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Risk Pooling, Leverage, and the Business Cycle

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In equilibrium, financial sector leverage also determines counter-cyclical Sharpe ratios and pro-cyclical risk-free interest rates. Last, we investigate the relationship between the size of the financial sector, leverage, and welfare. We show that limiting financial sector leverage determines a sub-optimal pooling of idiosyncratic risk but fosters the growth rate of the h/entrepreneurs' consumption. On the other side, when the financial sector is too large, it destroys too many resources after intermediation costs. Therefore, the h/entrepreneurs benefit the most when the financial sector is neither too small nor too big.

Keywords
Amplification, Business Cycle, Financial Frictions, Leverage, Risk Pooling

JEL Codes
E13, E32, E69, G12

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Risk Pooling, Leverage, and the Business Cycle*

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Abstract

This paper investigates the interdependence between the risk-pooling activity of the financial sector and: output, consumption, risk-free rate, and Sharpe ratio in a dynamic general equilibrium model of a productive economy. Due to their exposure to idiosyncratic shocks and market segmentation, heterogeneous households/entrepreneurs (h/entrepreneurs) are willing to mitigate their risk through a financial sector. The financial sector pools risky claims issued by different firms within its assets, faces an associated intermediation cost and, via leverage, provides a risk-free asset to h/entrepreneurs. Exogenous systematic shocks change the relative size of the financial sector, and thus the equilibrium amount of pooled risk, making financial leverage state-dependent and counter-cyclical. We study how this mechanism endogenously channels amplification of consumption and mitigation of output fluctuations. In equilibrium, financial sector leverage also determines counter-cyclical Sharpe ratios and pro-cyclical risk-free interest rates. Last, we investigate the relationship between the size of the financial sector, leverage, and welfare. We show that limiting financial sector leverage determines a sub-optimal pooling of idiosyncratic risk but fosters the growth rate of the h/entrepreneurs' consumption. On the other side, when the financial sector is too large, it destroys too many resources after intermediation costs. Therefore, the h/entrepreneurs benefit the most when the financial sector is neither too small nor too big.

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1 Introduction

There is widespread agreement that financial intermediation is not only a veil between savers and borrowers but it has a fundamental role to properly characterize the business and financial cycle (Adrian and Shin, 2010; Borio, 2014; Brunnermeier and Sannikov, 2014, 2016; He et al., 2017; He and Krishnamurthy, 2019).

There are several channels through which the financial sector may affect the business cycle. In this paper we focus on two specific functions of the financial sector: risk pooling and risk mitigation.

On the liability side, by issuing a risk-free bond (i.e. via leverage) the financial sector provides a risk-mitigation instrument to householders. This creates a risk-mismatch between the intermediaries' assets (risky stakes in firms) and liabilities (deposits or other low-risk assets).

Exogenous systematic shocks change the relative size of the financial sector, and so its provision of risk pooling and mitigation. In turn, this affects the macro-dynamics of the economy as a whole. The magnitude of the mismatch, and so the sensitivity of the intermediaries' balance sheet to exogenous systematic shocks, relates to the amount of financial sector leverage, itself dependent on the size of idiosyncratic risks within the economy. The higher the leverage the higher the impact.

The objectives of this paper are: a) Investigate the joint effect of systematic and idiosyncratic shocks on both the real and financial economic dynamics in a model where financial intermediaries pool idiosyncratic risk of productive activities and provide risk mitigation to the households/entrepreneurs (later on h/entrepreneurs); b) Highlight the important role played by the financial sector leverage and size in mitigating and amplifying consumption and output fluctuations.

We model the general equilibrium productive economy in continuous-time, and solve for the equilibrium joint dynamics of capital prices and size of the financial sector in closed-form.[1]

We assume that capital is held by equally risk averse heterogeneous h/entrepreneurs and an aggregate financial sector.[2] Financial frictions are introduced by

[1]From the methodological point of view, we follow the approach proposed by Brunnermeier and Sannikov (2014), however, the economic model is largely different focusing on the leverage of the financial sector rather than on leverage of non-financial firms. Morover, the role of the financial sector in our model is primarily to mitigate risk, and thus to provide a risk-free asset to the h/entrepreneurs rather than monitoring the borrowers.

[2]The objective function of the financial sector is to maximize the utility of the discounted stream of dividends (consumption) paid out to the bankers, who are risk-averse (log-utility). This

assuming *segmented financial markets*. In the spirit of Diamond (1984), the financial sector has a cost advantage at pooling idiosyncratic risks. It collects risky claims issued by different firms after the payment of an intermediation (monitoring) cost and offsets its assets with its own capital endowment plus bond issuance.

Due to market segmentation, each household is also an entrepreneur that invests in its own specific firm only, sustaining both systematic and idiosyncratic risks. Since h/entrepreneurs are not able to diversify idiosyncratic risks among themselves, they purchase a risk-free bond issued by the financial sector. The bond acts as an instrument of risk mitigation and allows h/entrepreneurs to smooth consumption.

In equilibrium, the h/entrepreneurs' demand for risk-free bonds is supplied by the financial sector, that uses it to leverage its balance sheet. Firms are therefore financed by both h/entrepreneurs and financial intermediaries that in our framework provide venture capital, that is, they bear a fraction of all the risk of the firm asset rather than financing them via debt securities. Therefore, firms neither do leverage nor default.

What is also important to stress is that, differently from Brunnermeier and Sannikov (2014), h/entrepreneurs do not leverage either.

We silence both the channels of firms' and households' leverage on purpose because we aim to focus on the effect that financial sector leverage has on the business cycle, without the indirect effects that leverage of non-financial firms' and householders' might generate (differently from Brunnermeier and Sannikov (2014) and Korinek and Simsek (2016), among others).

When an exogenous systematic shock hits the economy, the financial sector balance sheet is affected more than proportionally with respect to the one of h/entrepreneurs', thus affecting its (relative) capitalization as well as the amount of pooled idiosyncratic risk.

Because of this setting, we differentiate significantly from Brunnermeier and Sannikov (2014), since in our framework the role of the financial sector is twofolded. On the one side, it supplies the economy with risk-mitigation instruments. On the other side, it generates risk endogenously as its leverage over exposes the wealth dynamics to aggregate fluctuations. Therefore, financial sector leverage contributes both to the amplification and mitigation of exogenous systematic shocks to aggregate consumption and output fluctuations, respectively (differently than in Brunnermeier and Sannikov, 2014, where leverage of the non-financial firms

assumption prevents the financial sector from default. It also allows us to study the features of the equilibrium by its non-degenerate ergodic density.

only amplifies exogenous shocks).

Morover, it is interesting to observe that the relationship between financial sector leverage and exogenous systematic shocks is counter-cyclical as in Brunnermeier and Sannikov (2014) non-financial firm leverage, but for different economic reasons. In fact, in our framework, after a negative shock, the financial sector relative wealth share decreases, while its leverage increases further, to keep up with the h/entrepreneurs' higher demand for risk-free bonds. The opposite holds as a response to positive shocks: the financial sector increases its relative capitalization, its leverage reduces, and so does its supply of risk-free bonds.

The mechanism is consistent with recent empirical findings suggesting countercyclical financial leverage (see He et al., 2017).[3]

From the macroeconomic perspective, the financial sector capitalization channels exogenous systematic shocks because the width of aggregate output and consumption fluctuations largely depends on financial leverage. The dynamics of output is affected by intermediation costs both in its drift and volatility. Due to those costs, the output per unit of capital depends by a factor that negatively relates to size of the financial sector.[4]

The output drift is decreasing in the size of the financial sector due to a *pecuniary externality*: the larger the financial sector capitalization, the lower the aggregate productivity of capital (due to high intermediation costs per unit of capital), the lower the cost of capital, the lower the investment in new capital.

The output volatility is mitigated due to a *positive externality*: having a large fraction of idiosyncratic risks that are pooled by the financial sector implies that

[3] This stylized fact stays in stark contrast with previous evidence in Adrian et al. (2014) where leverage is pro-cyclical. This is due to our choice of considering financial intermediaries focusing on their activity as *central dealers* of idiosyncratic risky claims, and relates to the marginal value of the financial sector's aggregate wealth. Pro-cyclical leverage empirical evidence also features in Adrian and Shin (2010, 2013) and has a theoretical foundation in Adrian and Boyarchenko (2012). In this stream of the literature, pro-cyclical leverage is a consequence of pro-cyclical VaR leverage constraints. The problem of leverage cyclicality is also discussed in Adrian et al. (2016), where they consider the difference between market and book leverage. In particular, the paper shows that procyclical book leverage derives from financials reducing lending by reducing their debt, while countercyclical market leverage comes from the fact that more of the value of the financial firm is in the hands of the debt holders during downturns, as the share price of the bank falls.

[4] This result squares nicely with the empirical evidence in Philippon and Reshef (2012), claiming that the size of financial intermediaries relate to the remuneration of their executive; in fact, they show that the size distribution of financial firms explains about one fifth of the premium for their executives. This is relavant because financial services account for up to 25% of the overall increase in wage inequality since 1980. In particular, they argue that financiers may be overpaid from a social point of view.

capital is overall less productive (due to intermediation costs). Thus, being the size of the financial sector positively related to its own stock of capital, a negative relationship holds between aggregate financial capitalization and productivity. The latter decreases the size of capital growth rates fluctuations as driven by systematic shocks.

Another theoretical result of our model concerns the equilibrium risk-free interest rates; in our model, equilibrium risk-free rates turn negative when the financial sector is small.[5] This effect mirrors the h/entrepreneurs' higher demand of risk-free bonds, and it does not require a crisis situation to take place. Procyclical interest rates co-exists with counter-cyclical Sharpe ratios both for pooled and un-pooled risky claims.

In the last part of the paper we study agents welfare as related to the size of the financial sector.

For the h/entrepreneurs, the welfare benefit of risk mitigation is counterbalanced by the associated -indirect- cost, that is paid in terms of a lower individual and aggregate consumption growth.

Overall, we find that the h/entrepreneurs benefit the most when the financial sector is neither too small (offering too little -and costly- risk mitigation) nor too big (so that h/entrepreneurs have a lower level of capital).

Motivated by this finding, we investigate whether static leverage constraints and redistributive taxation policies could be welfare improving for the h/entrepreneurs. According to our model: a) A tax that redistributes wealth from the financial sector to the h/entrepreneurs' prevents the former from growing too large, and so to waste too many resources after intermediation costs; b) Leverage constraints contribute at reducing both drift and diffusion of relative wealth process between sectors, thus affecting the equilibrium macroeconomic dynamics. The direct effect of the prudential policy is to stabilise the dynamics of aggregate consumption, while it negatively affects the mitigation of the output fluctuations. When the constraint is binding, a further drawback consists of an impaired capability of intermediaries at rebuilding their wealth.

The welfare effect of the policy is thus two-sided: from the perspective of the financial sector, the result is a net welfare loss. This is because when the leverage constraints are binding, they keep the financial sector relative capitalization at a low(er) level with a high(er) probability. Conversely, leverage constraints may be welfare-improving for the h/entrepreneurs, as there exists a trade-off between the

[5] According to Gourinchas and Rey (2017), a weakened financial sector may lead to persistently low, or even negative, short term interest rates for an extended period of time.

gain from the higher growth rate of their consumption versus the loss due to a weakened financial sector (less mitigation).

This suggests that there exist leverage constraints and redistributive taxation policies such that the size of the financial sector remains within an "optimal" range in order to improve h/entrepreneurs' welfare.

The paper proceeds as follows. In Section 1.1, we frame our results as related to the incumbent literature. In Section 2, we outline the model micro-foundation. We start by introducing production technologies and returns on risky assets (2.1). Then, we introduce the agents and their optimization problem (2.2).

In Section 3, we derive the competitive equilibrium (3.1) and discuss the benchmark cases of full-risk-pooling and no-risk-pooling (3.2). In Section 4, we focus our analysis on the intermediate case where both classes of agents co-exist and characterise the link between financial sector leverage and the dynamics of real (4.2) as well as financial (4.3) macro-variables. Finally, in Section 5, we study the effects of risk pooling and mitigation on welfare and investigate the role of leverage constraints and redistributive taxation policies at increasing the h/entrepreneurs' welfare. Section 6 concludes.

1.1 Related Literature

This paper belongs to the body of literature describing the relationship between financial intermediation, macroeconomic dynamics, asset pricing, and their implications for welfare.

Methodologically, we are close to the seminal work of Brunnermeier and Sannikov (2014). However, we substantially diverge in several dimensions. While in their model the most productive agents (experts) leverage their balance sheet, in our model the h/entrepreneurs do not leverage, even if they are the most productive agents. Conversely, it is the financial sector that leverages up and sells to the h/entrepreneurs risk-free bonds in exchange of a fraction of risky equity capital of their firms.

What follows is that, in our model, more productive agents have extra risk exposure, and thus demand for mitigation instruments. On the contrary, financial intermediaries, even if less productive, provide risk-pooling (and mitigation) by buying risky claims from the h/entrepreneurs, and risk mitigation by issuing risk-free bonds that they sell to h/entrepreneurs and therefore leverage up.

It is also relevant to stress that, whereas in our model positive relative shocks favor the financial sector, in Brunnermeier and Sannikov (2014) they benefit the experts.[6]

Another important difference concerns the financial frictions. In Brunnermeier and Sannikov (2014) model, the friction is that the experts' consumption must hold positive, whereas the consumption of households' may be negative. In our case, the friction comes after the assumption of *market segmentation*. These differences lead to substantially opposite equilibrium dynamics and ergodic wealth share distribution. In these terms, our model is complementary to their.

More specifically, we can structure our contribution along the following dimensions: the role of exogenous (systematic and) idiosyncratic risks in a dynamic model with frictions (IR); the role of financial sector leverage (LV) and size in amplifying but also mitigating the propagation of exogenous idiosyncratic and systematic shocks (AM), as well as their effect over the business cycle, consumption, and their fluctuations (BC); how the allocation of risk and market segmentation relate to asset pricing (AP); the welfare implications of leverage and size of the financial sector (W).

An early approach connecting the allocation of risk to portfolio choices *(IR)* in a general equilibrium set-up is in Heaton and Lucas (2004). Their analysis builds on the observation that idiosyncratic risk is priced by the market, since agents are risk averse and unable to diversify idiosyncratic shocks by themselves. Nevertheless, they do not consider any financial sector, nor the connection between financial leverage and asset pricing.

By introducing *market segmentation*, our model also relates to the body of literature that studies *incomplete* markets and the role of aggregate uninsurable shocks in equilibrium dynamics. Seminal papers in this field are Aiyagari and Gertler (1991), Huggett (1993), as well as Aiyagari and Rao (1994), where precautionary savings and transaction costs are introduced in a general equilibrium exchange economy.[7] A cornerstone theoretical contribution that relates to the field is Allen and Gale (1994), where the relationship between incomplete market participation of households and asset pricing volatility is studied in a static setting.[8]

[6] This happens because, in both models, positive aggregate shocks structurally advantage the leveraged counterpart, since they increase more than proportionally its relative share of aggregate wealth.

[7] A comprehensive review of these models and applications in continuous time with a focus on their macroeconomic implications is in Achdou (2017).

[8] A recent contribution introducing a different form of *market segmentation* in a general equilibrium model with banks and aggregate risk is Gale et al. (2018). In their paper, segmented markets imply that the markets for capital and output are segregated from each other at different

In this paper, *endogenous risk* takes place as an amplification/mitigation (AM) of exogenous systematic shocks.[9] As in Brunnermeier and Sannikov (2016), our model features the so called *volatility paradox* (see also Adrian and Brunnermeier, 2016), i.e. lower exogenous risk may lead to higher endogenous volatility, especially when financial capitalisation is arbitrary low. Nevertheless, our model differs in several substantial ways.

First, we account for both systematic and idiosyncratic risks as determinants of aggregate fluctuations. This feature squares with empirical evidence suggesting a relationship between macroeconomic dynamics and the state of the financial system (Adrian et al., 2019). Second, in our model the effect of increasing idiosyncratic risk leads to further leverage. This is because, after *market segmentation*, the h/entrepreneurs increase their demand for risk-free bonds.[10] Another relevant element of our model is that equilibrium risk-free interest rates fluctuate over time (and may take negative values) instead of being constant.[11]

As for *LV*, our paper moves along the seminal stream accounting for financial frictions in general equilibrium (for a general discussion see Moritz and Taylor (2012) and Brunnermeier et al. (2012)) and, more specifically, to those known as *post-crisis* macro models (see Haven et al., 2016).

An important feature we share with this literature is the connection between financial leverage and the amplification of shocks, originally modelled as a *financial accelerator*.[12]

The idea of the financial cycle being determinants of the business cycle is introduced in Carlstrom and Fuerst (1997). A similar setting with adjustment costs on capital investment is in Kiyotaki and Moore (1997), and it is developed in a New Keynesian setting by Bernanke et al. (1999). A more recent contributions

consecutive times, and the banks are the only ones allowed to produce new capital.

[9]Endogenous risk is generated by the allocation of wealth between different classes of agents and channels by the leverage mechanism through which the financial sector pools risky claims. The mechanism is introduced in Brunnermeier and Sannikov (2014) and has its roots in Hayashi (1982), where the price of physical capital relates to investments.

[10]In Brunnermeier and Sannikov (2014), for instance, the opposite happens, because higher idiosyncratic risks increase the borrowing cost of experts, and thus decrease their equilibrium leverage.

[11]In Brunnermeier and Sannikov (2014) and Phelan (2016) instead, agents are risk neutral, and thus the equilibrium risk-free rate equals the discount rate of the most conservative class of agent.

[12]The *financial accelerator* mechanism works as follows: as investment demand increases, asset price increases. Since the agents may use assets as collateral, this improves their balance sheet condition, and so the external finance risk premium shrinks. This pushes forward demand for investments and so on, thus generating amplification.

is, among the others, Korinek and Simsek (2016). In their paper, they focus on the effect of macro-prudential policies on the household leverage. Conversely, we focus on the intermediaries side.[13]

Still concerning LV, the core difference between the aforementioned stream of the literature and our paper consists of the source of frictions, *market segmentation* in place of an agency problem, and thus of the nature of the amplification.

What follows is that the existence of uninsurable systematic risk is required for amplification to take place (the idea is introduced in a theoretical setting by Krishnamurthy, 2003). It is relevant to highlight that the core effect of introducing financial frictions through occasionally binding constraint is that central theorems of welfare do not hold, and the equilibrium risk allocation is inefficient (as for example in Mendoza and Bianchi, 2010; Bianchi, 2011).[14]

From the asset pricing perspective (AP), our contribution has common characteristics with the literature of general equilibrium models where financial cycles and constraints determine asset prices, as for example He and Krishnamurthy (2011, 2013).

These models are able to reproduce the observed rising Sharpe ratios and falling risk-less interest rates during crises, but do not generate negative risk-free interest rates.

Likewise, we match similar patterns when the financial sector is poorly capitalised. Besides, we consider a production rather then an exchange economy, thus we can associate Sharpe ratios and interest rates to the dynamics of financial leverage, as well as consumption and business cycle fluctuations. Another important difference in our model is that we do not need the constraint to be always binding in order to generate those effects. Moreover, we explicitly micro-found the demand for risk-free assets and show that in certain states of the world the real risk free interest rate could be negative. In our model, this happens because markets are incomplete. Accordingly, risk mitigation instruments are exclusively supplied by financial intermediaries.

When the relative capitalization of the financial sector is too small, i.e. its capital is scarce, intermediaries shall leverage heavily to supply risk-free assets

[13] Recent relevant papers in this field, that nevertheless do not consider idiosyncratic risk, are Angeloni and Faia (2013), Angeloni et al. (2015), Ansgar (2016), Nuno and Carlos (2017) and Rampini and Viswanathan (2017).

[14] We obtain a similar effect by *market segmentation*, yet suboptimal allocation is not contingent and happens also when the constraint is slack. An interesting exercise discussing how to counter pecuniary externality in a financial accelerator framework is in Korinek (2011). The paper proposes a tax based disincentive to extreme leverage.

to the households. For this reason, h/entrepreneurs are willing to pay -*negative* risk-free rates- in order to avoid their exposure to idiosyncratic and systematic shocks.

In light of the role of financial leverage and size as related to the business cycle, BC, and risk-free interest rates during the crisis (see He et al., 2010), our results relate to those papers at the intersection between finance and macroeconomics treating systemic risk, as for example Nuno and Rey (2017).

With Nuno and Rey (2017) we share the trade-off between economic growth and stability, although our mechanism of amplification is deeply different, and so it is our definition of systemic crisis.[15] In these terms, our model is similar to He and Krishnamurthy (2019) where systemic risk is the conditional probability of reaching binding constraint states. However, this is not the focus of our paper.

Finally, we investigate the relationship between the size of the financial sector, leverage, and welfare (W). We show that the h/entrepreneurs' welfare dynamically relates to the size of the financial sector, and so to its provision of risk-mitigation via leverage. However, we show that either a small size (and high leverage) or a large size of the financial sector could be detrimental for the welfare of h/entrepreneurs'. On this side, we are related to the work of Philippon (2010) that studies the interaction between the financial and non-financial sectors, and investigates whether it is optimal to subsidize or tax the former. Notwithstanding these few common elements, our model is largely different, and so it is the role of the financial sector. We also partially relate to the literature that investigates optimal financial leverage constraints, and in particular to Phelan (2016) and Pancost and Robatto (2018).

