilde{\theta}\right] + k\lambda\bar{\theta}_{\rm B}(\bar{\theta}_{\rm A} - g_{\rm A}\tilde{\theta})\right\} = 0.
\end{aligned}$$

If $\bar{\theta}_{\mathrm{A}} < g_{\mathrm{A}}\tilde{\theta}$, then $\bar{\theta}_{\mathrm{A}}(g_{\mathrm{A}} + g_{\mathrm{B}}^2) < g_{\mathrm{A}}\tilde{\theta}$ too, as $g_{\mathrm{A}} + g_{\mathrm{B}}^2 < 1$. I.e., each individual term in the sum in equation (D.7) would be negative, which contradicts the assumption that both $\bar{\theta}_{\mathrm{A}}$ and $\tilde{\theta}$ are steady-states. Similarly, if $\bar{\theta}_{\mathrm{A}} > \frac{g_{\mathrm{A}}}{1-g_{\mathrm{A}}g_{\mathrm{B}}}\tilde{\theta}$, then both terms in the sum in equation (D.7) would be positive, again contradicting the steady-state assumption. Due to symmetry, the result for $\bar{\theta}_{\mathrm{B}}$ follows, as do the bounds stated in Proposition 4.

Appendix E. Proof of Lemma 1

Existence of a steady-state in which $\bar{\theta}_{\mathrm{A}}^{\mathrm{A}} = \bar{\theta}_{\mathrm{A}}^{\mathrm{B}} = \bar{\theta}_{\mathrm{B}}^{\mathrm{A}} = \bar{\theta}_{\mathrm{B}}^{\mathrm{B}} = 0$ for the general system is trivial. The corresponding Jacobian evaluated at the zero steady-state is

$$\mathbf{J} = \begin{pmatrix} \beta_{\mathrm{A}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 & (1-\beta_{\mathrm{A}})\lambda\frac{\langle k^2\rangle}{\langle k\rangle} & 0 & 0 \\ (1-\beta_{\mathrm{B}})\lambda\frac{\langle k^2\rangle}{\langle k\rangle} & \beta_{\mathrm{B}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 & 0 & 0 \\ 0 & 0 & \beta_{\mathrm{A}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 & (1-\beta_{\mathrm{A}})\lambda\frac{\langle k^2\rangle}{\langle k\rangle} \\ 0 & 0 & (1-\beta_{\mathrm{B}})\lambda\frac{\langle k^2\rangle}{\langle k\rangle} & \beta_{\mathrm{B}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 \end{pmatrix} \tag{E.1}$$

Thus,

$$|\mathbf{J} - r\mathbf{I}| = \\ \left[\left(\beta_{\mathrm{A}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 - r\right)\left(\beta_{\mathrm{B}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 - r\right) - \lambda^2(1-\beta_{\mathrm{A}})(1-\beta_{\mathrm{B}})\left[\frac{\langle k^2\rangle}{\langle k\rangle}\right]^2\right]^2. \tag{E.2}$$

It follows that $|\mathbf{J} - r\mathbf{I}| = 0$ if r solves

$$\begin{aligned} \left(\beta_{\mathrm{A}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 - r\right)\left(\beta_{\mathrm{B}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - 1 - r\right) &= \lambda^2(1-\beta_{\mathrm{A}})(1-\beta_{\mathrm{B}})\left[\frac{\langle k^2\rangle}{\langle k\rangle}\right]^2 \Rightarrow \\ (1+r)\left[(1+r) - \beta_{\mathrm{A}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle} - \beta_{\mathrm{B}}\lambda\frac{\langle k^2\rangle}{\langle k\rangle}\right] &= \lambda^2\left[\frac{\langle k^2\rangle}{\langle k\rangle}\right]^2[1 - \beta_{\mathrm{B}} - \beta_{\mathrm{A}}]. \end{aligned} \tag{E.3}$$

Solving equation (E.3), we find that the two eigenvalues are

$$\begin{aligned} r_1 &= \lambda \frac{\langle k^2 \rangle}{\langle k \rangle} - 1, \\ r_2 &= \lambda \frac{\langle k^2 \rangle}{\langle k \rangle} \left[\beta_{\mathrm{A}} + \beta_{\mathrm{B}} - 1 \right] - 1. \end{aligned}$$

As the maximum value that $[\beta_{\mathrm{A}} + \beta_{\mathrm{B}} - 1]$ can take is 1, it is the case that $r_2 \leq r_1$ and therefore, also for all levels of homophily we find that the steady-state in which no information survives is stable if and only if $\lambda \leq \frac{\langle k \rangle}{\langle k^2 \rangle}$.

Appendix F. Proof of Theorem 2

Simple substitution shows that $\bar{\theta}^{\mathrm{A}}_{\mathrm{A}} = \bar{\theta}^{\mathrm{B}}_{\mathrm{A}} = 0$ is a solution to equations (15) and (16), while $\bar{\theta}^{\mathrm{A}}_{\mathrm{B}} = \bar{\theta}^{\mathrm{B}}_{\mathrm{B}} = 0$ is a solution of equations (17) and (18) respectively. This is true irrespective of the degree of homophily in the population.

Similarly, $\bar{\theta}^{\mathrm{B}}_{\mathrm{A}} = 0$ and $\bar{\theta}^{\mathrm{A}}_{\mathrm{A}} > 0$ solve equation (16) if and only if $\beta_{\mathrm{A}} = 1$. The same argument can be applied to all equations (15)-(18). Thus, if a meme exhibits a positive steady-state in the group that prefers it and groups interact to some extent, the same meme must also exhibit a positive steady-state in the complement group.

Appendix G. Proof of Theorem 3

Information loss due to segregation is $g_l \tilde{\rho} - \bar{\rho}_l$. For this term to be negative, it is sufficient that $g_l \tilde{\rho}(k) < \bar{\rho}_l(k)$ for all k. As the lower bound for $\bar{\theta}_l$ is $g_l \tilde{\theta}$, we know that

$$\bar{\rho}_l(k) > g_l \sum_k \tilde{P}(k) \frac{k \lambda \tilde{\theta}}{1 + g_l k \lambda \tilde{\theta}} > g_l \tilde{\rho}(k).$$

I.e., for agents of any degree, segregation leads to an information loss for each meme $l \in \{A, B\}$, and hence $g_l \tilde{\rho} - \bar{\rho}_l < 0$. Furthermore, a sufficient condition for $|g_{\mathrm{A}} \tilde{\rho} - \bar{\rho}_{\mathrm{A}}| > |g_{\mathrm{B}} \tilde{\rho} - \bar{\rho}_{\mathrm{B}}|$ is that $|g_{\mathrm{A}} \tilde{\rho}(k) - \bar{\rho}(k)| > |g_{\mathrm{B}} \tilde{\rho}(k) - \bar{\rho}_{\mathrm{B}}(k)|$. Note that

$$\frac{g_{\mathrm{A}} \tilde{\rho}(k) - \bar{\rho}_{\mathrm{A}}(k)}{g_{\mathrm{B}} \tilde{\rho}(k) - \bar{\rho}_{\mathrm{B}}(k)} = \frac{g_{\mathrm{A}}}{g_{\mathrm{B}}} \frac{\tilde{\rho}(k) - \frac{1}{g_{\mathrm{A}}} \frac{k \lambda \bar{\theta}_{\mathrm{A}}}{1 + k \lambda \bar{\theta}_{\mathrm{A}}}}{\tilde{\rho}(k) - \frac{1}{g_{\mathrm{A}}} \frac{k \lambda \bar{\theta}_{\mathrm{A}}}{1 + k \lambda \bar{\theta}_{\mathrm{B}}}} \tag{G.1}$$

which is larger than 1 if and only if $g_{\text{A}} > g_{\text{B}}$, as then both terms on the right hand side are larger than 1, while for $g_{\text{B}} > g_{\text{A}}$, they are both smaller than 1. This immediately shows that

$$|g_{\text{A}}\tilde{\rho}(k) - \bar{\rho}_{\text{A}}(k)| > |g_{\text{B}}\tilde{\rho}(k) - \bar{\rho}_{\text{B}}(k)| \tag{G.2}$$

if and only if $g_{\text{A}} > g_{\text{B}}$.