A common element between this work and Phelan (2016) is the relationship between financial leverage constraints and welfare. His paper suggests that a policy of recapitalizing banks, that mechanically decreases leverage to the optimal level, is welfare-improving. This relates to the concept of *welfare maximizing size* of the financial sector suggested by our model. Nevertheless, we strongly differ with respect to several aspects. First, our model depicts counter-cyclical rather then pro-cyclical leverage.[16] Second, we introduce *market segmentation* as

[15]To Nuno and Rey (2017) systemic risk is the probability of intermediaries default whereas in our model it can be interpreted as the probability of being below a certain capitalisation threshold.

[16]This results is due to the fundamentally different nature of the financial intermediaries in our models. Conversely, in Phelan (2016) the intermediaries are risk neutral and have a comparative advantage at producing capital goods, in our model they are risk-averse, and have the role of pooling idiosyncratic risky claims.

a friction that allows us to model the demand for risk-free assets. Moreover, our model displays a smooth dynamics rather then a step-wise process of aggregate consumption. This allows us to relate financial leverage to the economic macro-dynamics. Third, our equilibrium risk-free interest rate is not fixed. On the contrary, it is state-dependent and may take negative values.

Similar to our setting, Pancost and Robatto (2018) consider the role of the banks in providing risk pooling services as well as their role of supplying risk mitigation instruments through deposits.[17]

Moreover, both our papers draw their conclusions by focusing on the h/entrepreneurs' welfare, while do not (directly) consider the financial intermediaries' welfare. In this setting, we reach similar conclusion concerning the welfare improvement that may come after imposing leverage constraints. However, our papers differ substantially in several aspects. First, the focus of our paper is on the relationship of financial sector leverage with the business cycle; for this reason we consider both idiosyncratic and systematic shocks. Vice versa, Pancost and Robatto (2018) consider only idiosyncratic shocks. Second, in Pancost and Robatto (2018) the main objective is to determine the optimal capital requirement regulation by exploiting the trade-off between good and bad risk taking.

In summary, the strength of our model and of its theoretical predictions is its ability to jointly consider four very different dimensions: the role of systematic and idiosyncratic risks, how their allocation relates to asset pricing and stems into the amplification/mitigation of exogenous systematic shocks, the role of leverage constraints for the dynamics of the financial sector capitalisation, and their effect over the macroeconomic dynamics. In this term, the general equilibrium framework allows us to disentangle the interlinks along these four dimensions. For this reason, our model is unique in the literature.

2 The Model

In this section, we introduce the economic environment, together with the associated productive technologies, and describe the features of risky assets returns. We then discuss the agents' problem. We start the section with a narrative description of the model.

We consider a continuous-time infinite-horizon production economy with two non-fungible goods: physical capital (such as a tree) and output (perishable good,

[17]A similar argument has been proposed by DeAngelo and Stulz (2015).

such as apples). Each good is produced by a specific type of firms, the perishable good acts as numéraire. The capital producing business is inter-temporal and risky. The perishable good producing business is non inter-temporal and risk-free.

There are two types of assets: risky claims and risk-free bonds. Risky claims are written on the net revenues of capital-producing firms. The risk to which they are exposed is both systematic (economy-wide) and idiosyncratic (firm-specific). Risk-free bonds have value as risk-mitigation instruments and are in zero-net supply.

The economy is populated by two classes of agents: *h/entrepreneurs* and *financial intermediaries*. Their risky assets holdings differ as follows. Financial intermediaries are allowed to pool idiosyncratic risk and thus are exposed to systematic risk only. The expected return on their risky assets is reduced by a cost of intermediation for each unit of capital.[18] Conversely, each entrepreneur is allowed to invest in one capital producing firm only.

Since h/entrepreneurs do not pay the intermediation cost, they earn higher expected returns. However, their over-exposure to idiosyncratic risk generates positive demand for risk-mitigation instruments. As we shall see, in equilibrium, this demand will be satisfied by the financial sector through its short position in risk-free bonds, leading to leverage. This friction can be read as a market structure where the set of investment *opportunities available* to each agent is tied up to the class she belongs to.[19,20]

The share of risky claims that is left un-pooled, i.e. that remains in the hands of the h/entrepreneurs', determines the idiosyncratic risk allocation in the economy and with it consumption, output, risky assets, and their prices in equilibrium. Key parameters shall be the size of systematic and idiosyncratic risk, and intermediation costs.

[18] The intermediation cost can be thought as a reduced form that represents the administrative costs that the intermediaries bear for screening and monitoring the firm activity, that instead the entrepreneur observes, as well as for operational purposes.

[19] From the firms perspective, the limited access to financial markets is an exceedingly relevant topic. For instance, Davydiuka et al. (2018) provide a theoretical model that motivates the substantial decline of small firms going public in the last 20 years (as documented in Gao et al., 2013) by the presence of increasing financial frictions, such as IPO and regulatory-disclosure related costs.

[20] In Appendix B we show that, even if both h/entrepreneurs and financial intermediaries have full access to pooled and un-pooled risky assets, there exists *market segmentation* (incomplete market participation) as long as the intermediaries are more efficient at pooling claims than h/entrepreneurs.

2.1 Technologies and Risky Claims

There exist two types of firms: Type I has the inter-temporal role of generating new physical capital (trees) through a concave technology $\Phi(\cdot)$ that uses the perishable good (apples) as input. Let $[0,1]$ be a continuum of type I firms and let $dW_t \perp d\tilde{W}_t^i \perp d\tilde{W}_t^j \; \forall i \neq j, \{i,j\} \in [0,1]$ be independent standard Brownian motions defined on the filtered probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where $\{\mathcal{H}_t, t > 0\}$ is the natural filtration over the measurable space $(\Omega, \mathcal{H})$. The capital stock $k_t^i \in \mathbb{R}$ managed by firm $i \in [0,1]$ follows a bi-variate Itô diffusion

$$T_t^i: \quad \frac{dk_t^i}{k_t^i} = \left(\Phi(\iota_t^i) - \delta\right) dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t{}^i, \text{ with } \Phi(\iota) = \frac{1}{\theta} \log\left(1 + \theta\iota\right), \tag{1}$$

where δ is the depreciation rate, ι_t^i is the reinvestment rate as dependent on the concavity parameter θ, σ and $\tilde{\sigma}$ are constant systematic and idiosyncratic diffusion terms, respectively.[21] We use q_t to denote the equilibrium price of physical capital in terms of the perishable good.

Firms of type I earn revenues by renting capital to firms of type II at the instantaneous price p_t. Firms of type II, also a continuum, do not have an inter-temporal dimension, and produce perishable good y_t^i through a linear production function that has capital as input:

$$y_t^i = Ak_t^i. \tag{2}$$

The profit of the i^{th} firm of type II at time t is thus simply $(A - p_t)k_t^i$.

Note that since production technologies are linear, firms break even and earn no profits in equilibrium.[22] The ownership of both types of firms is thus irrelevant. In t, each firm of type I finances its activities by issuing risky claims with pay-off in $s = t + dt$ equal to its net revenues. The return of firm i risky claim between t and $s = t + dt$ is dR_t^i.

2.2 Financial Sector and h/entrepreneurs

The economy is populated by h/entrepreneurs and financial intermediaries. H/Entrepreneurs consist of a continuum of heterogeneous agents of unit mass

[21]The stochastic process driving the dynamics of physical capital can be interpreted as stochastic depreciation (see Wälde, 2011).

[22]Type I firm technology in non-linear in ι, however linearity in k is maintained through the identification of $c = \iota k$ as expense for the perishable consumption, see Appendix A.

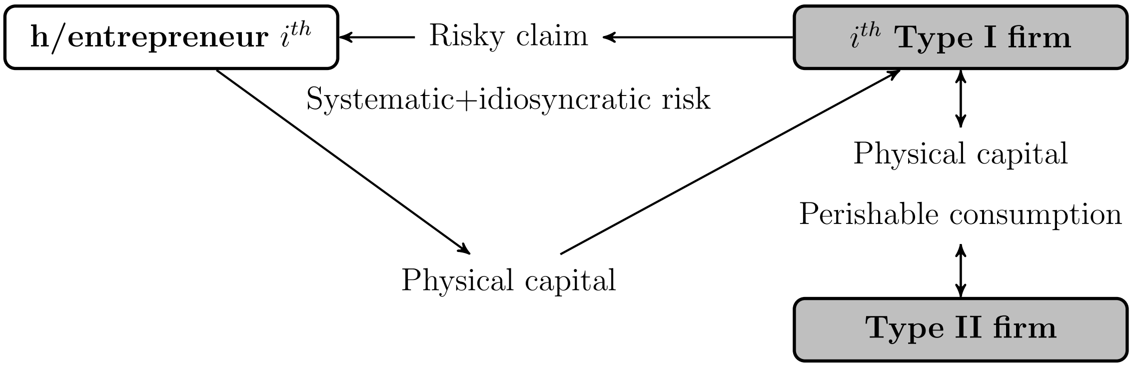

Figure 1: Micro-structure of production and risky claims for the h/entrepreneurs

$\mathbb{H} := [0,1]$ indexed by $h \in \mathbb{H}$. Similarly, the intermediaries belong to $\mathbb{F} := (1,2]$ and are indexed by $f \in \mathbb{F}$. Since the latter are homogeneous, they can be accounted for as a representative *financial sector*.

The i^{th} agent has an initial endowment $k_0^i \neq 0$ such that her wealth equals $e_0^i = k_0^i q_0$. In each time interval $[t, t+dt)$, agent i consumes at a rate $\frac{c_t^i}{e_t^i}$ and allocates a fraction ω_t^i of what is left to risky claims and a fraction $(1-\omega_t^i)$ to risk-less bonds. All agents have log utility and discount the future at the same rate ρ; they are infinitely lived and chose c_t^i and ω_t^i to maximize their objective function

$$V_0 = \max_{\left\{c_t^i, \omega_t^i\right\} \in B^i} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \ln c_t^i dt \right], \quad i = h, f \tag{3}$$

subject to

$$B_t^i: \quad \frac{de_t^i}{e_t^i} = \omega_t^i dR_t^i + \left(1 - \omega_t^i\right) r_t dt - \frac{c_t^i}{e_t^i} dt, \tag{4}$$

where r_t is the risk-free interest rate, and the i^{th} agent has access to a different risky portfolio with return dR_t^i.[23]

The financial sector can invest the stock of physical capital at its disposal across all firms, against the payment of an intermediation cost η per unit of capital.[24]

[23] The derivation of the solution is in Appendix C.1. With a slight abuse of notation we use dR_t^i to denote the return to the agent i of firm $i = h$, and dR_t^f to denote the return of the aggregate portfolio that pools risky claims issued by all firms $i \in [0,1]$.

[24] A seminal paper that develops a theoretical framework where financial intermediation costs associate to a net advantage due to diversification is Diamond (1984). In an economy where all the agents are risk averse, the paper shows that financial intermediaries must have lower deleg-

The intermediation cost can be thought as a reduced form that represents the administrative costs that the intermediaries bear for screening and monitoring each firm, which the entrepreneur instead observes, as well as for operational purposes.

Conversely, due to *market segmentation*, the h/entrepreneurs cannot diversify among firms (risky claims): the i^{th} entrepreneur can invest only in the i^{th} firm.[25]

Firms are therefore financed by both h/entrepreneurs and financial intermediaries that, in our framework, provide venture capital services.

Thus, the return on risky claim i, dR_t^i, has the following structure:

$$dR_t^i = \underbrace{\mu_t^i dt}_{\text{Expected return}} - \underbrace{[\mathbb{1}_{i=f}] \frac{\eta}{q_t} dt}_{\text{Intermediation cost}} + \underbrace{\sigma_t dW_t}_{\text{Systematic risk}} + \underbrace{[\mathbb{1}_{i=h}] \tilde{\sigma} d\tilde{W}_t^i}_{\text{Idiosyncratic risk}}, \tag{5}$$

where $\mathbb{1}_i$ is the indicator function. Note that the expected return μ_t^i and systematic risk σ_t shall be determined in equilibrium as dependent on the firms' optimizing behaviour.

As we shall see, the portfolio choices of h/entrepreneurs' and intermediaries' in equilibrium will differ substantially, due to their asymmetric exposure to idiosyncratic and systematic risks as well as to the expected returns of their risky assets.

In particular, each entrepreneur will finance capital producing firms proportionally to the share of its wealth that is not allocated in risk-free bonds. Antithetically, intermediaries will finance firms by means of their own endowment plus the stock of capital they acquire from the h/entrepreneurs versus the issuance of risk-free bonds.

To synthetically represent the relationship between the i^{th} entrepreneur and the i^{th} firm, in Figure 1 we depict the micro-structure of production from the h/entrepreneurs' perspective. Similarly, Figure 2 represents the mechanism by which the financial sector may purchase a fraction of the h/entrepreneurs' physical capital versus the issuance of risk-free bonds.

It is relevant to highlight that *market segmentation*, which we have assumed

ation costs than an entrepreneur to viably provide intermediation services. This intermediaries centralized monitoring structure will mean that there are not active markets for their pooled assets. This relates to the concept of segmented financial markets, being the aggregate financial sector the only one supplying risk-mitigation instruments. From an empirical perspective, the side effect of risk pooling at financial institutions is treated, among the others, in Wolf (2010) and van Oordt (2014).

[25] As we have already stressed, this assumption makes us closer to reality, where the majority of SMEs have limited access to capital markets.

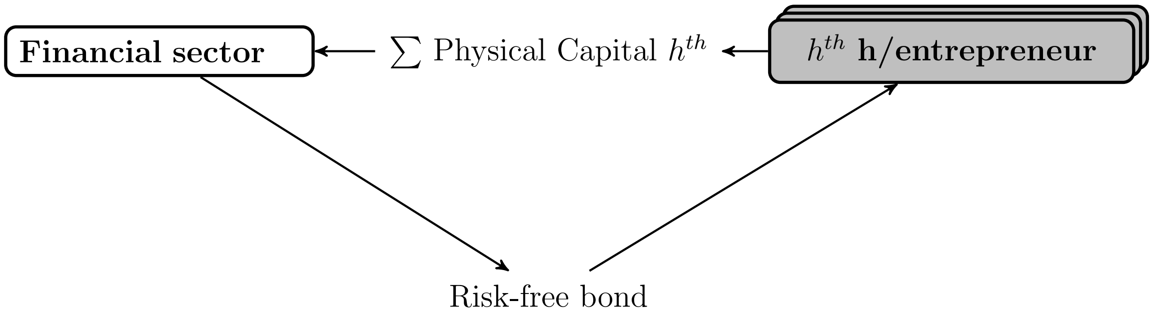

Figure 2: The financial sector and its purchase of a fraction of the h/entrepreneurs' physical capital versus the issuance of risk-free bonds.

to be an exogenous financial friction, may emerge in equilibrium in presence of *transaction costs* on the h/entrepreneurs' side to pool idiosyncratic risky assets. In Appendix B we show that, even if both h/entrepreneurs and financial intermediaries have full access to risk-free bonds, as well as pooled and un-pooled risky assets, there exists *market segmentation* as long as the intermediaries are more efficient at pooling claims than h/entrepreneurs, and the transaction cost is not too large.[26]

3 The Equilibrium Dynamics

In this section, we outline the main steps to derive the *competitive equilibrium* of this economy.

First, in Section 3.1, we define the equilibrium, specify the associated return on risky assets, and characterise the unique state variable: the relative capitalization of the financial sector.

Second, in Section 3.2, we introduce and discuss the benchmark cases of the *full-risk-pooling* economy, where the financial sector holds the whole capital stock, as well as the *no-risk-pooling* economy, where there is no financial sector.

Henceforth, we denote all the aggregate variables with a capital letter.

[26]It is relevant to stress that the presence of capital markets is not self-sufficient to solve the monitoring problem, as long as the transaction cost is higher for the h/entrepreneurs' than for the financial sector. See Diamond (1984).

3.1 Competitive Equilibrium

Informally, the equilibrium consists of maps from histories of systematic shocks to prices (capital prices, returns on risky claims, risk-free interest rates), production choices and consumption choices, as well as asset allocations such that firms maximise their profits, agents maximise their expected utility, and markets clear. Formally:

Definition 1. ***Competitive Equilibrium***
Conditional on an initial allocation of capital among the agents, an equilibrium is an adapted stochastic process that maps histories of systematic shocks $\{dW_t\}$ *to prices* $\{q_t\}$*, returns on risky claims* $\left\{dR_t^h, dR_t^f; h \in \mathbb{H}\right\}$*, risk free rates* $\{r_t\}$*, production choices* $\{k_t^i, \iota_t^i; i \in [0,1]\}$*, consumption choices* $\left\{C_t^h, C_t^f; h \in \mathbb{H}\right\}$*, and asset allocations* $\left\{\omega_t^h, \omega_t^f; h \in \mathbb{H}\right\}$ *such that:*

1. *Firms maximise their profits:*

   (a) Firms of type I

$$\left\{k_t^i, \iota_t^i\right\} \in \arg \max_{\left\{k_t^i, \iota_t^i\right\} \in T^i} \left\{\mathbb{E}_t \left[k_s^i q_s e^{\int_t^s \frac{p_u - \iota_u^i}{q_u} du}\right] - k_t^i q_t\right\}, \forall i \in [0,1]; \quad (6)$$

   (b) Firms of type II

$$k_t^i \in \arg \max_{k_t^i \geq 0} \left\{(A - p_t)\, k_t^i\right\}, \forall i \in [0,1]. \quad (7)$$

2. *Agents maximise their utility:*

$$\left\{c_t^i, \omega_t^i\right\} \in \arg \max_{\left\{c_t^i, \omega_t^i\right\} \in B^i} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \ln c_t^i dt\right], \forall i \in \mathbb{H} \cup \mathbb{F}.$$

3. *All markets clear:*

   (a) Risky asset

$$\int_{\mathbb{F}} \omega_t^f e_t^f df + \int_{\mathbb{H}} \omega_t^h e_t^h dh = K_t q_t; \quad (8)$$

   (b) Bond

$$\int_{\mathbb{F}} (1 - \omega_t^f) e_t^f df + \int_{\mathbb{H}} \left(1 - \omega_t^h\right) e_t^h dh = 0, \quad (9)$$

(c) Consumption

$$\int_{\mathbb{F}} \left(A - \iota_t^f - \eta\right) k_t^f df + \int_{\mathbb{H}} \left(A - \iota_t^h\right) k_t^h dh = C_t^f + C_t^h; \tag{10}$$

(d) Capital

$$\int_{\mathbb{F}} k_t^f df + \int_{\mathbb{H}} k_t^h dh = K_t. \tag{11}$$

An equilibrium snapshot of the agents' balance sheet at any instant of time t is in Figure 3. The dark grey boxes depict the asset allocation of each class of agents while the light grey boxes represent their liabilities.

The financial sector holds a long (leveraged) position in the aggregate portfolio of risky claims that is financed by both its own capital endowment $\int_{\mathbb{F}} e^f df = E^f$ plus a short position in risk-free bonds $\left(\omega^f - 1\right) E^f$ (the former left-hand side addend of Equation 9). Conversely, each entrepreneur allocates its wealth between a single risky asset and a risk-free bond (the former left-hand side addends of Equation 8 and 9, respectively). Market clearing conditions imply that risk-free bond is in zero net supply (Equation 9) and financial sector capital and aggregate h/entrepreneurs' wealth sum up to the aggregate wealth within the economy $K_t q_t$ at any t (Equation 8). Accordingly, the stock of wealth that belongs to the aggregate of entrepeneurs holds as $\int_{\mathbb{H}} e^h dh = E^h$.

To place the last piece of the jigsaw, Figure 4 shows the balance sheet of the j^{th} capital producing firms at any time t. As for the h/entrepreneurs' and financial sector balance sheets in Figure 3, the dark grey box represents the value of the firm's assets, whereas the light grey ones depicts the value of its liabilities. Basically, each capital producing firm is jointly financed by h/entrepreneurs' and financial intermediaries' capital stock, that is they bear a fraction of all the risk of the firm's asset rather than financing them via debt security. Therefore, firms neither do leverage nor default.

In summary, each firm collects physical capital from both h/entrepreneurs' and intermediaries' (straight arrows) versus the issuance of risky claims written on its net revenues (dashed arrows). In particular, the j firm gathers capital $\omega^h e^{h,j}$ from the j^{th} entrepreneur as well as from the financial sector, that evenly finances the continuum of capital producing firms, so that $\int_{\mathbb{F}} \omega^f e^{f,j} df = \int_{\mathbb{J}} \omega^f e^{f,j} dj = \omega^f E^f$.