Appendix H. Proof of Theorem 4

Point 1:

An agent of degree k and group l prefers a segregated society over an integrated one if

$$\frac{s}{h} < m_l(k),$$

where

$$m_l(k) = \frac{\tilde{\rho}(k) - \bar{\rho}_l(k)}{\bar{\rho}_{\text{-l}}(k)}.$$

For $l = A$,

$$\begin{aligned} \frac{d\ln(m_{\text{A}}(k))}{dk} &= \frac{\lambda\bar{\theta}_{\text{B}}}{1+k\lambda\bar{\theta}_{\text{B}}} - \frac{\lambda\tilde{\theta}}{1+k\lambda\tilde{\theta}} - \frac{\lambda\bar{\theta}_{\text{A}}}{1+k\lambda\bar{\theta}_{\text{A}}} \\ &= \frac{\lambda}{(1+k\lambda\bar{\theta}_{\text{B}})(1+k\lambda\bar{\theta}_{\text{A}})(1+k\lambda\tilde{\theta})} \cdot \\ &\quad \cdot \left[\bar{\theta}_{\text{B}} - \tilde{\theta} - \bar{\theta}_{\text{A}} - 2k\lambda\bar{\theta}_{\text{A}}\tilde{\theta} - k^2\lambda^2\bar{\theta}_{\text{A}}\bar{\theta}_{\text{B}}\tilde{\theta}\right] \end{aligned} \tag{H.1}$$

which is always negative, as $\tilde{\theta} > \bar{\theta}_{\text{B}}$. By symmetry, $\frac{d\ln(m_{\text{B}}(k))}{dk} < 0$ holds as well. It is therefore the case that for each group, agents that have more meetings per period prefer an integrated society for a broader range of $\frac{s}{h}$ than agents with fewer meetings.

Point 2:

The second claim of Theorem 4 is that for all k, $m_{\text{B}}(k) > m_{\text{A}}(k)$ if and only if $g_{\text{A}} > g_{\text{B}}$, i.e., for two agents with the same degree, the agent belonging to the minority group prefers segregation for a broader range of $\frac{s}{h}$. This holds if, for

$g_{\text{A}} > g_{\text{B}}$,

$$\frac{m_{\text{A}}(k)}{m_{\text{B}}(k)} = \frac{\tilde{\rho}(k) - \bar{\rho}_{\text{A}}(k)}{\tilde{\rho}(k) - \bar{\rho}_{\text{B}}(k)} \frac{\bar{\rho}_{\text{A}}(k)}{\bar{\rho}_{\text{B}}(k)} < 1. \tag{H.2}$$

This condition can be re-written as

$$\begin{aligned}
(\tilde{\rho}(k) - \bar{\rho}_{\text{A}}(k))\bar{\rho}_{\text{A}}(k) \quad &< \quad (\tilde{\rho}(k) - \bar{\rho}_{\text{B}}(k))\bar{\rho}_{\text{B}}(k) \\
\frac{k\lambda\bar{\theta}_{\text{A}}}{1 + k\lambda\bar{\theta}_{\text{A}}} \left[\frac{k\lambda\tilde{\theta}}{1 + k\lambda\tilde{\theta}} - \frac{k\lambda\bar{\theta}_{\text{A}}}{1 + k\lambda\bar{\theta}_{\text{A}}} \right] \quad &< \quad \frac{k\lambda\bar{\theta}_{\text{B}}}{1 + k\lambda\bar{\theta}_{\text{B}}} \left[\frac{k\lambda\tilde{\theta}}{1 + k\lambda\tilde{\theta}} - \frac{k\lambda\bar{\theta}_{\text{B}}}{1 + k\lambda\bar{\theta}_{\text{B}}} \right]
\end{aligned}$$