The market price of the risky claim q and the dynamics of its returns, as related to the firms' optimal policies in (6) and (7), are discussed at length in Appendix A. In the aggregate, the mass of capital under both types of firms' management

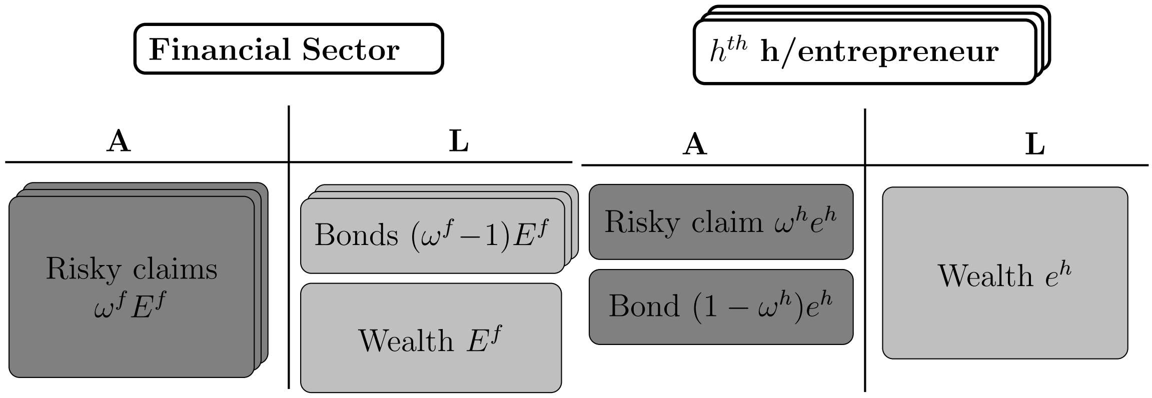

Figure 3: Synthetic agents' balance sheets at time t

sums up to the total stock of capital K within the economy, so that (11) holds.

To further characterise the dynamics of the competitive equilibrium in Definition 1, we shall restrict our search to the class of dynamically simple equilibria.[27] Moreover, we look for equilibria where the stochastic process that drives the price of physical capital q_t is an Itô diffusion.

Assumption 1. ***Price of Physical Capital Dynamics***
The price of physical capital evolves as an Itô diffusion:

$$dq_t := q_t\mu_t^q dt - q_t\sigma_t^q dW_t, \tag{12}$$

where μ_t^q and σ_t^q are $\mathcal{H}_t$-adapted functions.

According to Assumption 1, the dynamics of capital price is not affected by idiosyncratic shocks. Moreover, the minus sign to the diffusion term implies that positive shocks to capital stock affect negatively the price of unit of capital in consumption good.[28]

Before moving to the state variable and its dynamics, we characterize the equilibrium dynamics of risky assets returns dR_t^i and discuss the relationship between the expected returns of risky assets held by financial sector and h/entrepreneurs. To do so we derive the optimal decisions of firms.

[27]The equilibrium is dynamically simple, i.e. it is time homogeneous and Markov in the state variable and it is such there exists an associated ergodic distribution. For a formal definition see Duffie et al. (1994)

[28]This choice is fundamental, as we shall see in the proof of Theorem 1 there does not exist an equilibrium with $\mathbb{C}ov_t\left[dk_t^i, dq_t\right] > 0$.

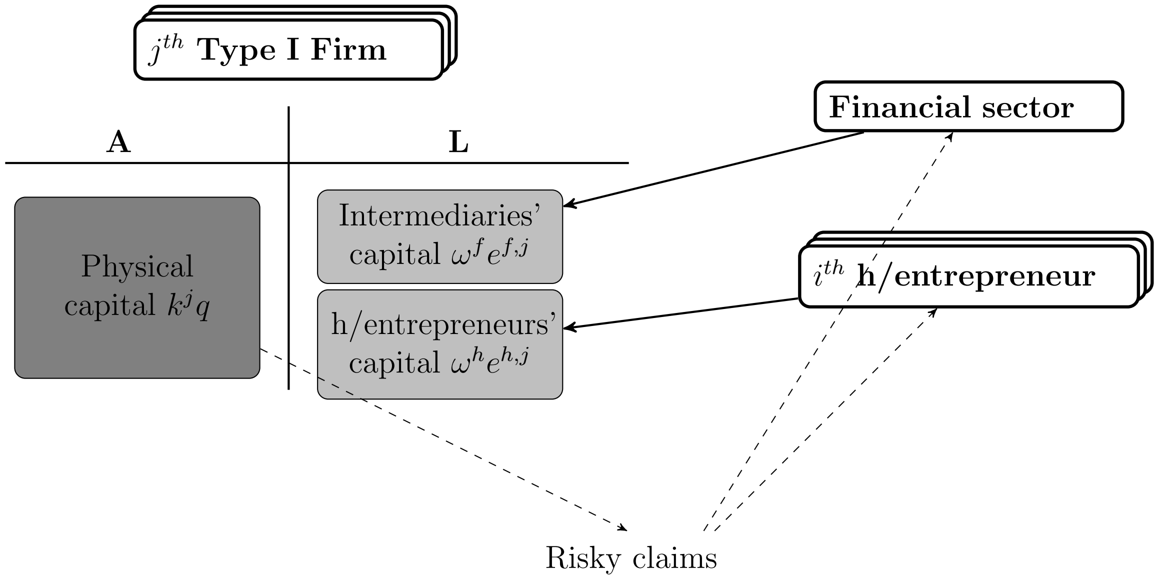

Figure 4: Synthetic balance sheet of the j^{th} capital producing firms at time t

As already introduced in Section 2.1, the firms in charge of producing new capital dispose of the physical stock of agents k_t^i, and rent it to the firms of type II that use it to produce output by the technology in Equation (2) at a price p_t. Since the marginal productivity of type II firms is constant, the equilibrium rental rate is also constant, $p_t = A$. Firms of type I decide how much perishable good Y_t to demand for generating new capital. Both components gives rise to the firm i *dividend yield* term of μ_t^i:

$$\frac{A - \iota_t^i}{q_t}.$$

The dividend yield represents the share of the asset expected return paid in value of consumption and not re-invested to generate new capital at $t + dt$.

Aside the dividend yield there is the capital gain obtained by the resale value of capital. By Itô's Lemma, given the law of motion of k_t^i in (1) and the dynamics of q_t in (12), the gain on capital stock value of the i^{th} agent evolves according to:

$$\frac{d\left(k_t^i q_t\right)}{k_t^i q_t} = \left(\Phi(\iota_t^i) - \delta + \mu_t^q - \sigma_t^q \sigma\right) dt + \left(\sigma - \sigma_t^q\right) dW_t + \tilde{\sigma} d\tilde{W}_t^i,$$

where ι_t^i is the fraction of total output demanded to generate new capital and $\Phi(\cdot)$ is the investment function. Firm type-I profit maximization implies that ι_t^i and q_t

satisfy the first order condition

$$\Phi'(\iota_t^i) = \Phi'(\iota_t^j) = \frac{1}{q_t}, \tag{13}$$

for every couple of firms i, j.

Wrapping up, it can be shown that (see Appendix A) the total return on the i^{th} claim follows the dynamics in equation (5) with

$$\mu_t^i = \mu_t := \underbrace{\frac{A - \iota_t}{q_t}}_{\text{Dividend Yield}} + \underbrace{\Phi(\iota_t) - \delta + \mu_t^q - \sigma\sigma_t^q}_{\text{Capital Gain}}, \tag{14}$$

and

$$\sigma_t := \sigma - \sigma_t^q.$$

By equations (5) and (14), the expected return on risky assets of h/entrepreneurs is higher than for the financial sector, and so it is the associated risk. The difference is the intermediation cost ηK_t^f that the financial sector must pay in order to pool idiosyncratic risks $\tilde{\sigma}$ from different type I firms. The result is summarised in the following Lemma:

Lemma 1. ***Expected Return Gap***
In equilibrium, the expected return on risky assets of h/entrepreneurs and financial sector, μ_t^h and μ_t^f respectively, are related by

$$\underbrace{\frac{\mathbb{E}_t[dR_t^h]}{dt}}_{\mu_t^h} = \underbrace{\frac{\mathbb{E}_t[dR_t^f]}{dt}}_{\mu_t^f} + \frac{\eta}{q_t}.$$

The gap is increasing in the intermediation η and decreasing in the price level q_t.

In order to derive the equilibrium, we express optimal portfolios, drift, and diffusion of the stochastic process in (12) as functions of financial sector relative wealth share ψ_t, defined as follows:

Definition 2. ***Relative Financial Capitalization***
Let ψ_t be the financial sector's share of total capital value. Conversely, $(1 - \psi_t)$

represents the h/entrepreneurs' share of total capital value:

$$\psi_t := \frac{E_t^f}{K_t q_t}, \quad 1 - \psi_t := \frac{E_t^h}{K_t q_t}.$$

As we shall see, all relevant equilibrium quantities can be written as a function of ψ_t (see Appendix C.2).

We now have all the ingredients to derive the dynamics of the state variable ψ_t at a competitive equilibrium. Moreover, we outline the conditions such that both classes of agents survive, i.e. there exists an ergodic density of the financial sector relative wealth share. Our results are summarised in the following theorem:

Theorem 1. ***Relative Capitalization Dynamics***
Given the law of motion of q in (12), there exists a unique (Markov) competitive equilibrium and it is characterized by the following:

1. *The relative capitalization dynamics follows the diffusion process*

$$d\psi_t = \underbrace{\psi_t \sigma_t^2 \left[\left(\omega_t^f - 1 \right)^2 - \frac{\psi_t}{\sigma_t^2} \frac{\eta}{q_t} \right]}_{\psi_t \mu_t^\psi(\psi_t, q_t)} dt + \underbrace{\psi_t \sigma_t \left(\omega_t^f - 1 \right)}_{\psi_t \sigma_t^\psi(\psi_t, q_t)} dW_t. \tag{15}$$

2. *The associated dynamics of price $q_t(\psi_t)$ satisfies Assumption 1 with*

$$\begin{cases} \sigma_t^q = -\epsilon_{q,\psi} \sigma_t^\psi \\ \mu_t^q = \mathcal{A} q(\psi) \end{cases}, \tag{16}$$

   where $\epsilon_{q,\psi}$ is the physical capital price elasticity to financial sector relative wealth share and $\mathcal{A}$ is the characteristic operator.

3. *As long as the intermediation cost η is positive and not too high, the left-hand side and right hand side boundaries, $\psi = 0$ and $\psi = 1$, are never attainable,*

$$\eta \in \left(0, \tilde{\sigma}^2 \frac{1 + \theta A}{1 + \theta\rho + \theta\tilde{\sigma}^2} \right) \quad \Rightarrow \psi_t \in (0, 1) \ \forall t \in (0, \infty), \tag{17}$$

   and there exists a unique (non trivial) ergodic density $\pi(\psi)$.

4. *When the intermediation cost η lays outside the interval in (17), then the economy drifts either to the right-hand or to the left-hand side boundary, respectively. In particular:*

(a) Full-risk-pooling economy,

$$\eta = 0 \Rightarrow \mu_t^\psi > 0 \Rightarrow \lim_{t\to\infty} \psi_t = 1 \ \mathbb{P} - a.s.;$$

(b) No-risk-pooling economy,

$$\eta > \tilde{\sigma}^2 \frac{1+\theta A}{1+\theta\rho+\theta\tilde{\sigma}^2} \Rightarrow \mu_t^\psi < 0 \Rightarrow \lim_{t\to\infty} \psi_t = 0 \ \mathbb{P} - a.s.$$

Proof. Points 1 and 2 are proved in Appendix C.3. The characteristic operator $\mathcal{A}$ is defined in Øksendal (2003). Points 3, together with the characterization of the ergodic density, are discussed in Appendix C.4. Point 4 (a) is proved by setting $\eta = 0$ in the consumption market clearing condition (42). It follows that $\mu_t^q = \sigma_t^q = 0$. By point 1, $\mu_t^\psi > 0$ and thus, $\psi_t \to 1$ when $t \to \infty$. Point 4 (b) is proved similarly. □

The core implication of Theorem 1 is that we are able to express the dynamics of all relevant equilibrium variables in the model as a function of the dynamics of the relative capitalization ψ_t as expressed in point 1. Such dynamics depends also on the dynamics of the price of physical capital, q_t, as assumed in Assumption 1. Given the relation between the two dynamics in point 2, we can solve for their drift and diffusion numerically.[29]

Another important result is that, provided intermediation costs are neither too low nor too high, the relative capitalization keeps floating *around* its long-run average where both classes of agents have positive relative capitalization (point 3). In this sense, heterogeneity is *persistent*.[30]

Instead, when intermediation costs are either null or too high (depending on the size of idiosyncratic volatility), the economy collapses in one of two "extreme" cases: the *full-risk-pooling* economy and the *no-risk-pooling* economy (point 4).

In the intermediate case where both classes of agents coexist, henceforth an economy with *partial risk pooling*, it is interesting to outline the way exogenous systematic shocks affect equilibrium prices and relative wealth share dynamics altogether.

[29]See Appendix C.2 for details.

[30]Note that the long-run dynamics of ψ_t does not necessary coincide with the associated deterministic steady state where the drift is null. A discussion upon the relationship between steady-state and long-term average of the stochastic process describing the equilibrium is in Klimenko et al. (2017).

To illustrate this relationship, in Figure 5 we plot the price level q_t (left) as well as the associated drift (centre) and diffusion (right) of the financial sector relative capitalization (blue) and prices (red) dynamics as a function of the relative financial capitalization $\psi \in (0,1)$.[31]

As far as the capital prices level is concerned, the larger the relative size of the financial sector the lower the price of physical capital (Figure 5, left). This negative relation is due to the higher incidence of the costs of intermediation on the average productivity of capital when the financial sector is large. For example, in the extreme case where the financial sector manages all the capital, the intermediation cost is paid on all units of capital.

In general, positive exogenous systematic shocks shift the size ψ_t (and thus q_t) to the right towards one (minimum $q(1)$), because in equilibrium, due to leverage and risk pooling, the total return of the financial sector portfolio is higher than the return of the h/entrepreneurs. The opposite occurs for negative shocks. Importantly, the response of the relative size and capital price dynamics to exogenous shocks is state dependent, especially in the neighbourhood of the boundaries.[32] When the financial sector capitalization is small, its drift is positive (negative for q_t). As we shall see this is the result of high leverage and high Sharpe ratio. When the financial sector capitalization is big enough, instead, the drift of its relative capitalization is negative (positive for q_t). This is because the benefit of leverage are reduced (lower Sharpe ratio) while the costs associated to intermediation (proportional to η/q_t) are higher. Overall, the relative capitalization of the financial sector shrinks (while q_t increase). The central panel of Figure 5 provides an illustration.

Such dynamics is associated with a volatility that is maximal for quite low level of capitalization. This is a direct effect of leverage which amplifies external shocks.

[31] We solve the model numerically by assuming the following baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$. According to Ang et al. (2006) and Fangjian (2009), reasonable values for the annualized systematic and idiosyncratic volatilities are approximately 0.2 and 0.55, respectively. The remaining parametric specification is close to the one in Brunnermeier and Sannikov (2016) Despite these choices do not come after calibration, they produce reasonable qualitative outcomes. To verify the model robustness, in Appendix E we discuss the changes of equilibrium dynamics with respect to the key parameters in the model, namely the size of systematic and idiosyncratic risk as well as intermediation costs.

[32] It is important to stress that in our economy both classes of agents are equally risk averse and discount the future at the same rate. These assumptions, together with bonds in zero net supply and h/entrepreneurs' idiosyncratic risk exposure, imply that there does not exist an equilibrium where the financial sector purchases risk-free bonds, i.e. $\omega_t^f \geq 1$ (the financial sector is always leveraged).

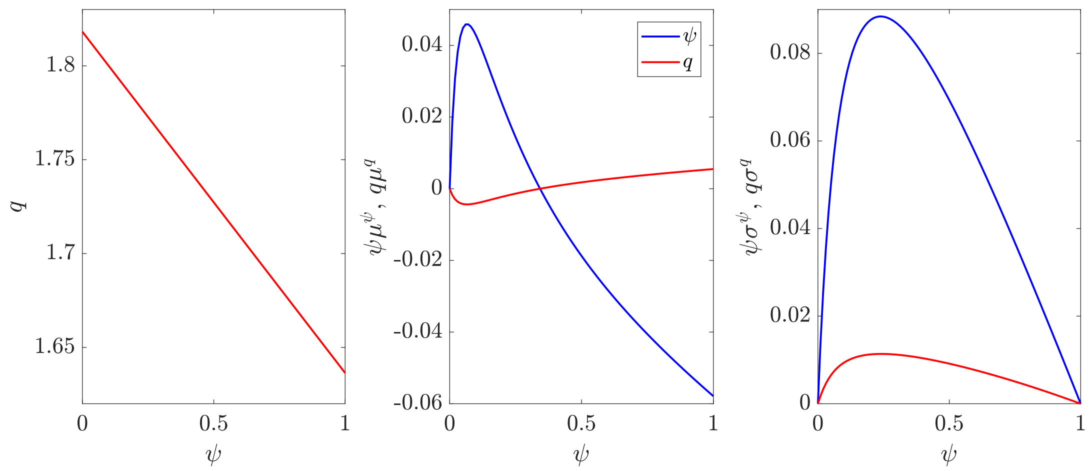

Figure 5: Left: Price level $q(\psi)$. Centre, right: Drift (centre) and diffusion (right) of the financial sector relative capitalization (blue) and prices (red) dynamics as a function of $\psi \in (0,1)$. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

Figure 5, right panel. In Appendix E, we shall also show how drift and diffusion change with idiosyncratic risk $\tilde{\sigma}$ and systematic risk σ. When the financial sector is small (high leverage) they both increase with $\tilde{\sigma}$ (the higher the risk, the higher the demand for risk mitigation, the higher the leverage) and decrease with σ (the higher the systematic risk, the lower the Sharpe ratio, the lower the leverage). The last result is consistent with the *volatility paradox*: due to leverage, a lower systematic risk increase fluctuations.

3.2 The Benchmarks

In this section, we introduce the two extreme cases that act as the benchmarks of our analysis. The former is the *no-risk-pooling* economy, where the h/entrepreneurs hold all the capital and market segmentation plays a big role. The latter is the *full-risk-pooling* economy, where the financial sector holds the whole stock of physical capital and market segmentation plays no role.

No-risk-pooling The equilibrium at the left-hand side boundary ($\psi = 0$) implies a constant price of physical capital $q(0)$ (in fact $\mu^q(0) = \sigma^q(0) = 0$), investment $\iota(0)$, risk-free interest rates $r(0)$, risky claim return $\mu^h(0)$, and their Sharpe ratio

$\xi^h(0)$. In particular,

$$q(0) = \frac{1+\theta A}{1+\theta\rho}, \quad \iota(0) = \frac{q(0)-1}{\theta} = \frac{A-\rho}{1+\theta\rho}, \quad r(0) = \rho + \Phi(\iota(0)) - \delta - \sigma^2 - \tilde{\sigma}^2,$$

$$\mu^h(0) = \frac{A-\iota(0)}{q(0)} + \Phi(\iota(0)) - \delta, \quad \xi^h(0) = \frac{\frac{A-\iota(0)}{q(0)} + \sigma^2 + \tilde{\sigma}^2 - \rho}{\sqrt{\sigma^2+\tilde{\sigma}^2}}.$$

In this economy markets are utterly *segmented.* Financial intermediaries do not supply any risk-mitigation instrument to the economy and each entrepreneur has full exposure to its idiosyncratic shocks. The equilibrium interest rate is lower than how it would be with a financial sector, and it is such that, in absence of risk mitigation assets, agents are happy to invest their wealth in risky claims only. High value of q decrease the dividend yield (but increase the capital gain due to higher investment) and decrease also the Sharpe ratio. The latter depends also on systematic and idiosyncratic risk. Although both increase risk, they also decrease the risk-free rate and thus, overall, increase the equilibrium Sharpe ratio.

In this benchmark, the aggregate output (perishable good) Y_t follows a Geometric Brownian Motion (GBM). The same holds for the capital stock (due to the linearity of type II technology) and for aggregate consumption:

$$\left.\frac{dY_t}{Y_t}\right|_{\psi=0} = \left.\frac{dC_t}{C_t}\right|_{\psi=0} = \frac{dK_t}{K_t} = [\Phi(\iota(0)) - \delta]\, dt + \sigma dW_t.$$

Although aggregate output and consumption are moved only by the systematic shocks, each entrepreneur individual consumption bears its uninsured idiosyncratic risk leading to a low welfare.

Full-risk-pooling The *full-risk-pooling* economy is reachable when the cost of intermediation equals zero, unless the obvious case when the financial sector is endowed with the whole aggregate wealth at $t = 0$ so that $\psi_0 = 1$. Also this equilibrium implies a constant price of physical capital $q(1)$, investment $\iota(1)$, risk-free interest rates $r(1)$, risky claim return $\mu^h(1)$, and their Sharpe ratio $\xi^h(1)$.[33]

[33] Also this equilibrium is a special case of John Cox and Ross (1985).