and through collecting terms, re-arranging, and making use of the fact that $\bar{\theta}_{\text{B}} = \frac{g_{\text{B}}}{g_{\text{A}}}\bar{\theta}_{\text{A}}$, it can be simplified to

$$\tilde{\theta}\left[1 - k^2\lambda^2\bar{\theta}_{\text{A}}\bar{\theta}_{\text{B}}\right] < \frac{1}{g_{\text{A}}}\bar{\theta}_{\text{A}} + 2k\lambda\bar{\theta}_{\text{A}}\bar{\theta}_{\text{B}}. \tag{H.3}$$

This is satisfied, as we know that $\bar{\theta}_{\text{A}} > g_{\text{A}}\tilde{\theta}$, i.e., $\frac{1}{g_{\text{A}}}\bar{\theta}_{\text{A}} > \tilde{\theta}$.

Appendix I. Proof of Proposition 5

For $n_l(k)$, we know from equation (I.3) and the definitions of $\hat{\theta}_l$ and $\hat{\theta}_{\text{-l}}$ that

$$\begin{aligned}
n_{\text{A}}(k) \quad &= \quad \frac{(1 + k\lambda\bar{\theta}_{\text{B}})(1 + k\lambda(\bar{\theta}_{\text{B}} - g_{\text{B}}\bar{\theta}_{\text{AB}}))}{(1 + k\lambda\bar{\theta}_{\text{A}})(1 + k\lambda(\bar{\theta}_{\text{A}} + g_{\text{B}}\bar{\theta}_{\text{AB}}))}, \qquad (\text{I.1}) \\
n_{\text{B}}(k) \quad &= \quad \frac{(1 + k\lambda\bar{\theta}_{\text{A}})(1 + k\lambda(\bar{\theta}_{\text{A}} - g_{\text{A}}\bar{\theta}_{\text{AB}}))}{(1 + k\lambda\bar{\theta}_{\text{B}})(1 + k\lambda(\bar{\theta}_{\text{B}} + g_{\text{A}}\bar{\theta}_{\text{AB}}))}. \qquad (\text{I.2})
\end{aligned}$$

We also have the following relationships:

- $\bar{\theta}_{\text{A}} > \bar{\theta}_{\text{B}}$ if and only if $g_{\text{A}} > g_{\text{B}}$.
- $\bar{\theta}_{\text{A}} + g_{\text{B}}\bar{\theta}_{\text{AB}} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{A}}(k)$ and $\bar{\theta}_{\text{B}} + g_{\text{A}}\bar{\theta}_{\text{AB}} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{B}}(k)$, i.e., $\bar{\theta}_{\text{A}} + g_{\text{B}}\bar{\theta}_{\text{AB}} > \bar{\theta}_{\text{B}} + g_{\text{A}}\bar{\theta}_{\text{AB}}$ if and only if $g_{\text{A}} > g_{\text{B}}$.
- $\bar{\theta}_{\text{A}} - g_{\text{A}}\bar{\theta}_{\text{AB}} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{A}}(k)[1 - \bar{\rho}_{\text{B}}(k)]$ and $\bar{\theta}_{\text{B}} - g_{\text{B}}\bar{\theta}_{\text{AB}} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{B}}(k)[1 - \bar{\rho}_{\text{A}}(k)]$, i.e., $\bar{\theta}_{\text{A}} - g_{\text{A}}\bar{\theta}_{\text{AB}} > \bar{\theta}_{\text{B}} - g_{\text{B}}\bar{\theta}_{\text{AB}}$ if and only if $g_{\text{A}} > g_{\text{B}}$.

This implies that if $g_{\text{A}} > g_{\text{B}}$, the numerator of equation (I.1) is smaller than the numerator of equation (I.2), while the denominator of (I.1) is larger than the denominator of (I.2). Thus, if $g_{\text{A}} > g_{\text{B}}$, then $n_{\text{A}}(k) < n_{\text{B}}(k)$.

The fact that integration is not a Nash equilibrium whenever segregation is preferred by some agents follows directly from the following Lemma:

Lemma 6. *For both l and all k, λ and $P(k)$, it is the case that $n_l(k) \geq m_l(k)$, with strict inequality for finite λ.*

Proof. Employ the definitions of $\hat{\rho}_l(k)$ and $\hat{\rho}_{\text{-}1}(k)$ to re-write $n_l(k)$ as

$$n_l(k) = \frac{\hat{\theta}_l - \bar{\theta}_l}{\bar{\theta}_{\text{-}1} - \hat{\theta}_{\text{-}1}} \frac{(1 + k\lambda\bar{\theta}_{\text{-}1})(1 + k\lambda\hat{\theta}_{\text{-}1})}{(1 + k\lambda\bar{\theta}_l)(1 + k\lambda\hat{\theta}_l)}. \tag{I.3}$$

We define

$$\bar{\theta}_{\text{AB}} \equiv \sum_k \tilde{P}(k) \frac{k\lambda\bar{\theta}_{\text{B}}}{1 + k\lambda\bar{\theta}_{\text{B}}} \frac{k\lambda\bar{\theta}_{\text{A}}}{1 + k\lambda\bar{\theta}_{\text{A}}} \tag{I.4}$$

which allows us to write

$$\hat{\theta}_l = \sum_k \tilde{P}(k)\bar{\rho}_l(k) = \bar{\theta}_l + g_{\text{-}1}\bar{\theta}_{\text{AB}}, \tag{I.5}$$

$$\hat{\theta}_{\text{-}1} = \sum_k \tilde{P}(k)\bar{\rho}_{\text{-}1}(k)[1 - \bar{\rho}_l(k)] = \bar{\theta}_{\text{-}1} - g_{\text{-}1}\bar{\theta}_{\text{AB}}, \tag{I.6}$$

and equation (I.3) becomes

$$n_l(k) = \frac{(1 + k\lambda\bar{\theta}_{\text{-}1})(1 + k\lambda\hat{\theta}_{\text{-}1})}{(1 + k\lambda\bar{\theta}_l)(1 + k\lambda\hat{\theta}_l)}. \tag{I.7}$$

In fact,

$$m_l(k) = \frac{\tilde{\rho}(k) - \bar{\rho}_l(k)}{\bar{\rho}_{\text{-}1}(k)} = \frac{1 + k\lambda\bar{\theta}_{\text{-}1}}{1 + k\lambda\bar{\theta}_l} \frac{\tilde{\theta} - \bar{\theta}_l}{\bar{\theta}_{\text{-}1}(1 + k\lambda\tilde{\theta})}, \tag{I.8}$$

which shows that $n_l(k) \geq m_l(k)$ if

$$\frac{1 + k\lambda\hat{\theta}_{\text{-}1}}{1 + k\lambda\hat{\theta}_l} \geq \frac{\tilde{\theta} - \bar{\theta}_l}{\bar{\theta}_{\text{-}1}(1 + k\lambda\tilde{\theta})}. \tag{I.9}$$

We know from Proposition 4 that $\bar{\theta}_l > g_l\tilde{\theta}$. Therefore, the numerator of the right-hand side of equation (I.9) is smaller than $g_{\text{-}1}\tilde{\theta}$, while $\bar{\theta}_{\text{-}1} > g_{\text{-}1}\tilde{\theta}$, i.e., the denominator in equation (I.9) is larger than $g_{\text{-}1}\tilde{\theta}(1 + k\lambda\tilde{\theta})$. By applying these bounds, we know that the right-hand side of equation (I.9) is strictly less than