In particular, we have:

$$q(1) = \frac{1+\theta(A-\eta)}{1+\theta\rho}, \quad \iota(1) = \frac{A-\rho-\eta}{1+\theta\rho}, \quad r(1) = \rho + \Phi(\iota(1)) - \delta - \sigma^2,$$

$$\mu^f(1) = \frac{A-\iota(1)-\eta}{q(1)} + \Phi(\iota(1)) - \delta, \quad \xi^f(1) = \frac{\frac{A-\iota(1)-\eta}{q(1)} + \sigma^2 - \rho}{\sigma}.$$

Note that when $\eta > 0$ capital prices and investment are lower: $q(1) < q(0)$ implies $\iota(1) < \iota(1)$. Interest rates are higher, $r(1) > r(0)$, due to the fact that the financial sector can diversify all the idiosyncratic risk and thus has a zero demand/supply of risk mitigation for higher rates than when h/entrepreneurs are alone. Positive intermediation costs imply instead that capital is less productive (some resources are lost by the intermediation process) and its equilibrium prices is lower. Lower prices imply also lower investment and thus lower drift, a *pecuniary externality* of the high intermediation costs. Risk premia, and so Sharpe ratios, are also a function of capital prices. A low capital price implies a higher dividend yield and a lower capital gain (lower investment). The Sharpe ratio is also lower due to higher interest rates.

Also in this benchmark the output follows a GBM, the same process followed by total consumption and the capital stock:

$$\left.\frac{dY_t}{Y_t}\right|_{\psi=1} = \left.\frac{dC_t}{C_t}\right|_{\psi=1} = \frac{dK_t}{K_t} = [\Phi(\iota(1)) - \delta]\, dt + \sigma dW_t.$$

With positive intermediation cost, $\eta > 0$, the growth rate of output, capital, and consumption is lower in the *full-risk-pooling* economy than in the *no-risk-pooling* case. Nevertheless, in both cases the aggregate volatility is state independent and equals σ. The same process is followed also by the disposable output $\tilde{Y}$, defined as the output net of intermediation costs: $\tilde{Y}_t = Y_t - \eta K_t = (A-\eta)K_t$.

In the next section we focus to the dynamics of the macro-variables in the intermediate case of *partial-risk-pooling* where the h/entrepreneurs and the financial sector coexist.

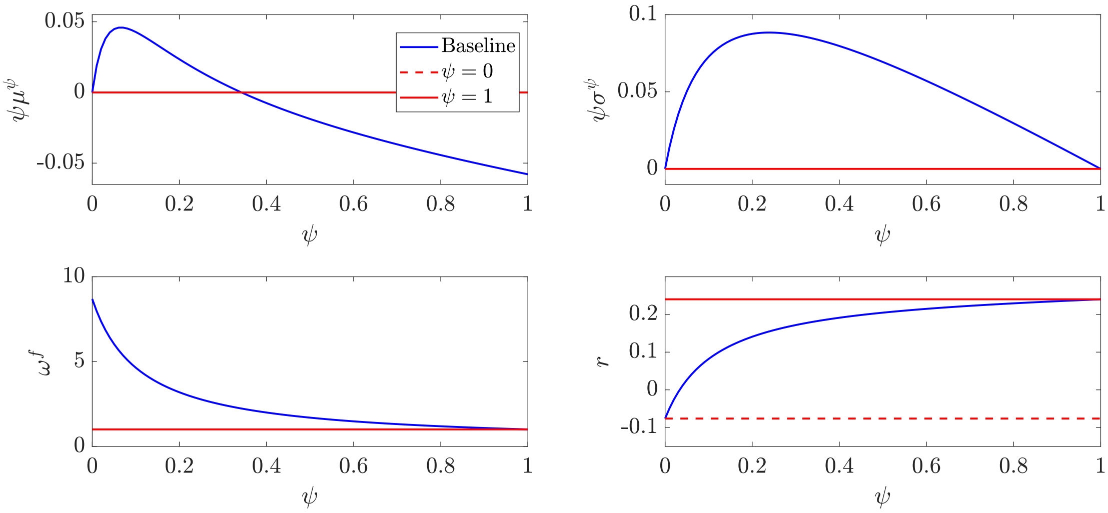

Figure 6: Top: Drift (left) and diffusion (right) of the equilibrium process $d\psi$ as a function of ψ. Bottom: Equilibrium leverage ω^f and risk-free interest rates r as a function of ψ. In red, the benchmark cases of *full-risk-pooling* (solid) and *no-risk-pooling* (dashed). Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$ $\eta = 0.1$, $\theta = 2$ and $\rho = 0.05$.

4 Real and Financial Macro-dynamics

In this section, we describe the equilibrium dynamics of leverage and macro-variables when intermediation costs are positive, so that the financial sector does not dominate, but not too high, so that the condition (17) of Theorem 1 is satisfied and the financial sector capitalization stays positive. We shall characterise this case of *partial-risk-pooling* as a deviation from the benchmarks of full-risk-pooling (and positive intermediation cost) and no-risk-pooling discussed in the previous section.

The discussion is structured as follows. First, in Section 4.1 we investigate the mechanism that links the fluctuations of the financial sector relative wealth share to financial leverage and risk-free interest rates. Second, in Section 4.2 we study how those fluctuations affect real macro-variables, such as aggregate consumption and disposable output. More specifically, we focus on the relationship between the volatility of those variables and the exogenous systematic volatility σ, conditional on the financial sector relative capitalization ψ.

Finally, in Section 4.3, we discuss the relationship between leverage, Sharpe ratios, and risk-free interest rates.

4.1 Leverage and Risk-free Rates

Having solved for the competitive equilibrium, we are able to address several questions upon the theoretical implications of our model, namely: how does financial leverage react and modifies positive and negative exogenous systematic shocks, respectively? What is the relationship connecting the financial sector relative wealth share, and thus financial leverage, to risk-free interest rates?

Figure 6 illustrates the drift (top-left) and diffusion (top-right) of the equilibrium relative wealth share level $d\psi_t$ over the state-space $\psi \in (0, 1)$. In red, we show the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. In the same figure (bottom row), we plot the financial leverage ω^f (right) and the risk-free interest rate r (left) as a function of ψ.

Financial leverage The financial sector leverage, ω^f, is a decreasing function of ψ because the smaller the financial sector, the higher the demand of risk mitigation, the larger the leverage (Figure 6, bottom left). As shown in Appendix C.2, in equilibrium it holds

$$\omega_t^f = \frac{1}{\psi_t}\left(1 - (1 - \psi_t)\frac{\mu_t - r_t}{\sigma_t^2}\right). \tag{18}$$

Since in equilibrium leverage cannot be larger than $\frac{1}{\psi}$, which occurs when the financial sector holds all risky claims, the diminished financial sector leverage reflects its risk aversion. Note that, despite leverage is decreasing in ψ, the total holding of the financial sector, $\omega^f\psi$ is increasing in ψ, consistently with the equilibrium nature of the model.

How does leverage changes with exogenous shocks? As confirmed by Theorem 1, positive (negative) exogenous systematic shocks deteriorate the financial sector assets and move its size towards one (zero). Stated differently, the diffusion term σ^ψ contributes positively to the size law of motion, see the top-right panel of Figure 6. The latter together with the fact that ω^f is a decreasing function of ψ, imply that negative shocks increase equilibrium leverage. This is the result of a relatively higher demand for risk-mitigation instruments by the h/entrepreneurs. The opposite holds as a response to positive shocks: when the financial sector increases its relative wealth share, its leverage reduces, and its supply of risk-mitigation instruments decreases. Since financial leverage is convex in ψ, conditional on the state, increments of leverage are more than proportional with respect to reductions.

Overall, the size of the financial sector is pro-cyclical and financial leverage is *counter-cyclical* as also suggested by the recent empirical findings in Yepez (2017) and He et al. (2017).

Given that σ_t increases in σ, leverage in (18) is decreasing in the systematic volatility σ -the lower the systematic risk, the higher the risky claim demand of the financial sector, the higher the equilibrium leverage- and increasing in the idiosyncratic risk volatility $\tilde{\sigma}$ -the higher the idiosyncratic risk, the higher the demand for risk mitigation, the higher the leverage. Appendix E provides a graphical representation of the result. Given that financial leverage is associated with high fluctuations of the financial sector size, decreasing systematic risks also increases the relative size fluctuations, an effect consistent with the *volatility paradox*.

As we shall see, leverage is the core driver of several of our results. In fact, due to its one-to-one relationship with the relative size of the financial sector, its is equivalent to express the equilibrium as a function of either variable.

Risk-free interest rates As far as the equilibrium risk-free interest rate r_t is concerned (see Figure 6, bottom row, right), the risk-free return on bonds is increasing in financial sector wealth share, due to a declining demand/increasing supply of mitigation instruments, making interest rates are pro-cyclical.

For low value of financial sector capitalization, r turns negative. Since both sides are equally risk-averse, with a high demand/low supply of bonds, h/entrepreneurs are willing to pay the financial sector to offload some of their risky claims to its balance sheet. The effect does not require any "crisis" contingency to take place, rather it is generated by market segmentation jointly with the allocation of capital and risk among heterogeneous classes of agents.

Finally note that the full-risk-pooling $r(1)$ and the no-risk-pooling $r(0)$ act as the upper and lower bound for the risk-free rate dynamics (bottom right panel of Fig. 6).

4.2 Consumption and the Business Cycle

The relationship between the size of the financial sector, its leverage, and the business cycle is a long-standing issue. In particular, there are several studies exploring the connection between the size of the financial system and intermediation to output and consumption growth and growth volatility. In Denizer et al. (2002), for example, countries with more developed financial sectors are shown to experience less fluctuations in output, consumption, and investment growth. More recently, Beck et al. (2014) show that intermediation activities increase growth and reduce volatility in the long-run. Nevertheless, they argue that an over-sized financial sector could result in miss-allocation of resources. What follows is that the over-development of auxiliary financial services may lead the financial sector to grow too large relative to its *social optimum*.

In the light of these empirical findings, we use our theoretical results to highlight the mechanism that relates the size of the financial sector to the equilibrium behaviour of real macro-variables such as aggregate consumption and disposable output.

In equilibrium, the aggregate output Y_t can be decomposed as the sum of consumption C_t, investments I_t, and what is spent as intermediation costs due to pooling, G_t. We denote as *disposable output* $\tilde{Y}_t$ the fraction of total output that is either consumed or invested to generate new capital, $\tilde{Y}_t = C_t + I_t$, or, equivalently, $\tilde{Y}_t = Y_t - G_t$. $\tilde{Y}_t$ is the share of output that contributes at generating welfare. The dynamics of total output is

$$dY_t = AdK_t = \underbrace{dC_t + dI_t}_{d\tilde{Y}_t} + dG_t.$$

where $G_t = \eta\psi_t K_t$, $I_t = \iota_t K_t$ and thus $C_t = (A - \iota_t - \eta\psi_t)K_t$.

For the purpose of our analysis, we focus on consumption and disposable output only.[34] In particular (see Appendix C.7), it is possible to show that the dynamics of disposable output and consumption growth rates evolve as Itô's processes whose drifts ($\mu_t^{\tilde{Y}}$ and μ_t^C) as well as diffusions ($\sigma_t^{\tilde{Y}}$ and σ_t^C) are function of both state ψ_t and prices q_t.

Disposable output In Figure 7 we plot the drift (left) and the normalized diffusion (right) of the (aggregate) disposable output growth process $\frac{d\tilde{Y}}{\tilde{Y}}$. In red, we depict the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

Despite both drift and diffusion depend on the financial sector relative wealth share and they always remain within the bounds set by the two benchmarks. In particular, $\mu^{\tilde{Y}}$ is decreasing in ψ (increasing in financial leverage), whereas the (normalized) diffusion term $\frac{\sigma^{\tilde{Y}}}{\sigma}$ is a convex function of the financial relative capitalization ψ.

In our model, the output drift is decreasing in the relative size of the financial sector due to a *pecuniary externality*: the larger the financial sector capitalization, the lower the aggregate productivity of capital (due to high intermediation costs

[34]The dynamics dI_t and dG_t are reported in Appendix C.7.

per unit of capital), the lower the cost of capital, the lower the investment in new capital.

The reduction of output volatility implies that $\sigma^{\tilde{Y}}$ can be read as a mitigation with respect to the width of exogenous fluctuations due to the volatility of capital stock σ. This feature highlights a *positive externality* of the financial sector activity: having a large fraction of idiosyncratic risks that are pooled by the financial sector implies that capital is less productive (due to intermediation costs) and thus, being the size of the financial sector positively related to capital, a negative relationship between capital and its productivity. The latter decreases the size of capital growth rates fluctuations as driven by systematic shocks.

Lemma 2. ***Mitigation*** *The diffusion terms of disposable output growth can be written as mitigation with respect to the exogenous systematic shocks volatility σ. In particular*

$$\sigma_t^{\tilde{Y}}(\psi_t) = \sigma \left[1 - \underbrace{\left(\frac{\sigma_t^{\psi}}{\sigma} \right) \frac{\eta \psi_t}{A - \psi_t \eta}}_{Mitigation} \right]. \tag{19}$$

In Figure 7 we plot the drift (left) and the normalized diffusion (right) of the (aggregate) disposable output growth process $\frac{d\tilde{Y}}{Y}$. In red, we depict the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. As far as the mitigation is concerned, the right panel shows that the mitigation is a concave function of ψ. Indeed, the negative correlation between $(A - \eta\psi)$ and K implies that the mitigation is maximal when the volatility σ^{ψ} is high, i.e., for relative small values of the financial sector size. As far as output fluctuations are concerned, this suggests that there exists an optimal size of the financial sector.

The mitigation of disposable output volatility $\sigma^{\tilde{Y}}$ with respect to σ is in line with the empirical findings in Beck et al. (2014) suggesting that, in the long-run, intermediation-based services positively associate with growth and negatively with growth volatility (see Figure 7).[35]

Consumption To understand the connection between financial relative wealth share and the dynamics of consumption, in the top panels of Figure 8 we plot

[35]Conversely, non-intermediation services increase the output volatility of high income countries. Nevertheless, the role that intermediation and non-intermediation financial activities play in the growth process of countries is not yet fully disentangled.

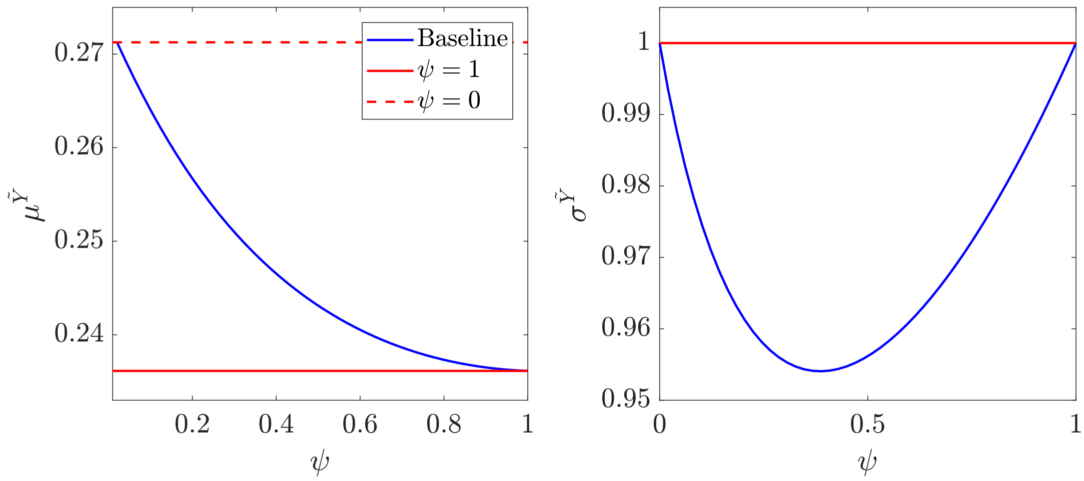

Figure 7: Diffusion (left) and normalized diffusion (right) of the (aggregate) disposable output growth rate $\frac{d\tilde{Y}}{\tilde{Y}}$. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

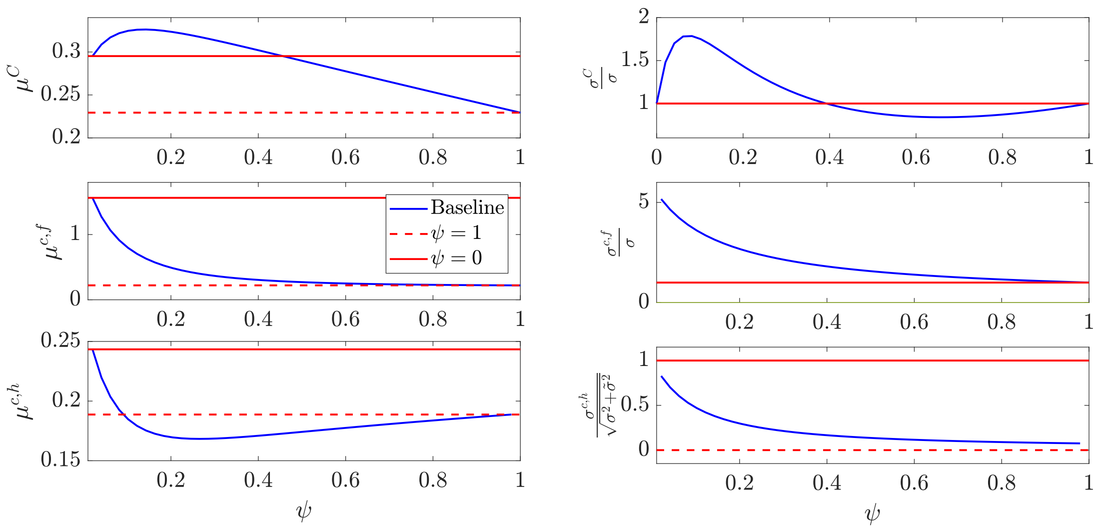

Figure 8: Top: Drift (left) and normalized diffusion (right) of the equilibrium aggregate consumption growth rate $\frac{dC}{C}$ as a function of ψ. Middle: Drift (left) and normalized volatility (right) of the financial sector consumption growth rate $\frac{dc^f}{c^f} \propto \frac{de^f}{e^f}$ as a function of ψ. Bottom: Drift (left) and normalized volatility (right) of the h/entrepreneurs' consumption growth rate $\frac{dc^h}{c^h} \propto \frac{de^h}{e^h}$ as a function of ψ. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$ $\eta = 0.1$, $\theta = 2$ and $\rho = 0.05$.

the drift (left) and the normalized diffusion (right) of the aggregate consumption growth rate $\frac{dC_t}{C}$ as a function of the state $\psi \in (0,1)$. In the same figure, we plot the drift and the normalized diffusion of the consumption growth rate process for the financial sector (centre panel) and the h/entrepreneurs' (bottom panel), respectively. In red, we depict the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy.

In the aggregate, the financial sector relative wealth share negatively affects the drift consumption μ^C. Moreover, as long as ψ is small enough, the consumption drift lays above the upper benchmark where $\psi = 0$ (Figure 8, top left panel). This result is mainly due the *pecuniary externalities*: when the financial sector manages capital, it reduces aggregate productivity, making physical capital relatively cheaper. The fact, that the dynamics of physical capital prices q inversely relates to the dynamics of ψ implies that lower financial relative wealth share (higher financial leverage) relates to higher prices, investments, and thus consumption growth.[36]

As far as consumption volatility is concerned, as long as ψ is small enough, consumption volatility is *amplified* whereas it is *mitigated* when the financial sector is relatively well capitalised.

Lemma 3. ***Amplification and Mitigation*** *The diffusion terms of aggregate consumption growth can be written as amplification and mitigation with respect to the exogenous systematic shocks volatility σ. In particular*

$$\sigma_t^C(\psi_t) = \sigma \left[1 + \underbrace{\frac{1}{\theta}\frac{\sigma_t^q}{\sigma}\frac{q_t}{A - \iota_t - \psi_t \eta}}_{Amplification} - \underbrace{\frac{\sigma_t^\psi}{\sigma}\frac{\eta \psi_t}{A - \iota_t - \psi_t \eta}}_{Mitigation} \right]; \tag{20}$$

Perhaps the most compelling feature is that σ^C can be decomposed as the sum of an *amplification* plus a *mitigation* term. It follows that the magnitude of consumption volatility with respect to σ depends on what component dominates: the endogenous term in Equation (20) amplifies the consumption fluctuations after exogenous shocks when ψ is low, i.e. financial leverage is high. The fluctuations of aggregate consumption capture here the fluctuations of investment (through capital prices). At the same time, aggregate consumption fluctuations are mitigates them when the financial sector is well capitalised (top-right panel), an effect of reduced volatility of output. This result is consistent with Denizer et al.

[36]We remind that in our model $\iota \propto q$.

(2002), whose empirical findings suggest that risk management services provided by financial intermediaries may be particularly important in reducing consumption volatility.

The other panels consider separately the growth rates of h/entrepreneurs and financial sector consumption. Both drift and volatility of financial sector consumption growth declines with its size. The larger the financial sector, the lower the drift, implying a decrease of its relative wealth growth rate (affecting the drift), and a decrease of fluctuations (affecting the volatility). A somehow similar effect occurs for the volatility of consumption growth rates of h/entrepreneurs consumption. Here, however, idiosyncratic risks play a big role: the larger the financial sector, the lower the share of idiosyncratic risk not pooled by it, the lower the entrepreneur consumption growth rate volatility. The drift is first sharply declining in the financial sector size, reflecting the shape of h/entrepreneurs wealth drift when the financial sector is small, and the slowly increasing when the financial sector is too large.

4.3 Financial Leverage and Asset Pricing

Now that we have drawn our main theoretical results concerning the relationship between the financial sector relative wealth share and the macro dynamics, we focus on the interlink between financial leverage and and financial assets returns, their Sharpe ratios, and risk-free rates. We discuss our results in light of some stylized facts: i) Risk-free rates are pro-cyclical (Fatih Guvenen, 2006) ii) As long as agents are able to adjust their leverage, Sharpe ratios are counter-cyclical, i.e. assets that covary with leverage are riskier and earn a proportionally larger risk premium (Adrian et al., 2014; Dell'Ariccia et al., 2014).[37]

To study how financial leverage relates to asset pricing, in Figure 9 we plot the financial sector risky assets expected returns $\frac{1}{dt}\mathbb{E}\left[dR^f\right] = \mu^f$ (bottom, left) and volatility $\frac{1}{dt}\sqrt{\mathbb{V}ar\left[dR^f\right]} = \sigma^f$ (bottom, right) as a function of ω^f. In red, we show the benchmark cases of maximum (solid) and minimum (dashed) leverage.

In the same Figure (top, right) we plot the Sharpe ratios of the financial sector ξ^f (blue, solid) and of the h/entrepreneurs' ξ^h (blue, dashed) as a function of ω^f. What stands out is that Sharpe ratios are increasing with financial leverage.

Accordingly to what we discussed in Section 4.1, that there exists a negative relationship between the financial sector relative wealth share and its leverage, since financial leverage is counter-cyclical, so it is the corresponding Sharpe ratio.

[37] A similar argument, from a theoretical perspective, is in Brunnermeier and Pedersen (2008).

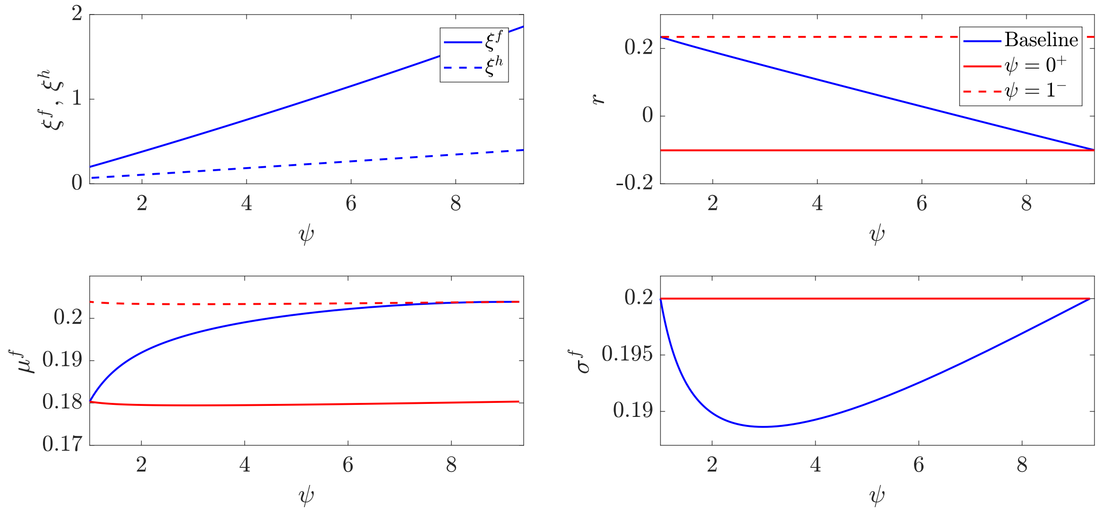

Figure 9: Top left: Sharpe ratios (left) financial sector ξ^f (blue, solid) and of the h/entrepreneurs' ξ^h (blue, dashed). Top right: Risk-free interest rate (right) as a function of financial leverage ω^f. Bottom: Financial risky assets expected return μ^f (left) and diffusion σ^f (right) as a function of financial leverage ω^f. Bottom: Sharpe ratio ξ^R (left) and risk-free interest rates (right) as a function of financial leverage. In red, the benchmark cases of maximum (solid) and minimum (dashed) leverage. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

This is because, as long as the financial sector is free to adjust its leverage, its assets covary with leverage, they are riskier, and thus earn a larger risk premium. The plot also clarifies that Sharpe ratios faced by the entrepreneur (and including idiosyncratic risk) are lower than those faced by the financial sector, consistently with the opposite position that they have in the bond market.

Accordingly, risk-free interest rates being decreasing in ω^f (Figure 9, top, right), are pro-cyclical. This is because, in our model, high leverage corresponds to scarce financial capitalization, and so a scarce supply of risk-free bonds.

In this term, the link between financial leverage Sharpe ratios, and interest rates strictly relates to the pooling capacity of the financial sector, and can be decomposed into two different components. First, higher financial leverage corresponds to lower (even negative, depending on the parameters) interest rates. Second, higher leverage corresponds to higher aggregate marginal productivity, and thus higher risky assets returns, since a smaller share of aggregate wealth is spent after pooling.

Note that the size of idiosyncratic risks contribute also to the financial sector risk premia despite the fact they can be pooled, and therefore eliminated via diversification. This is due to the assumption of *market segmentation* as well as to agents' risk aversion. In fact, the h/entrepreneurs exposure to idiosyncratic risk, jointly with their share of the aggregate wealth, determines the aggregate demand of risk-free bonds, and so the equilibrium financial leverage. As long as there exists residual (un-pooled) idiosyncratic risk, that is accounted for in the equilibrium risk-free rates.

A further interesting implication of our model is that, unlike He and Krishnamurthy (2013), there is no need of binding constraints to link financial leverage to Sharpe ratios: in this terms, it is an inherent effect of financial intermediation.

The connection between higher risk premia and *limited participation* models is well known (see Fatih Guvenen, 2006), and dates back to Basak and Cuoco (1998).[38] Our contribution is to implement the aforementioned mechanism in a fully-fledged general equilibrium model of a production economy and, in particular, to draw the relationship between financial and real macro-dynamics.

5 Leverage and Welfare

In this section, we study how the relative capitalization of the financial sector, and so its leverage, relates to the agents' welfare. First, in Section 5.1 we derive the welfare of the h/entrepreneurs' as well as of the financial sector, both conditional on an initial financial sector size and unconditionally, using the ergodic distribution of the state to weight the possible outcomes.

Second, in Section 5.2 we explore the effect of a static leverage constraint on the macroeconomic dynamics, and so on both classes of agents' as well as on the aggregate welfare.

Finally, being the leverage constraint related to the minimal size of the financial sector, it is relevant to explore the role of a redistributive taxation policy that progressively limits its size.

In Section 5.3, we study the relationship between the h/entrepreneurs' welfare, leverage constraint, and such a redistributive taxation.

Our purpose is to investigate whether a too small (or too big) financial sector is detrimental for the h/entrepreneurs' welfare; this would suggest that there exists

[38] In the original model the *limitation* is extreme, since the households have access to risk-free assets only. As a result, the equilibrium the interest rate adjusts such that stockholders borrow the *entire* wealth owned by non-stockholders and make interest payments every period, which sustains the consumption of the latter group.

a *"welfare optimal"* size of the financial sector, and so that leverage constraints and redistributive taxation may be welfare improving.

5.1 Welfare Analysis

In this section, we introduce the measure of the agents' welfare. In general, the welfare W^i of the i^{th} agent equals its value function V_t^i.[39] Our result is summarised in Proposition 1:

Proposition 1. ***Conditional Welfare***
The conditional *welfare of the i sector, for unitary capital, can be expressed as*

$$V_t^i := W^i(\psi_t) = \underbrace{\frac{\ln \rho q_t v(\psi_t)^i}{\rho}}_{Static} + \underbrace{\frac{1}{\rho} H(\psi_t)^i}_{Dynamic}, \quad i := \{h, f\}, \tag{21}$$

where $v(\psi)^i$ is the i^{th} class relative wealth share and H_t^i accounts for the i^{th} agents' dynamics of wealth.

Proof. See Appendix C.5. □

The welfare function in (21) is the sum of two components: the former is *static*, and accounts for the current benefit due to the ownership of a certain share of the aggregate wealth. The latter is *dynamic*, and summarizes the discounted benefit of future consumption (for further details, see Appendix C.5).

In the left panel of Figure 10 we show the welfare of h/entrepreneurs' contingent to relative wealth share ψ (blue line). In red, we display the benchmark case $\psi = 0$.

What stands out is that the h/entrepreneurs' conditional welfare W^h is an *inverted U-shaped* function of the financial relative capitalization ψ. For low level of ψ, W^h is increasing: the larger the financial relative capitalization, the higher the equilibrium risk-free interest rate (see Figure 6, bottom right), the more risk mitigation of aggregate output fluctuations is provided (see Figure 7, right), the higher h/entrepreneurs' welfare. Conversely, the W^h turns decreasing when the financial sector is relatively too large. In such a case, even if the supply of risk mitigation is quite large, the small relative size of the h/entrepreneurs diminishes their consumption growth rate (see also Figure 8, bottom panel), since a greater share of wealth is spent -destroyed- after the payment of intermediation costs.

[39] Since the model is scale invariant in K_t, it is possible to write the welfare of both h/entrepreneurs' and financial sector for unit of physical capital.

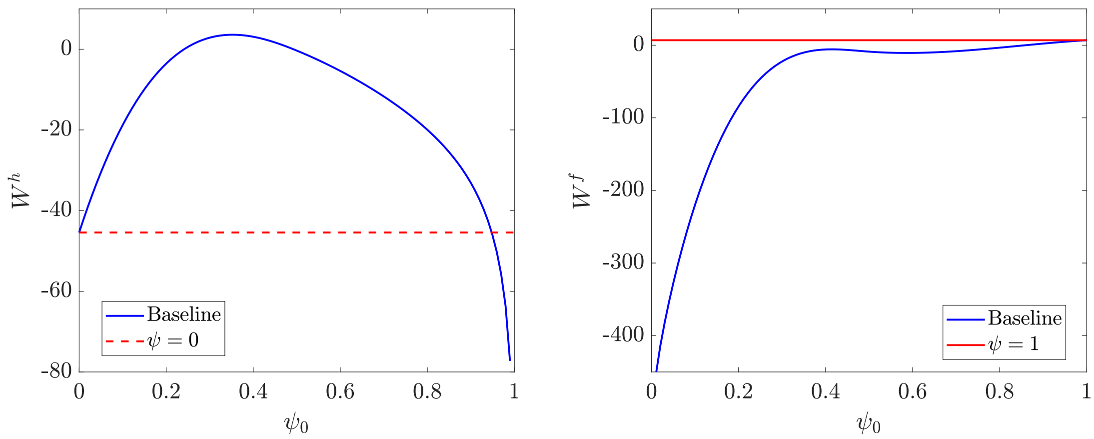

Figure 10: Conditional welfare of h/entrepreneurs' (left) and financial sector (right). In red, the benchmark cases $\psi = 0$ (dashed) and $\psi = 1$ (solid).

What is also interesting is that the level of relative financial capitalization such that $W^h(\psi)$ is maximal roughly coincides with the level of maximum mitigation of aggregate output fluctuations (see Figure 7, right). This result suggests that there exists a *"welfare optimal"* size of the financial sector.

In the right panel of Figure 10 we repeat the same exercise with respect to financial sector welfare W^f. In blue, we plot W^f contingent to the relative financial capitalization ψ. In red, we display the benchmark case when $\psi = 1$.

Overall, the financial sector conditional welfare is increasing in its own relative capitalization, and it is maximal when its relative capitalization approaches one. This is due to relative wealth share effect on the price of physical capital (the *static* terms of Equation 21). Conversely, the hump shape for middle capitalization states is due to the *pecuniary externality*: a higher share of capital held by the h/entrepreneurs maps into a higher value of physical capital stock, since fewer resources are destroyed after the payment of the intermediation costs, and hence to higher investments and output.

Unconditional and aggregate welfare In order to evaluate the welfare effect of imposing leverage constraints to the financial sector, rather than of having a redistributive policy, we may build an *unconditional* (expected) measure of welfare by weighting the conditional value of (21) by the associated ergodic density $\pi(\psi)$:

$$\mathbb{E}_0 \left[W^i(\psi) \right] = \int_0^1 W^i(\psi)\pi(\psi)d\psi. \tag{22}$$

Accordingly, the *aggregate* welfare equals sum of expected constrained welfare of h/entrepreneurs and financial sector weighted by a function $\Gamma(\psi)$, thus

$$W^{\Gamma} = \sum_i \mathbb{E}_0\left[W^i(\psi)\right]\Gamma(\psi)^i. \tag{23}$$

In Table 1, we compute the aggregate welfare in (23) for different weighting functions: in the first, second, and third rows, it is constant $\Gamma^i \in \{0; 1; 0.5\}$, whereas in the forth row it is proportional to each class relative wealth share.[40]

Weights, $\Gamma(\psi)^i$	Aggregate welfare
$\Gamma(\psi)^f = 0$; $\Gamma(\psi)^h = 1$	-4.7371
$\Gamma(\psi)^f = 1$; $\Gamma(\psi)^h = 0$	-61.12
$\Gamma(\psi)^f = \Gamma(\psi)^h = 0.5$	-32.93
$\Gamma(\psi)^f = \psi$; $\Gamma(\psi)^h = 1 - \psi$	-30.11

Table 1: Unconditional aggregate welfare for different weighting functions.

5.2 Leverage Constraints

The analysis of the previous section suggests that having a lower bound on the size of the financial sector may be welfare improving for the h/entrepreneurs. When leverage is counter-cyclical, as captured by our model, this can be achieved by imposing a static a leverage constraint. Our contribution is to provide theoretical evidence of the role that such constraints may have at determining the fluctuation of disposable output, consumption, and thus welfare.

Hereafter, we solve the model assuming an additional constraint to the financial sector leverage. Then, we discuss the effect of such a constraint over the equilibrium dynamics and the associated ergodic density $\pi(\psi)^{LC}$. The financial sector optimization problem is now written to take into account the additional constraint $\omega_t^f \leq LC$. Finally, we compute the welfare in presence of a static Leverage Constraint (LC).

[40] We compute the aggregate welfare in (23) numerically. In particular, we approximate the unconditional welfare over an evenly spaced grid $[0, 0.25, 0.5, 0.75, 1]$ and interpolate it by multinomial splines. Then, we integrate by trapezoid method $W^h(\psi)$ over the an evenly matched ergodic density $\pi(\psi)$ weighted by $\Gamma(\psi)$.

With a static LC, the HJB equation of the financial sector becomes

$$\rho V_t = \max_{\left\{\omega_t^f, c_t\right\}} \left\{ \ln c_t^f + \frac{1}{dt}\mathbb{E}_t\left[dV_t\right] - \lambda_t\left(\omega_t^f - LC\right)\right\},$$

where λ_t is the KKT multiplier and the boundary condition $\lim_{s\to\infty} e^{-\rho s}V(e_s) = 0$. The problem is solved in Appendix C.6. It is relevant to highlight that, since agents are risk averse, the LC is not always binding. It follows that the motion through which the equilibrium shifts in and out the constrained area is state contingent: both its speed and fluctuations depend on how restrictive the LC is. We now discuss the effect of LC over the equilibrium dynamics as well as on the agents' welfare.

Constrained dynamics In Figure 11 (top), we show the drift (left) and diffusion (right) of the (constrained) equilibrium relative wealth share process. In particular, we consider bounded (green) and unbounded (blue) LC. In red, we plot the benchmark cases of the *full risk pooling* (solid) and the *no risk pooling* (dashed) economy.[41] What stands out is that, when the constraint binds, it reduces both drift and diffusion of the state process. This result is intuitive since, when the financial sector leverage is exogenously capped by the prudential policy, so it is the supply of risk mitigation instruments to the h/entrepreneurs when financial capitalization is scarce. This can be seen through the portfolio choices of the agents in the states where the LC is binding.

In the same Figure (bottom), we repeat a similar analysis with respect to both equilibrium financial sector leverage ω^f (left) and h/entrepreneurs sector portfolio share in risky claims ω^h (right). What is relevant is that binding constraints oblige h/entrepreneurs to keep a higher share of their wealth allocated in risky assets. Accordingly, the speed at which the system drifts back towards the high capitalization phase is weakened: intermediaries cannot leverage themselves out of low capitalisation contingencies.

The last result is particularly clear if we look at the constrained ergodic density of the financial sector relative wealth share $\pi(\psi)^{LC}$. In Figure 12 we plot the ergodic density (left) and the cumulated density $\Pi(\psi)^{LC}$ (right) for bounded (green) and unbounded (blue) LC.

Similarly to Phelan (2016), the effect of constraint is not uniform over $\psi \in$

[41] Of course, in the benchmark cases LCs are utterly not relevant since the equilibrium leverage is fixed and equal to 1.

$(0, 1)$, and it consists of a higher ergodic probability for the economy to remain stuck in a low capitalisation state, i.e. a fatter left-tailed distribution. The higher the capital constraint, the higher the mass of the left-hand side tail: the density of the relative financial capitalization shifts to the left. Accordingly, the probability mass of states where the financial sector is better capitalized (lower leverage) is proportionally reduced.

Disposable output If we look to the effect of leverage constraints through the lenses of the business cycle (see Figure 13), we find out that bounded LC to the financial sector slightly increases the drift $\mu^{\tilde{Y}}$ (left) of disposable output growth as well as its volatility $\sigma^{\tilde{Y}}$ (right). This is because, in our model, the productivity of h/entrepreneurs is higher then the financial sector's (due to intermediation costs). Naturally, by imposing a maximum leverage to the financial sector, the aggregate wealth in the h/entrepreneurs' (risky) portfolio is increased, and so it is the aggregate disposable output.

What is also relevant is that the constraint impairs the mitigation effect of systematic shocks which is indirectly generated by the financial sector (Figure 13, left).

Consumption If it is not considered jointly with the dynamics of aggregate consumption, the pattern of aggregate output may be misleading.

As we show in Figure 14 (top panel), bounded LC reduces both the drift and diffusion aggregate consumption process. This is because, notwithstanding a higher level of disposable output, it is relatively costlier for the agents to generate capital for the future (the price of physical capital q is higher, and so it is the re-investment rate ι), thus the increase the fraction of disposable output which is spent for investment.

From the perspective of the financial sector, biding constraints reduce the growth rate of its consumption as well of its volatility, due to the limited leverage.

Conversely, the growth rate of the h/entrepreneurs' consumption is higher, due to the price effect of a higher share of risky capital in their portfolio (Figure 14, middle panel, left). At the same time, the h/entrepreneurs suffer for a scarce supply of risk-free bonds when they are needed the most; when financial capitalization is scarce, and the leverage constraint is binding, there is an extra exposure to idiosyncratic risks (bottom panel, right).

In summary, the most relevant pattern is that imposing limits to leverage

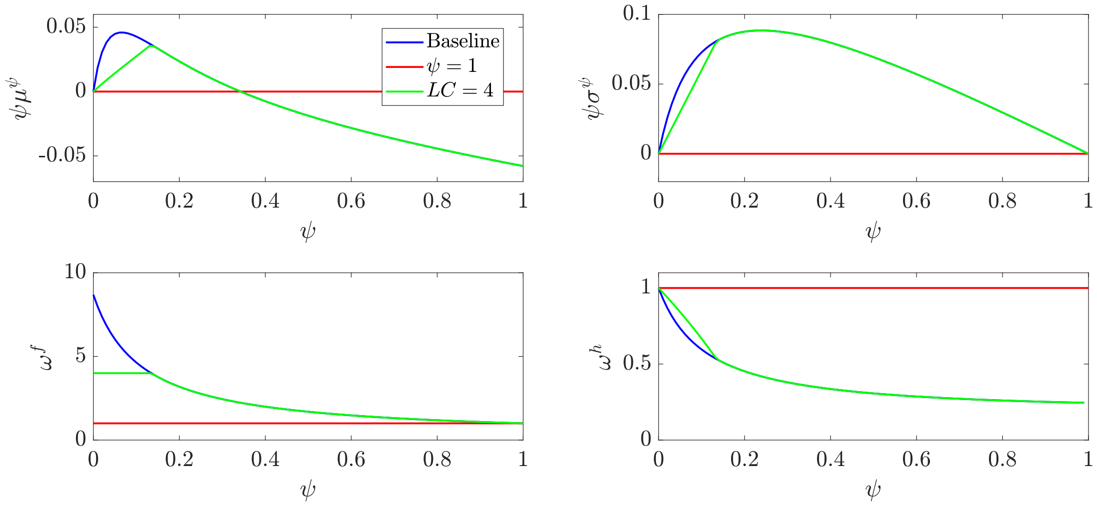

Figure 11: Top: Drift (left) and diffusion (right) of the process $d\psi_t$ for bounded (green) and unbounded (blue) constraints LC. Bottom: Equilibrium financial sector's (right) and h/entrepreneurs' (left) portfolio shares for bounded (green) and unbounded (blue) LC. In red, the benchmark case of the *full-risk-pooling* economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.1$, $\theta = 2$, $\sigma = 0.2$, and $\rho = 0.05$.