$\frac{1}{1+k\lambda\tilde{\theta}}$. Consequently, equation (I.9) is satisfied for sure if

$$\frac{1+k\lambda\hat{\theta}_{\text{-l}}}{1+k\lambda\hat{\theta}_l} \geq \frac{1}{1+k\lambda\tilde{\theta}} \Rightarrow \tag{I.10}$$

$$(1+k\lambda\hat{\theta}_{\text{-l}})(1+k\lambda\tilde{\theta}) \geq 1+k\lambda\hat{\theta}_l. \tag{I.11}$$

This, however, is always satisfied as $\tilde{\theta} > \hat{\theta}_l$ and $1 + k\lambda\hat{\theta}_{\text{-l}} > 1$. Finally, we know that $\lim_{\lambda\to\infty} m_l(k) = 0$. As $\lim_{\lambda\to\infty} \bar{\rho}_l(k) = 1$ for both l, we find that $\lim_{\lambda\to\infty} \hat{\theta}_{\text{-l}} = 0$ and applying this we find that $\lim_{\lambda\to\infty} n_l(k) = 0$, thus $m_l(k) = n_l(k)$ if and only if $\lambda \to \infty$. $\square$

Lemma 7. *For both l, and all k, λ and $P(k)$, it is the case that $\eta_l(k) \geq m_l(k)$, with strict inequality for finite λ and k. $\eta_l(k)$ is decreasing in k, λ, and $\tilde{\theta}$.*

Proof. Lemma 7 is easiest proven by focusing on the fact that segregation is preferred to integration if

$$h\bar{\rho}_l(k) + s\bar{\rho}_{\text{-l}}(k) < h\tilde{\rho}(k), \tag{I.12}$$

while segregation is a Nash equilibrium if

$$h\check{\rho}_l(k) + s\check{\rho}_{\text{-l}}(k) < h\tilde{\rho}(k). \tag{I.13}$$

By the definition of $\check{\rho}_l(k)$ and $\check{\rho}_{\text{-l}}(k)$, the fact that $\bar{\theta}_l > g_l\tilde{\theta}$ implies immediately that $\bar{\rho}_l(k) > \check{\rho}_l(k)$ and $\bar{\rho}_{\text{-l}}(k) > \check{\rho}_{\text{-l}}(k)$. It is therefore true that whenever segregation is preferred by all members of group l, it is also a Nash equilibrium. It is also immediate that the numerator of $m_l(k)$ is smaller than the numerator of $\eta_l(k)$, while the denominator of $m_l(k)$ is larger than the one of $\eta_l(k)$.

Finally, we know that $m_l(k)$ goes to zero if either k or λ go to infinity. It is straightforward to take the same limits directly in equation (23) to establish that also $\eta_l(k)$ goes to zero if either k or λ go to infinity. Similarly, from equation (23) it is obvious that changes in k, λ, and $\tilde{\theta}$ have the same qualitative effect on $\eta_l(k)$. If we denote $k\lambda\tilde{\theta} \equiv x$, we find that

$$\frac{d\eta_l(k)}{dk} = \frac{-g_l}{(1+x)^2(1+g_l x)^2}\left[2+2x+g_{\text{-}1}x^2\right]\frac{dx}{dk} < 0, \tag{I.14}$$

$$\frac{d\eta_l(k)}{d\lambda} = \frac{-g_l}{(1+x)^2(1+g_l x)^2}\left[2+2x+g_{\text{-}1}x^2\right]\frac{dx}{d\lambda} < 0, \tag{I.15}$$

$$\frac{d\eta_l(k)}{d\tilde{\theta}} = \frac{-g_l}{(1+x)^2(1+g_l x)^2}\left[2+2x+g_{\text{-}1}x^2\right]\frac{dx}{d\tilde{\theta}} < 0. \tag{I.16}$$

As a FOSD shift in the degree distribution implies an increase in $\tilde{\theta}$, the last derivative proves that $\eta_l(k)$ is indeed decreasing in such a shift. □