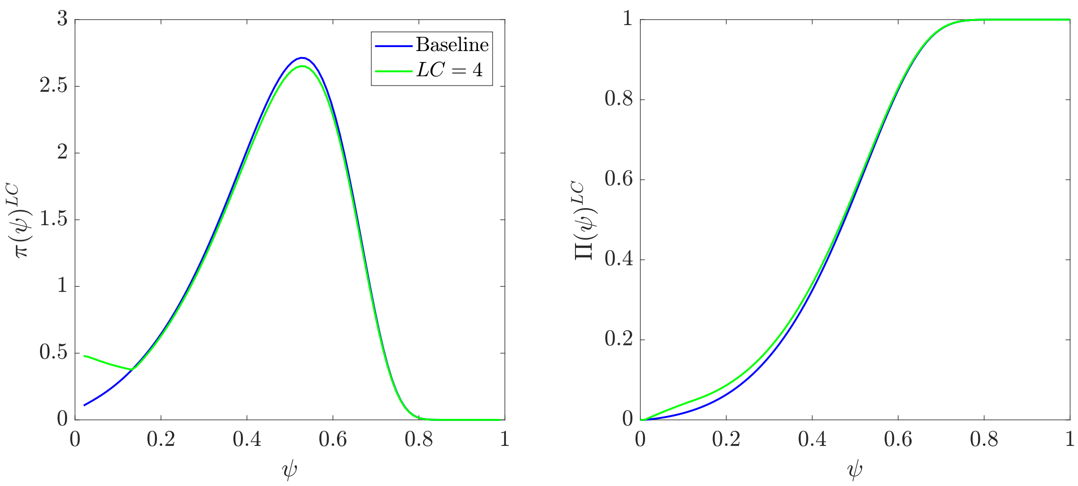

Figure 12: Constrained ergodic density (left) and cumulative density (right) functions of the relative wealth share ψ for bounded (green) and unbounded (blue) LC. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.1$, $\theta = 2$, $\sigma = 0.2$, and $\rho = 0.05$.

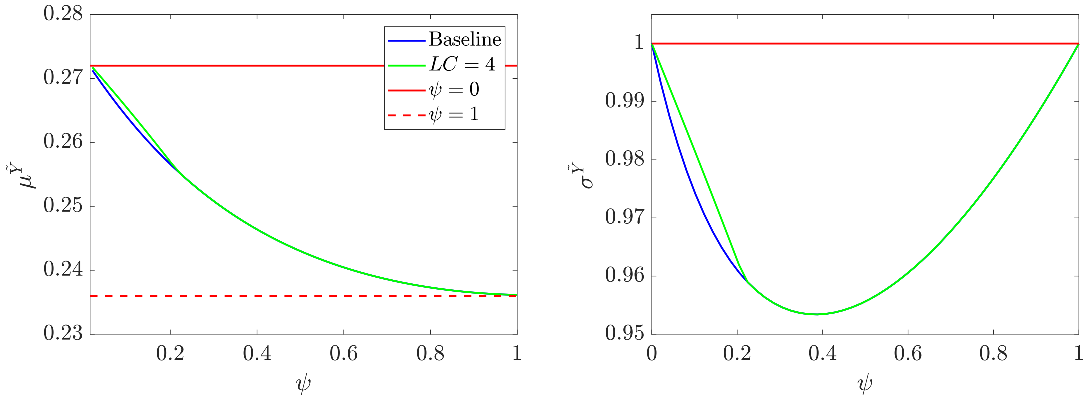

Figure 13: Diffusion (left) and normalized diffusion (right) of the (aggregate) disposable output growth rate $\frac{d\tilde{Y}}{\tilde{Y}}$ for bounded(blue) and unbounded *LCs*. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

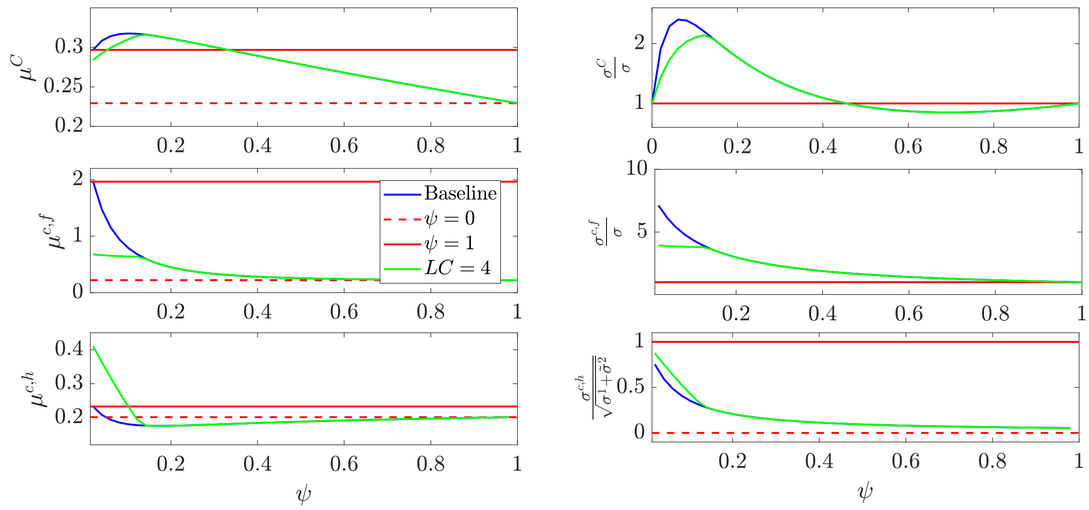

Figure 14: Consumption growth dynamics for bonded (green) and unbounded (blue) *LC*. Top: Drift (left) and normalized diffusion (right) of the equilibrium aggregate consumption growth rate $\frac{dC}{C}$ as a function of ψ. Middle: Drift (left) and normalized volatility (right) of the financial sector consumption growth rate $\frac{dc^f}{c^f} \propto \frac{de^f}{e^f}$ as a function of ψ. Bottom: Drift (left) and normalized volatility (right) of the h/entrepreneurs' consumption growth rate $\frac{dc^h}{c^h} \propto \frac{de^h}{e^h}$ as a function of ψ. In red, the benchmark cases of the *full-risk-pooling* (solid) and the *no-risk-pooling* (dashed) economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\sigma = 0.2$ $\eta = 0.1$, $\theta = 2$ and $\rho = 0.05$.

stabilizes the dynamics of the aggregate consumption by reducing its fluctuations, since it reduces the volatility of relative wealth share. However, at the same time, it reduces the aggregate consumption growth due to higher price of physical capital q, since higher investments are required to generate new capital in the future.

On the other hand, the LC contribute at reducing the mitigation of the disposable output fluctuations, since the constraint limit the positive externality due to the financial sector. Moreover, a restrictive policy hinders the optimal allocation of risk by setting an upper bound to the equilibrium supply of risk-mitigation instruments.

Welfare after LCs The welfare analysis with LCs is the same as proposed in Section 5.1, provided that a static upper bound for the financial sector leverage is imposed. Accordingly, in Table 1 we report the constrained and unconstrained *unconditional* (aggregate) welfare for different weighting functions Γ^i.

We start our analysis by focusing on the h/entrepreneurs' and financial sector welfare apart from each other: once we look at the unconditional welfare of the h/entrepreneurs' before and after LCs, we find out that the constraints may be welfare improving ($\Gamma^h = 1$ and $\Gamma^f = 0$, Table 1, second row). Conversely, when only the financial sector is considered, we find LCs to be welfare detrimental ($\Gamma^f = 1$ and $\Gamma^h = 0$, Table 1, third row).

These results may be better understood by looking at the agents' conditional welfare functions along with the ergodic distribution of the state variable (see Figure 12), left panel. In Figure 15 (left panel) we show the welfare of h/entrepreneurs' contingent to relative wealth share ψ (blue line) or subject to a LC (green line). In red (dashed line), we display the benchmark case $\psi = 0$.

What stands out is that LCs increase the h/entrepreneurs' conditional welfare when the financial sector is either under or over-capitalized. Conversely, W^f decreases for intermediate values of ψ.

In the same figure (right panel), we show the welfare of financial sector contingent to relative wealth share ψ for bounded (green) and unbounded (blue) LC. In red (solid), we display the benchmark case when $\psi = 1$.

What is interesting is that the benefit of the constraint for the financial sector holds only in case of extremely low capitalisation, whereas it reduces its conditional welfare for intermediate states where it is better capitalized. What follows is that, in general, leverage constraints are welfare detrimental for the financial sector. This is because, when binding, LCs keep financial sector relative capitalization ψ at a lower level with a higher probability (see Figure 12).

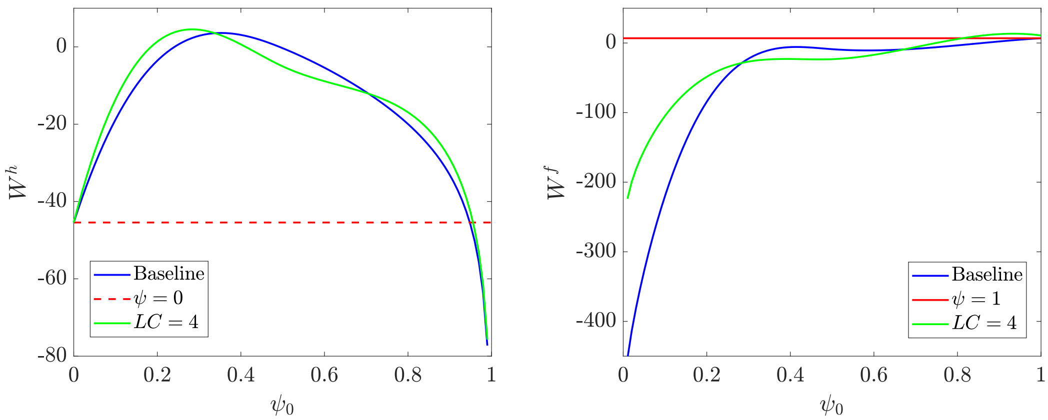

Figure 15: Conditional welfare of the h/entrepreneurs' (left) and of the financial sector (right) for bounded (green) and unbounded (blue) LC. In red, the benchmark cases of the *no-risk-pooling* (dashed) and the *full-risk-pooling* (solid) economy

Finally, in the last two rows of Table 1 we report the aggregate when the weighting function Γ is either constant and even, or proportional to each class relative wealth share. In either cases, the leverage constraints are welfare improving. In the former case this mean that, at this level of leverage constraint, the welfare gain of the h/entrepreneurs' more than compensate the welfare loss of the financial sector. Not surprisingly, the same result holds when the weighting function is proportional to the agents' relative share of wealth; in such a case, the welfare loss of the financial sector in states of low capitalization is considerably under weighted and the gain of the h/entrepreneurs' is overweighted.

Weights, $\Gamma(\psi)^i$	Aggregate Welfare, LC = Unbounded	W^{Γ}_{LC} $LC = 4$
$\Gamma(\psi)^f = 0; \Gamma(\psi)^h = 1$	-4.7371	-2.58
$\Gamma(\psi)^f = 1; \Gamma(\psi)^h = 0$	-61.121	-62.86
$\Gamma(\psi)^f = \Gamma(\psi)^h = 0.5$	-32.93	-32.37
$\Gamma(\psi)^f = \psi; \Gamma(\psi)^h = 1 - \psi$	-30.11	-28.64

Table 2: Unconditional aggregate welfare for different weighting functions.

5.3 Constraints, Redistributive Taxation, and Welfare

Now that we have pointed out how leverage constraints influences the h/entrepreneurs' as well as on the financial sector welfare, we conclude by addressing two further issues, namely: a) Since LCs may be beneficial, how does the h/entrepreneurs' unconditional welfare change for different levels of constraints? b) What is the role of a *redistributive taxation* that contributes at reducing the relative capitalization of the financial sector?

In Figure 16, we plot the unconditional welfare of the h/entrepreneurs' as a function of the leverage constraint LC.[42]

What stands out is that, according to our previous results, constraints to the financial sector leverage may be welfare improving for the h/entrepreneurs. In particular, the effect on W^h is positive as long as LC is not too high. Conversely, the level of the constraint compromises the equilibrium supply of risk mitigation instruments to the economy, and so the *positive externality* of the financial sector activity, when excessively high.

There is a growing literature regarding this aspect (see Blum and Hellwig, 1995; Blum, 2008; DeAngelo and Stulz, 2015; Myerson, 2014, among the others), however, the arguments considered for a lower leverage are based on either partial equilibrium models or focusing on information asymmetries. To our knowledge, our paper is the first that stresses the connection between leverage and the real as well as financial macro-dynamics, and that explicitly highlight the mechanism that links the agents' welfare to the size of the financial sector in a general, although extremely stylized, equilibrium model.

Redistributive taxation Having established the way leverage constraints affect the agents' welfare, we now investigate how *tax transfers* from the financial sector to the h/entrepreneurs' alter their welfare.

This is relevant because, being the LC related to the minimal size of the financial sector only, it does not prevent it to grow too large when the constraint is slack. In this term, the role of a *redistributive taxation* is to reduce the relative financial capitalization, and so the amount of resources it destroys after the payment of intermediation costs.

Let τ be the constant (tax) rate at which the stock of wealth is evenly redistributed from the financial sector to all the h/entrepreneurs'. It is possible to

[42] The welfare is approximated numerically with $T = 200$, and $N = 5,000$ over a evenly 8-spaced grid over $LC \in [2, 10]$ (green diamonds). We then interpolate the obtained points by multinomial splines (blue, solid line).

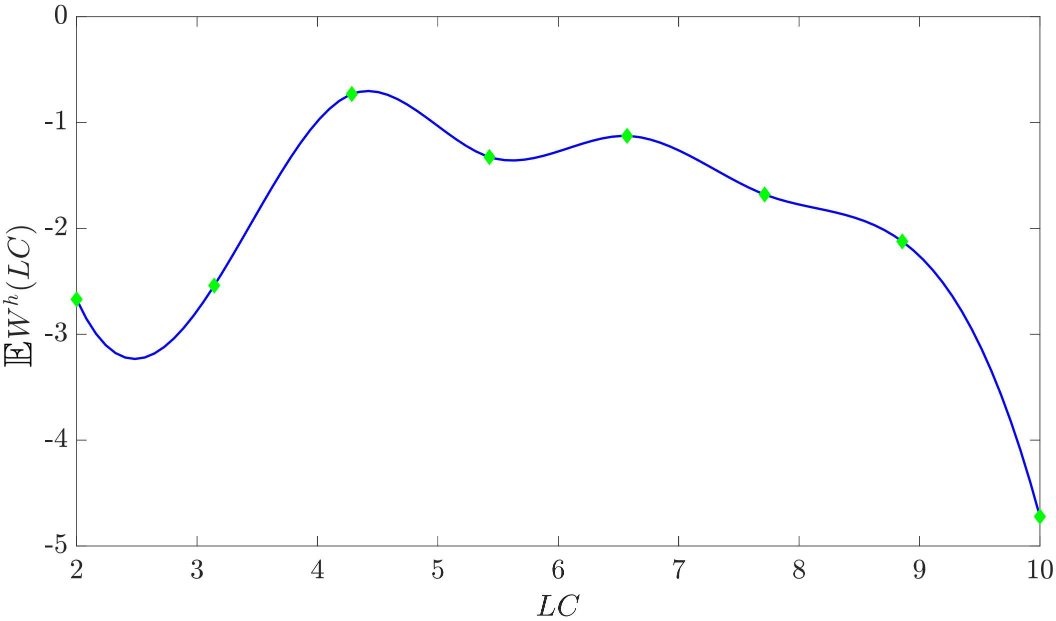

Figure 16: Unconditional h/entrepreneurs' welfare as a function of the leverage constraint. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.1$, $\theta = 2$, $\sigma = 0.2$, and $\rho = 0.05$.

show (the derivation is in Appendix D) that, accounting for the policy, the state variable ψ -financial relative capitalization- evolves as

$$\frac{d\psi_t^\tau}{\psi_t^\tau} = \frac{d\psi_t}{\psi_t} - \tau \frac{\psi_t}{1-\psi_t} dt,$$

where the term $\frac{d\psi_t}{\psi_t}$ has dynamics as in (15).

For our purposes, we look at the effect of different tax rates τ on the h/entrepreneurs' conditional welfare and on the ergodic density of the state. In Figure 17 (left panel), we plot the ergodic density $\pi(\psi)^\tau$ for no (blue) and positive tax rate (green). In the same Figure (right panel), we show the h/entrepreneurs' unconditional welfare W^h for an increasing level of τ.[43]

Not surprisingly, the redistributive policy affects the ergodic distribution by shifting it to the right, where the financial sector has a lower relative capitalization. In general, the higher ψ, the more effective the policy, since the redistribution is hyperbolically increasing in the state. This is because, the absolute redistribution is directly proportional to the wealth stock of the financial sector.[44]

As far as the h/entrepreneurs' welfare is concerned (Figure 17, right panel), we find that the redistributive taxation may be welfare improving for a moderate tax rate τ. This is because the financial capitalization is more likely to float through states where the *mitigation externality* of the financial sector is maximal, and fewer resources are destroyed after intermediation costs. Conversely, when τ is too high, the tax negatively affects the h/entrepreneurs' welfare since the financial sector is hindered from growing big enough, and so from supplying instruments of risk mitigation to the economy.

6 Conclusions

In this paper, we investigate the mechanism through which the risk pooling capacity of an aggregate financial sector, and thus its leverage, relates to the economic macro-dynamics. We do so in a theoretical framework where we impose financial frictions by *segmented financial markets*.

In order to mitigate the risk in their portfolios, h/entrepreneurs exchange physical capital versus risk-free bonds issued by the financial sector, who finances

[43]As for the results in Figure 16, the welfare function is approximated numerically over an evenly spaced grid (green diamond) and interpolated by multinomial splines (blue, solid line).

[44]Note that, at the boundaries: $\lim_{\psi \to 1} \frac{\partial}{\partial \psi} \left(\tau \frac{\psi_t}{1-\psi_t} \right) = \infty$, while $\lim_{\psi \to 0} \frac{\partial}{\partial \psi} \left(\tau \frac{\psi_t}{1-\psi_t} \right) = \tau$.

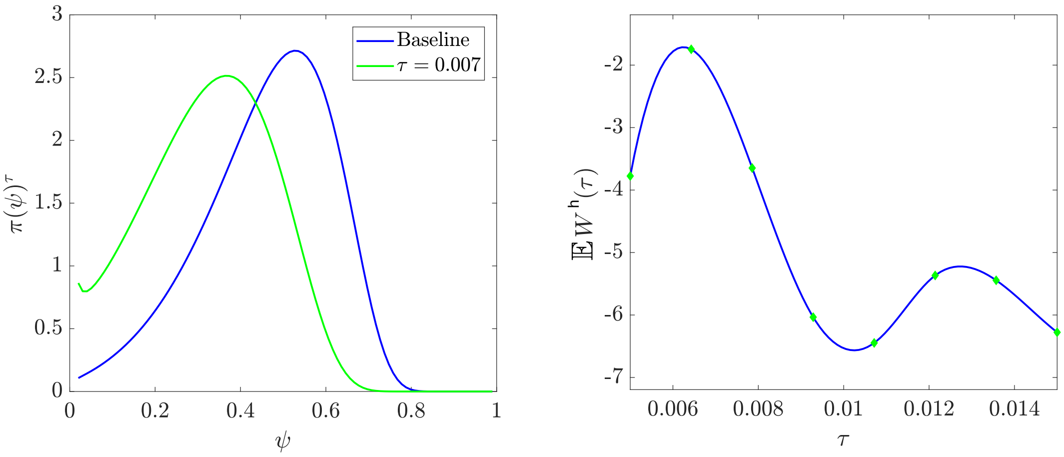

Figure 17: Left: Ergodic state density $\pi(\psi)^\tau$ before (blue) and after (green) a redistributive taxation policy. Right: Unconditional h/entrepreneurs' welfare as a function of the tax rate τ. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 0.6$, $\eta = 0.1$, $\theta = 2$, $\sigma = 0.2$, and $\rho = 0.05$.

its risky assets by leveraging its balance sheet. Intermediaries' mismatch of risk between assets and liabilities, together with a positive cost of risk pooling, stems into an equilibrium where heterogeneity is persistent.

The equilibrium allocation of risk generates an endogenous asset prices dynamics, counter-cyclical leverage, as well as amplification or mitigation of aggregate consumption, depending on the financial sector capitalization. Conversely, it mitigates disposable output fluctuations with respect to exogenous systematic shocks. Such mechanism roots in the assumption of *market segmentation* and does no require binding constraints. The endogenous dynamics of financial leverage stems from the agents' homogeneous preferences jointly with their asymmetric exposure to risk, which generates, in turn, structural demand for risk-mitigation instruments. In this terms, amplified (mitigated) macro-dynamics are inherent to the risk pooling activity of the financial sector.

In this setting, the existence of un-pooled idiosyncratic risk contributes to asset prices proportionally to the relative wealth share of the h/entrepreneurs sector, i.e. Sharpe ratios are increasing in financial leverage. On the other hand, risk-free interest rates are decreasing in financial leverage, and negative rates are associated to high leverage.

Finally, we investigate the relationship between the size of the financial sector, leverage, and welfare.

We find that limiting financial sector leverage contributes at smoothing the aggregate consumption fluctuations. From this perspective, our model suggests that there exists a trade-off between the welfare gain from aggregate consumption growth and the cost from its fluctuations when leverage is limited, and so that imposing leverage constraints may be welfare-improving for the h/entrepreneurs. Nevertheless, the stabilizing effects of constrained leverage associates to a suboptimal risk allocation in the economy as a whole. Moreover, binding constraints contingencies, where the financial sector is low-capitalised and risk-free bonds are scarcely supplied, are more persistent.

On the other side, we show that also preventing the financial sector to grow too large, and so to destroy too many resources after intermediation costs, may be welfare improving for the h/entrepreneurs.

These results suggest that there exist welfare improving leverage constraint and redistributive taxation policies such that the size of the financial sector remains within an " optimal" range.

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A Micro-foundation

The micro-foundation structure proposed in this section is the continuous-time equivalent of the one proposed in Ljungqvist and Sargent (2012), Chapter 12.

Firms of type II There exists a continuum of unitary mass of type II firms. Those firms produce output at a rate A. At each instant of time t, the i^{th} productive firm chooses the physical capital k_t^i in order to solve a static problem

$$\max_{k_t^i \geq 0} \left\{ y_t^i - p_t k_t^i \right\},$$

s.t.

$$y_t^i \leq A k_t^i, \tag{24}$$

where p_t is the rental rate of physical capital. Given linearity, the above has an interior solution only when the following zero-profit condition is satisfied:

$$p_t = A. \tag{25}$$

If (25) holds, the size of the i^{th} firm is indeterminate, and it is willing to supply any market demand.

Firms of type I There exists a continuum of unitary mass of type I firms. Those firms transform output into capital, store capital, and earn revenues by renting capital to type II firms at the equilibrium rate $p_t = A$. At each instant of time t, the i^{th} productive firm chooses how much value of capital $k_t^i q_t$ to store in order to earn stochastic returns dR_t^i per unitary capital, and how much numéraire $\iota_t^i k_t^i$ to purchase to generate new capital $\Phi(\iota_t^i) k_t^i$. The i^{th} firm finances itself by issuing state-contingent debt to the agent who supplies the capital stock.

It follows that, between t and s, the i^{th} firm solves the following problem

$$\max_{\left\{k_t^i, \iota_t^i\right\}} \left\{ \mathbb{E}_t^{\mathbb{Q}^i} \underbrace{\left[v_s e^{-\int_t^s r_s du} \right]}_{\text{Discounted "Net" Revenues}} - \underbrace{k_t^i q_t}_{\text{Cost of Capital}} \right\},$$

s.t.

$$T^i: \quad \frac{d\left(k_t^i q_t\right)}{k_t^i q_t} = \left(\Phi(\iota_t^i) - \delta + \mu_t^q - \sigma_t^q \sigma\right) dt + \left(\sigma - \sigma_t^q\right) dW_t + \tilde{\sigma} d\tilde{W}_t^i, \tag{26}$$

where $\mathbb{Q}^i$ is the risk neutral measure. The revenues v_s are "net" the cost of purchasing the "input", which in returns equals $e^{-\int_t^s \frac{\iota_u^i}{q_u}du}$ for unit of capital. By Equation (26), we know that

$$v_s = \underbrace{k_t^i q_t e^{\int_t^s \left(\Phi(\iota_u^i)-\delta+\mu_u^q-\sigma_u^q\sigma\right)du-\frac{1}{2}\|\Sigma_t^i\|^2du+\int_t^s \Sigma_t^i d\mathbf{W}_t}}_{k_s q_s} e^{\int_t^s \frac{p_u-\iota_u^i}{q_u}du},$$

where $\Sigma_t^i = \left[\begin{array}{cc} \sigma_t & [\mathbb{1}_{i=p}]\,\tilde{\sigma} \end{array}\right]$ and $d\mathbf{W}_t = \left[\begin{array}{c} dW_t \\ d\tilde{W}_t \end{array}\right]$.

The FOC on ι_t^i requires that

$$\Phi'(\iota_u^i) = \frac{1}{q_u}, \quad \forall u \in (t,s)\,.$$

By Type II firms optimality condition in (25), the FOC on k_t^i implies a zero-profit condition such that

$$\mathbb{E}_t^{\mathbb{Q}^i}\left[e^{\int_t^s \left(\mu_u-\frac{1}{2}\|\Sigma_t^i\|^2-r_u\right)du+\int_t^s \Sigma_u^i d\mathbf{W}_u}\right] = 1. \quad \forall i, \tag{27}$$

Note that the zero profit condition is consistent with the equilibrium return on the i^{th} risky claim dR_t^i. In fact,

$$\mu_t := \frac{\mathbb{E}_t\left[dR_t^i\right]}{dt} = \frac{A-\iota_t^i}{q_t} + \Phi(\iota_t^i) + \mu_t^q - \delta - \sigma_t^q\sigma,$$

$$\| \Sigma_t \|^2 = (\sigma-\sigma_t^q)^2 + \tilde{\sigma}^2 = \frac{\mathbb{V}ar_t\left[dR_t^i\right]}{dt} \Longrightarrow \sigma_t := \sigma - \sigma_t^q.$$

Condition (27) is equivalent to a non-arbitrage condition: the return on risky claims issued by type I firms (equity), must be such that their present discounted value equals the current value of physical capital stock $k_t^i q_t$ supplied by the agents. If such a condition is satisfied, the firm breaks even for each k_t^i, its size is indeterminate, and it is willing to supply each market demand.

To grant the existence (and uniqueness) of the competitive equilibrium, condition (27) must be consistent with the no-arbitrage condition for the aggregate portfolio held by the financial sector. The result is summarised in the following Proposition:

Proposition 2. ***Risk Neutral Measure***
Given the zero-profit condition in (27) and the no arbitrage condition for the ag-

gregate portfolio, the market price of systematic risk equals $\xi_t = \frac{\mu_t^f - r_t}{\sigma_t}$*. The latter implies that there exists a unique* $\mathbb{Q}^i$ *such that the price kernel is well defined,*[45] *and the price of idiosyncratic risk* $\tilde{\xi}_t$ *satisfies*

$$\tilde{\xi}_t = \frac{\mu_t^h - \mu_t^f}{\tilde{\sigma}} \geq 0 \Longleftrightarrow \eta \geq 0.$$

Proof. Given the zero-profit condition in (27), by *Girsanov Theorem III* (see Øksendal, 2003), the correspondent Radon-Nykodym derivative equals

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} = \exp\left\{-\int_t^s \xi_u dW_u - \int_t^s \tilde{\xi}_u d\tilde{W}_u - \frac{1}{2}\int_t^s \left(\xi_u^2 + \tilde{\xi}_u^2\right) du\right\}.$$

where $\mathbb{P}$ is the real probability measure, while ξ_t and $\tilde{\xi}_t$ represent the market prices of systematic and idiosyncratic risk respectively. Given the no-arbitrage condition for the aggregate portfolio:

$$\mathbb{E}_t^{\mathbb{Q}^f}\left[e^{\int_t^s \left(\mu_u^f - \frac{1}{2}\sigma_u^2 - r_u\right)du + \int_t^s \sigma_u^2 dW_u}\right] = 1,$$

it follows that

$$\frac{d\mathbb{Q}^f}{d\mathbb{P}} = \exp\left\{-\int_t^s \xi_t du - \frac{1}{2}\int_t^s \xi_t^2 dW_u\right\} \Longleftrightarrow \xi_t = \frac{\mu_t^f - r_t}{\sigma_t}.$$

The latter implies that the martingale measure for the $i^t h$ firm $\mathbb{Q}^i$ satisfies

$$d\mathbf{W}_t^{\mathbb{Q}^i} = \begin{bmatrix} \xi_t \\ \tilde{\xi}_t \end{bmatrix} dt + d\mathbf{W}_t.$$

where $\tilde{\xi}_t = \frac{\mu_t^h - \mu_t^f}{\tilde{\sigma}} = \frac{1}{\tilde{\sigma}}\frac{\eta}{q_t}$ and, thus

$$k_s^i q_s e^{-\int_t^s \left(r_u - \frac{A - \iota_u}{q_u}\right)du} = k_t^i q_t e^{-\int_t^t \left(r_u - \frac{A - \iota_u}{q_u}\right)du} + \int_t^s \Sigma_t' d\mathbf{W}_t^{\mathbb{Q}^i}.$$

[45] When the intermediation costs are null $\eta = 0$, it follows that $\tilde{\xi}_t = 0$ and, in turn, $\mathbb{Q}^i = \mathbb{Q}^f$. This case is consistent with the benchmark where markets are complete.

By taking the expected value under the probability measure $\mathbb{Q}^i$, it follows that

$$\mathbb{E}_t^{\mathbb{Q}^i}\left[k_s^i q_s e^{-\int_t^s \left(r_u - \frac{A-\iota_u}{q_u}\right)du}\right] = k_t^i q_t + \underbrace{\mathbb{E}_t^{\mathbb{Q}^i}\left[\int_t^s \Sigma_t d\mathbf{W}_t^{\mathbb{Q}^i}\right]}_{0},$$

is a martingale under $\mathbb{Q}^i$. □

B Limited Participation and Transaction Costs

In this appendix we consider the generalisation of the competitive equilibrium in Section 3 where both classes of agents, h/entrepreneurs and financial intermediaries respectively, have full access to risk-free bonds and pooled (p) as well as un-pooled (n) risky claims. In particular we as assume that, in order to pool risky claims from different firms, the h/entrepreneurs have to pay a transaction cost ε. In this terms we show that *market segmentation* arises naturally when the transaction cost is big enough with respect to the financial intermediation cost η.

Given the problem in (3), the optimal pooled and un-pooled portfolio choices of both classes of agents satisfy the following:

$$\omega_t^{i,n} = \frac{\mu_t - r_t}{\sigma_t^2 + \tilde{\sigma}^2}, \quad i = h, f; \tag{28}$$

$$\omega_t^{f,p} = \frac{\mu_t - \frac{\eta}{q_t} - r_t}{\sigma_t^2}, \quad \omega_t^{h,p} = \frac{\mu_t - \frac{\varepsilon}{q_t} - r_t}{\sigma_t^2}. \tag{29}$$

In equilibrium, the whole amount of wealth invested in risky claims, whether it is pooled or not, must equal the aggregate amount of physical capital, whereas the risk-free bonds must be in zero net supply. By market clearing conditions, it follows that

$$\left(\omega_t^{h,n} + \omega_t^{h,p}\right)(1-\psi_t) + \left(\omega_t^{f,p} + \omega_t^{f,n}\right)\psi_t = 1, \tag{30}$$

$$\left(1 - \omega_t^{h,n} - \omega_t^{h,p}\right)(1-\psi_t) + \left(1 - \omega_t^{f,p} - \omega_t^{f,n}\right)\psi_t = 0. \tag{31}$$

By matching equations (28) and (29), the market clearing conditions (30) and

(31), it is possible to show that:

$$\omega_t^f = \omega_t^{f,n} + \omega_t^{f,p} = 1 + \frac{\varepsilon - \eta}{q_t} \frac{1 + \psi_t \frac{\tilde{\sigma}^2}{\sigma_t^2}}{2\sigma_t^2 + \tilde{\sigma}^2}, \tag{32}$$

and

$$\omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} = 1 - \frac{\frac{\varepsilon - \eta}{q_t}}{\sigma_t^2} \psi_t. \tag{33}$$

In this terms, we are looking for those parametric conditions such that there exists *limited market participation*, i.e. the financial sector always leverages its balance sheet by issuing risk-free bonds. Conversely, h/entrepreneurs smooth consumption by allocating their wealth into both risky and risk-free claims in positive amounts whatever share of total wealth. The aforementioned conditions are satisfies if the following holds:

$$\begin{cases} \omega_t^f = \omega_t^{f,p} + \omega_t^{f,n} > 1 \\ \omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} > 0, \\ \omega_t^h = \omega_t^{h,p} + \omega_t^{h,n} < 1. \end{cases} \tag{34}$$

By matching equations (32) and (33) with system (34), we find that the following conditions must hold

$$\varepsilon > \eta \Rightarrow \omega_t^h < 1, \omega_t^f > 1,$$

while

$$\varepsilon < \eta + \sigma_t^2 \frac{q_t}{\psi_t} \Rightarrow \omega_t^h > 0.$$

Is summary, the transition cost for h/entrepreneurs ε is required to be bounded:

$$\eta < \varepsilon < \eta + \min_{\psi_t} \left\{ \sigma_t^2(\psi_t) \frac{q_t(\psi_t)}{\psi_t} \right\}.$$

The lower bound grants a comparative advantage to the financial sector at pooling risk, whereas the upper bound prevents the h/entrepreneurs to short the un-pooled security in equilibrium.

C Proofs

C.1 The Agents' Problem

Given a generic CRRA utility function, we know that

$$\lim_{\gamma \to 1} \frac{c_t^{1-\gamma} - 1}{1 - \gamma} = \ln c_t. \tag{35}$$

To our purpose we consider the affine transform

$$u(c_t) \coloneqq \frac{c_t^{1-\gamma}}{1-\gamma}, \tag{36}$$

since (35) and (36) have the same maximizer. Given the agents' problem, the Hamiltonian satisfies

$$\rho V_t = \max_{\{\omega_t, c_t\}} \left\{ u(c_t) + \frac{1}{dt}\mathbb{E}_t \left[dV_t\right] \right\},$$

subjected to the terminal condition $\lim_{t \to \infty} e^{-\rho t} V(e_t^i) = 0$. Given the generic motion of wealth stock of the $i^t h$ agent,

$$\frac{de_t^i}{e_t^i} = \left[r_t + \omega_t \left(\mu_t^i - r_t \right) - \frac{c_t}{e_t} \right] dt + \omega_t \Sigma_t' \mathbf{dW}_t,$$

it is possible to demonstrate that the first order conditions satisfy

$$\frac{c_t}{e_t} = A^{-1} \tag{37}$$

and

$$\omega_t = \frac{\mu_t^i - r_t}{\gamma \left(\sigma_t^i\right)^2} + (A\Sigma_t')^{-1} A_z' \Omega_t^z, \tag{38}$$

where $\mu_t^<$ and σ_t^z represent the state drift and diffusion, respectively. By taking the limit for $\gamma \to 1$, it follows that the HJB for log-utility can be then written as

$$1 + A_t' + A_z' \mu_t^z + \frac{A_{zz}'' \left(\Omega_t^z\right)' \left(\Omega_t^z\right)}{2} = \rho A. \tag{39}$$

By Feynman-Kač Theorem (see Huyên (2009)), the solution of Equation (39) satisfies

$$A(t, z_t) = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} ds \right] = \frac{1}{\rho}.$$

From (37) and (38), it follows that

$$c_t = \rho n_t,$$

and

$$\omega_t = \frac{\mu_t^i - r_t}{\left(\sigma_t^i\right)^2}, \tag{40}$$

respectively. The result is equivalent to (Merton, 1969; Samuelson, 1969).

C.2 Equilibrium Portfolios, Leverage, and Prices

According to Definition 2, the market clearing conditions for physical capital and risk-free bonds in Equations (8) and (9) can be written in terms of relative wealth share as:

$$\omega_t^f E_t^f + \omega_t^h E_t^h = K_t q_t \Longleftrightarrow \omega_t^h (1 - \psi_t) + \omega_t^f \psi_t = 1,$$

$$\frac{E_t^f (1 - \omega_t^f) + E_t^h (1 - \omega_t^h)}{K_t q_t} = 0 \Longleftrightarrow \left(1 - \omega_t^h\right)(1 - \psi_t) = (\omega_t^f - 1)\psi_t. \tag{41}$$

The clearing on the consumption good market (10) holds by Walras Law and equals:

$$(A - \iota_t - \eta) K_t \psi_t + (A - \iota_t) K_t (1 - \psi_t) = \rho K_t q_t. \tag{42}$$

By matching the market clearing condition on capital (42) and the optimal portfolios (40), we obtain

$$\omega_t^f = \frac{1}{\psi_t} - \frac{\mu_t - r_t}{\sigma_t^2} \frac{(1 - \psi_t)}{\psi_t}$$

which, by Lemma 1, can be written as

$$\omega_t^f = \frac{1}{\psi_t} - \frac{\frac{\eta}{q_t}}{\sigma_t^2 + \tilde{\sigma}^2} \frac{(1-\psi_t)}{\psi_t} - \frac{(1-\psi_t)}{\psi_t} \underbrace{\frac{\left(\mu_t^f - r_t\right)}{\sigma_t^2}}_{\omega_t^f} \frac{\sigma_t^2}{\sigma_t^2 + \tilde{\sigma}^2}.$$

Rearranging we find that

$$\omega_t^f = \frac{\sigma_t^2 + \tilde{\sigma}^2 - \frac{\eta}{q_t}(1-\psi_t)}{\psi_t \tilde{\sigma}^2 + \sigma_t^2}. \tag{43}$$

By substituting (43) into the market clearing for the risk-free bond (41), it is straightforward to find ω_t^h. Similarly, the equilibrium interest rate r_t can be obtained from (40). The results are summarized in the following Proposition:

Proposition 3. ***Equilibrium Portfolios and Interest Rate***
Equilibrium portfolio shares ω_t^f, ω_t^h and the interest rate r_t depend on relative wealth share ψ_t only:

$$\omega_t^f = \frac{\tilde{\sigma}^2 + \sigma_t^2 - \frac{\eta}{q_t}(1-\psi_t)}{\psi_t \tilde{\sigma}^2 + \sigma_t^2}, \quad \omega_t^h = 1 - \left(\omega_t^f - 1\right) \frac{\psi_t}{1-\psi_t}.$$

$$r_t = \frac{\rho}{\psi_t} - \left(\frac{1 + \theta A - q_t}{\theta q_t}\right) \frac{1-\psi_t}{\psi_t} + \Phi(\iota_t) - \delta + \mu_t^q - \sigma_t^q \sigma - (\sigma_t)^2 \omega_t^f.$$

We assumed both classes of agents have the same preferences. It follows that the portfolio share of the financial sector must be greater than or equal to 1. This is because, since the risk-free bond is in zero net supply, a positive portfolio share in bonds by the financial sector must be supplied by h/entrepreneurs. In equilibrium, this is not possible due to h/entrepreneurs assets exposure to idiosyncratic risk. Under the assumption that the idiosyncratic volatility $\tilde{\sigma}^2$ is greater then the intermediation cost rate $\frac{\eta}{q_t}$, the financial sector portfolio share ω_t^f is strictly greater then 1. The result is summarised in the following Corollary of Proposition 3 :

Corollary 1. ***Financial Leverage***
When the idiosyncratic volatility is greater then the intermediation cost rate, the financial sector holds a leveraged position, while the h/entrepreneurs hold positive portfolio shares in both risky and risk-free claims:

$$\tilde{\sigma}^2 > \frac{\eta}{q_t}, \quad \Rightarrow \omega_t^f > 1, \quad \omega_t^h \in (0,1),$$

Proof. The result comes after solving $\omega_t^f > 1$. □

Under the assumption of log investment function $\Phi(\iota_t) = \frac{\ln(\theta\iota_t+1)}{\theta}$, where the parameter θ represents the cost of technical non-fungibility between physical capital and consumption good, the price of physical capital q_t and the rate of reinvestment ι_t are affine transforms of the state ψ_t. In fact, by matching the consumption market clearing condition in (42) with the *Tobin's Q* in Equation (13), it follows that:

$$q_t = \frac{1+\theta(A-\eta\psi_t)}{1+\theta\rho}, \quad \iota_t = \frac{q_t-1}{\theta}. \tag{44}$$

C.3 Proof of Theorem 1, points 1 and 2

Given the state

$$\psi_t := \frac{E_t^f}{K_t q_t},$$

by Itô's lemma,

$$d\psi_t = \frac{\partial\psi_t}{\partial E_t^f} dE_t^f + \frac{\partial\psi_t}{\partial K_t q_t} dK_t q_t + \frac{1}{2}\frac{\partial^2\psi_t}{\partial\left(E_t^f\right)^2}\left(dE_t^f\right)^2 + \\ + \frac{1}{2}\frac{\partial^2\psi_t}{\partial\left(K_t q_t\right)^2} d\left(K_t q_t\right)^2 + \frac{\partial^2\psi_t}{\partial\left(K_t q_t\right)\partial E_t^f} dK_t q_t dE_t^f.$$

By substituting the optimal portfolio in the budget constraint of the financial sector we have

$$\frac{dE_t^f}{E_t^f} = \left(1-\omega_t^f\right) r_t dt - \rho dt + \mu_t^f \omega_t^f dt + \omega_t^f \sigma_t dW_t, \tag{45}$$

while the aggregate wealth evolves as

$$\frac{dK_t q_t}{K_t q_t} = \mu_t^f dt + \psi_t \frac{\eta}{q_t} dt + \sigma_t dW_t - \rho dt. \tag{46}$$

Given Equations (45) and (46) it follows that

$$d\psi_t = \psi_t \frac{dE_t^f}{E_t^f} - \psi_t \frac{dK_t q_t}{K_t q_t} + \psi_t \sigma_t^2 dt - \psi_t \sigma_t^2 \omega_t^f dt.$$

By considering Proposition 3 and rearranging,

$$\frac{d\psi_t}{\psi_t} = \underbrace{\sigma_t^2 \left[1 + \omega_t^f \left(\omega_t^f - 2\right) - \frac{\psi_t}{\sigma_t^2}\frac{\eta}{q_t}\right]}_{\mu_t^\psi} dt + \underbrace{\sigma_t \left(\omega_t^f - 1\right)}_{\sigma_t^\psi} dW_t.$$

Point 2 can be proved by looking for a Markov equilibrium in the state variable ψ_t. Similarly to Haven et al. (2016), if such an equilibrium exists, one must be able to express both drifts and diffusion in Equation (12) as a function of ψ_t only. By Itô's lemma,

$$dq_t = q'(\psi_t)\psi_t\mu_t^\psi dt + \frac{1}{2}q''(\psi_t)\psi_t^2 \left(\sigma_t^\psi\right)^2 dt - q'(\psi_t)\psi_t\sigma_t^\psi dW_t. \tag{47}$$

By matching drifts and diffusions of the dynamic Equations (47) and (12) we obtain the system in (16).

C.4 Proof of Theorem 1, points 3 and 4

Persistent heterogeneity In the neighbourhood of the right-hand side boundary, $\lim_{\psi\to 1^-} \sigma_t^q = 0$ implies, by continuity, that

$$\lim_{\psi\to 1^-} \omega_t^f = 1 \Rightarrow \lim_{\psi\to 1^-} \sigma_t^\psi = 0.$$

By the latter,

$$\lim_{\psi\to 1^-} \mu_t^\psi = -\left(\frac{\eta\,(1+\theta\rho)}{1+\theta(A-\eta)}\right) < 0 \iff \eta > 0. \tag{48}$$

Similarly, in the neighbourhood of the left-hand side boundary, $\lim_{\psi_t\to 0^+} \sigma_t^q = 0$. The latter implies that

$$\lim_{\psi\to 0^+} \mu_t^\psi = \Delta^2, \quad \lim_{\psi\to 0^+} \sigma_t^\psi = \Delta,$$

where, by (44),

$$\Delta = \tilde{\sigma}^2 - \underbrace{\left(\frac{\eta\,(1+\theta\rho)}{1+\theta(A-\eta)}\right)}_{\frac{\eta}{\bar{q}}}$$

is a positive constant. It follows that, in the surroundings of the left-hand side boundary, the dynamics of ψ_t behaves as a geometric Brownian motion with positive drift:

$$\psi_t^\epsilon = \epsilon \exp\left\{ \left(\frac{1}{2}\Delta^2 \right) t + \Delta W_t \right\}, \tag{49}$$

where ϵ is a positive number arbitrary close to 0. Hence, the process never reaches the absorbing state $\psi = 0$.

Given the Markov equilibrium in Theorem 1, and conditions (48) and (49), we know that state drift μ_t^ψ has positive sign at the left-hand side boundary whereas it is negative sign at the right-hand side one. It suffices to prove its derivative negative along the whole domain to grant a unique $\hat{\psi} \in (0,1)$ such that $\mu_t^\psi(\hat{\psi}) = 0$. In this fashion

$$\frac{\partial}{\partial \psi_t}\mu_t^\psi < 0, \forall \psi_t \in (0,1), \tag{50}$$

which leads to,

$$\underbrace{2\left(\omega_t^f\right)'(\sigma - \sigma_t^q)^2\left(\omega_t^f - 1\right)}_{A} - 2\left(\omega_t^f\right)^2(\sigma - \sigma_t^q)(\sigma_t^q)' +$$

$$\underbrace{-\frac{\eta\left(\rho + \frac{1}{\theta}\right)}{\frac{1}{\theta} + \eta\psi_t + A} - \psi_t \frac{\eta^2\left(\rho + \frac{1}{\theta}\right)}{\left[\frac{1}{\theta} + \eta\psi_t + A\right]^2}}_{B} +$$

$$-2(\sigma - \sigma_t^q)(\sigma_t^q)' + 4(\sigma - \sigma_t^q)(\sigma_t^q)'\omega_t^f < 0.$$

and, after some algebra,

$$(\sigma_t^q)'(\sigma - \sigma_t^q)\left(\omega_t^f - 1\right)^2 > \frac{A+B}{2}. \tag{51}$$

Provided that we assume (and numerically check) $(\omega_t^f)' < 0$, $\sigma > \sigma_t^q > 0$ and $(\sigma_t^q)' \geq 0$, condition (51) is always satisfied, since $A, B < 0$ and $\omega_t^f > 1$. Moreover, by Theorem 1 $\sigma_t^\psi \propto \sigma_t^q$, it follows that $\sigma_t^q > 0 \Rightarrow \sigma_t^\psi > 0$.

By considering the dynamics of $d\psi_t$ in Theorem 1, a unique ergodic distribution $\pi(\psi)$ exists as long as the first two moments of ψ_t exist and are finite. A rigorous discussion of the sufficient conditions of existence of the ergodic for Ito's Processes is in Zhenzhong and Chen (2013). Although we cannot derive closed-form solution for ψ_t, its first moment can be determined as

$$d\left(e^{-\int_0^t \mu_s^\psi ds}\psi_t\right) = -e^{-\int_0^t \mu_s^\psi ds}\mu_s^\psi \psi_t dt + e^{-\int_0^t \mu_s^\psi ds} d\psi_t = e^{-\int_0^t \mu_s^\psi ds}\psi_t \sigma_t^\psi dW_t.$$

If we integrate both sides and take expected value, we have

$$\mathbb{E}_0\left[\psi_t\right] = \psi_0 \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] + \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\int_0^t e^{-\int_0^s \mu_u^\psi du}\psi_s \sigma_s^\psi dW_s\right].$$

Since the term in dW_s is an Itô integral, it has expected value equals zero and thus

$$\mathbb{E}_0\left[\psi_t\right] = \psi_0 \mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] \tag{52}$$

where ψ_0 is an arbitrary starting point. Thus, the first moment of the distribution is defined as long as $\mathbb{E}_0\left[e^{\int_0^t \mu_s^\psi ds}\right] < \infty$. We prove it numerically by simulation. Similarly we can derive the variance as

$$\mathbb{V}ar_0\left[\psi_t\right] = \mathbb{E}_0\left[\psi_t^2\right] - \mathbb{E}_0\left[\psi_t\right]^2. \tag{53}$$

The first term of (53) we can be found by solving

$$d(x^2) = 2xdx + 2dx^2$$

where $x = e^{-\int_0^t \mu_s^\psi ds}\psi_t$, which leads to

$$d\left(e^{-2\int_0^t \mu_s^\psi ds}\psi_t^2\right) = 2e^{-\int_0^t \mu_s^\psi ds}\psi_t e^{-\int_0^t \mu_s^\psi ds}\psi_t \sigma_t^\psi dW_t + e^{-2\int_0^t \mu_s^\psi ds}\left(\psi_t \sigma_t^\psi\right)^2 dt.$$

It follows that

$$\mathbb{E}_0\left[\psi_t^2\right] = \psi_0^2 \mathbb{E}_0\left[e^{2\int_0^t \mu_s^\psi ds}\right] + \mathbb{E}_0 E\left[e^{2\int_0^t \mu_s^\psi ds}\int_0^t e^{-2\int_0^t \mu_s^\psi ds}\left(\psi_s \sigma_s^\psi\right)^2 ds\right]$$

and thus

$$\mathbb{V}ar_0\left[\psi_t\right] = 2\mathbb{E}_0\left[e^{2\int_0^t \mu_s^\psi ds}\int_0^t e^{-2\int_0^s \mu_u^\psi du}\left(\psi_s \sigma_s^\psi\right)^2 ds\right].$$

Thus, the second (central) moment of the distribution is defined as long as $\mathbb{V}ar_0\left[\psi_t\right] < \infty$. We prove it numerically by simulation.

The Ergodic Density The Fokker-Plank equation for the ergodic density satisfies

$$\frac{\partial}{\partial t}\pi(\psi,t) = -\frac{\partial}{\partial \psi}\left\{\psi\mu^{\psi}\pi\left(\psi,t\right) - \frac{1}{2}\frac{\partial}{\partial \psi}\left[\psi^{2}\left(\sigma_{t}^{\psi}\right)^{2}\pi\left(\psi,t\right)\right]\right\} = 0. \tag{54}$$

By integrating over $(0,\psi)$ and rearranging, we can write (54) as the following ODE

$$d\ln h\left(\psi\right) = 2\frac{\mu^{\psi}}{\psi\left(\sigma^{\psi}\right)^{2}},$$

where

$$h\left(\psi\right) = \pi\left(\psi\right)\psi^{2}\left(\sigma^{\psi}\right)^{2}.$$

By integrating one more time, given a boundary condition $h(0) = h_0$, we obtain the density function of ψ_t as

$$\pi\left(\psi\right) = \frac{h_0 e^{\int_0^{\psi}\frac{2\mu^{\psi}(s)}{s\left(\sigma^{\psi}(s)\right)^{2}}ds}}{\psi^{2}\left(\sigma^{\psi}\right)^{2}},$$

where h_0 is such that $\int_0^1 \pi(\psi)d\psi = 1$.

C.5 Proof of Proposition 1

By following the approach in Haven et al. (2016), we know that the optimal consumption of the log agent satisfies $\ln c_t^i = \ln \rho e_t^i$. By considering the dynamics of aggregate wealth

$$\frac{de_t^i}{e_t^i} = \mu_t^{e,i}dt + \sigma_t^{e,i}dW_t,$$

it follows that, by Itô's Lemma,

$$d\ln\rho e_t^i = \frac{1}{\rho}\left[\mu_t^{e,i} - \frac{1}{2}\left\|\Sigma_t^i\right\|^2\right]dt + \frac{1}{\rho}\Sigma_t^i\mathbf{dW}_t.$$

By integrating over $(0,s)$ and multiplying by $e^{-\rho s}$

$$e^{-\rho s}\left(\ln\rho e_s^i - \ln\rho e_0^i\right) = e^{-\rho s}\left\{\int_0^s\left[\mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2\right]du + \int_0^s \Sigma_t^i\mathbf{dW}_u\right\}.$$

By integrating over $(0,\infty)$ and taking expected value

$$\underbrace{\mathbb{E}_0\left[\int_0^\infty e^{-\rho s}\ln\rho e_s^i ds\right]}_{W^i(\psi)} = \frac{\ln\rho e_0^i}{\rho} + \mathbb{E}_0\left[\int_0^\infty e^{-\rho s}\int_0^s \mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2 duds\right].$$

By changing variable

$$\int_0^\infty e^{-\rho s}\left[\int_0^s \mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2 du\right] ds = \int_0^\infty\int_u^\infty e^{-\rho s}\left[\mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2 du\right] dsdu$$

$$\int_0^\infty \left\{-\frac{1}{\rho}e^{-\rho s}\left[\mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2 du\right]\right\}_u^\infty du = \frac{1}{\rho}\int_0^\infty e^{-\rho u}\left[\mu_u^{e,i} - \frac{1}{2}\left\|\Sigma_u^i\right\|^2 du\right] du.$$

Thus, for a unitary aggregate capital $K_t = 1$, considering agents of the class h

$$W^h(\psi_0) = \frac{\ln\rho q_0(1-\psi_0)}{\rho} + \frac{1}{\rho}\mathbb{E}_0\left[\int_0^\infty e^{-\rho s}\underbrace{\mu_s^{e,h} - \frac{1}{2}\left(\omega_s^h\right)^2\left(\sigma_s^2 + \tilde{\sigma}^2\right)}_{f(\psi_s)} ds\right]. \quad (55)$$

By Feynman-Kač Theorem (see Huyên (2009)),

$$W^h(\psi_t) = \frac{\ln\rho q_t(1-\psi_t)}{\rho} + \frac{1}{\rho}H^h(\psi_t).$$

where $H(\psi_t)^p$ solves the associated ODE

$$\rho H(\psi)^h = f(\psi) + H(\psi)_\psi^h \mu^\psi \psi + \frac{H(\psi)_{\psi\psi}^h\left(\sigma^\psi\psi\right)^2}{2}.$$

We compute the value of H^i conditional on the state ψ_0 by numerical simulation.

C.6 Constrained Portfolios

By considering the constrained version of the problem in Appendix C.1, by standard dynamic programming the HJB satisfies

$$\rho V_t = \max_{\{\omega_t, c_t\}}\left\{\ln c_t + \frac{1}{dt}\mathbb{E}_t\left[dV_t\right] - \lambda_t(\omega_t - LC)\right\}$$

where λ_t is the Lagrangian multiplier associated to the constraint

$$\omega_t \leq LC.$$

By taking FOCs and considering complementary slackness, given the dynamics of V_t, the optimal portfolio share ω_t^C satisfies the following system:

$$\begin{cases} \omega_t^U - \omega_t^C = \frac{\lambda_t}{\rho \sigma_t^2}, \\ \lambda_t \left(\omega_t^C - LC \right) = 0, \\ \lambda_t \geq 0, \\ \omega_t^C - LC \leq 0, \end{cases} \tag{56}$$

where ω_t^U is the unconstrained solution. The possible couples $\left\{ \omega_t^C, \lambda_t \right\}$ that satisfy (56) are:

$$\begin{cases} \omega_t^C = \omega_t^U, \lambda_t = 0 & \omega_t^U < LC \\ \omega_t^C = LC, \lambda_t = \rho \sigma_t^2 \left(\omega_t^U - LC \right) & \omega_t^U \geq LC. \end{cases}$$

C.7 Macro-dynamics

By Itô's lemma, the dynamics of aggregate consumption $C_t = (A - \iota_t - \psi_t \eta) K_t$ is given by

$$dC_t = (A - \iota_t - \psi_t \eta) dK_t - K_t d\iota_t - \eta K_t d\psi_t - \mathbb{C}ov \left[d\iota_t, dK_t \right] - \mathbb{C}ov \left[d\psi_t, dK_t \right].$$

By considering the stochastic processes dK_t and $d\psi_t$ and

$$d\iota_t = \frac{1}{\theta} \left(q_t \mu_t^q - q_t \sigma_t^q dW_t \right),$$

we obtain, by substitution and rearranging

$$\begin{aligned} \frac{dC_t}{C_t} = (\Phi(\iota_t) - \delta) dt - \frac{1}{\theta} \frac{q_t \mu_t^q + \theta \eta \psi_t \mu_t^\psi + q_t \sigma_t^q \sigma - \theta \psi_t \left(\sigma - \sigma_t^q \right) \left(\omega_t^f - 1 \right) \sigma}{A - \iota_t - \psi_t \eta} dt + \\ + \sigma \left(1 - \left(1 - \frac{\sigma_t^q}{\sigma} \right) \eta \psi_t \frac{\omega_t^f - 1}{A - \iota_t - \psi_t \eta} + \frac{q_t}{\theta} \frac{\sigma_t^q}{\sigma} \frac{1}{A - \iota_t - \psi_t \eta} \right) dW_t. \end{aligned}$$

By Itô's lemma, the dynamics of aggregate investment $I_t = \iota_t K_t$ is given by

$$dI_t = d\left(\iota_t K_t\right) = K_t d\iota_t + \iota_t dK_t + \mathbb{C}ov\left[d\iota_t dK_t\right],$$

and, after substituting and rearranging,

$$\frac{dI_t}{I_t} = \left[\Phi(\iota_t) - \delta + \frac{q_t}{\theta \iota_t}\left(\mu_t^q - \sigma_t^q \sigma\right)\right] + \sigma\left(1 - \frac{1}{\theta}\frac{\sigma_t^q}{\sigma}\frac{q_t}{\iota_t}\right) dW_t.$$

Similarly, the dynamics of aggregate intermediation costs $G_t = \eta\psi_t K_t$ is given by

$$dG_t = d\left(\eta\psi_t K_t\right) = \eta\left[\psi_t dK_t + K_t d\psi_t + \mathbb{C}ov\left(d\psi_t, dK_t\right)\right],$$

thus

$$\frac{dG_t}{G_t} = \left[\Phi(\iota_t) - \delta + \mu_t^\psi + \sigma\sigma_t^\psi\right] dt + \sigma\left[1 + \left(1 - \frac{\sigma_t^q}{\sigma}\right)\left(\omega_t^f - 1\right)\right] dW_t.$$

It follows that

$$\frac{d\tilde{Y}_t}{\tilde{Y}_t} = \Phi(\iota_t) - \delta + \psi_t\eta\frac{\mu_t^\psi + \sigma\sigma_t\left(\omega_t^f - 1\right)}{A - \psi_t\eta} dt + \sigma\left[1 - \psi_t\left(1 - \frac{\sigma_t^q}{\sigma}\right)\frac{\left(\omega_t^f\right)\eta}{A - \psi_t\eta}\right] dW_t.$$

D Redistributive Taxation

In this appendix, we describe the equilibrium dynamics of the relative financial capitalization ψ when an exogenous taxation evenly redistributes resources at a rate τ from the financial sector to the h/entrepreneurs. In this setting, we consider the case where the taxation is constant and equals τ for every value of the state $\psi \in (0, 1)$.

Since all the agents have log preferences and the tax transfer is proportional to their whole stock of wealth, it does not directly affect their portfolio and consumption choices. It does instead affect their conditional and unconditional welfare.

Let the dynamic budget constraint of the h/entrepreneurs' and of the financial sector evolve as

$$dE_t^h = E_t^h\left(\mu_t^{e,h} dt + \sigma_t^{e,h} dW_t\right) + \underbrace{\tau E_t^f dt}_{\text{(Positive) Tax}}, \tag{57}$$

$$dE_t^f = E_t^f \left(\mu_t^{e,f} dt + \sigma_t^{e,f} dW_t\right) - \underbrace{\tau E_t^f dt}_{\text{(Negative) Tax}} \quad , \tag{58}$$

respectively, where the drift and diffusion terms $\mu_t^{e,i}, \sigma_t^{e,i}$ $i \in \{h, f\}$ are defined in Equation (4). The *tax* terms in Equations (57) and (58) represent the redistribution effect of wealth between sectors by mean of the taxation policy. Note that the absolute value of the tax, τE_t^f, is directly proportional to the financial sector stock of wealth E_t^f; as such it proportionally enters the h/entrepreneurs' dynamic budget constraint.

By Itô's Lemma, the level of relative financial capitalization evolves as

$$d\left(\frac{E_t^b}{E_t^b + E_t^h}\right) = \frac{E_t^h}{\left(E_t^b + E_t^h\right)^2} dE_t^b - \frac{E_t^b}{\left(E_t^b + E_t^h\right)^2} dE_t^h + $$
$$+ \frac{\partial \psi}{\partial E^h \partial E^b} dE_t^h dE_t^b + \frac{1}{2} \frac{\partial^2 \psi}{\partial^2 E^h} \left(dE_t^h\right)^2 + \frac{1}{2} \frac{\partial^2 \psi}{\partial^2 E^b} \left(dE_t^b\right)^2 ,$$

where the dynamics of wealth follow the processes in (57) and (58). By substituting, rearranging, and considering: $\frac{E_t^h}{E_t^f} := \frac{1}{\psi_t} - 1$:

$$\frac{d\psi_t^\tau}{\psi_t^\tau} = \frac{d\psi_t}{\psi_t} - \tau \frac{\psi_t}{1 - \psi_t} dt,$$

where the process $\frac{d\psi_t}{\psi_t}$ is defined as in (15).

E Comparative Dynamics

In this Appendix, we discuss the changes of equilibrium dynamics with respect to the key parameters in the model, namely the size of systematic and idiosyncratic risk as well as intermediation costs.

Figure 18 shows the drift (left) and diffusion (right) of the process $d\psi_t$ as a function of the state $\psi \in (0, 1)$ for different values of systematic diffusion σ. In Figure 19, we perform the same comparative statics for equilibrium portfolio shares ω^f and ω^h. In the bottom graphs, we consider two sections of the upper ones for increasing levels of $\sigma = 0.2$ (blue) and 0.6 (green). In red, we plot the benchmark case of the *full risk pooling* economy.

With reference to Figure 18, when the financial sector is arbitrary well capitalised (ψ is high), decreasing systematic risk σ has the effect of reducing σ^ψ:

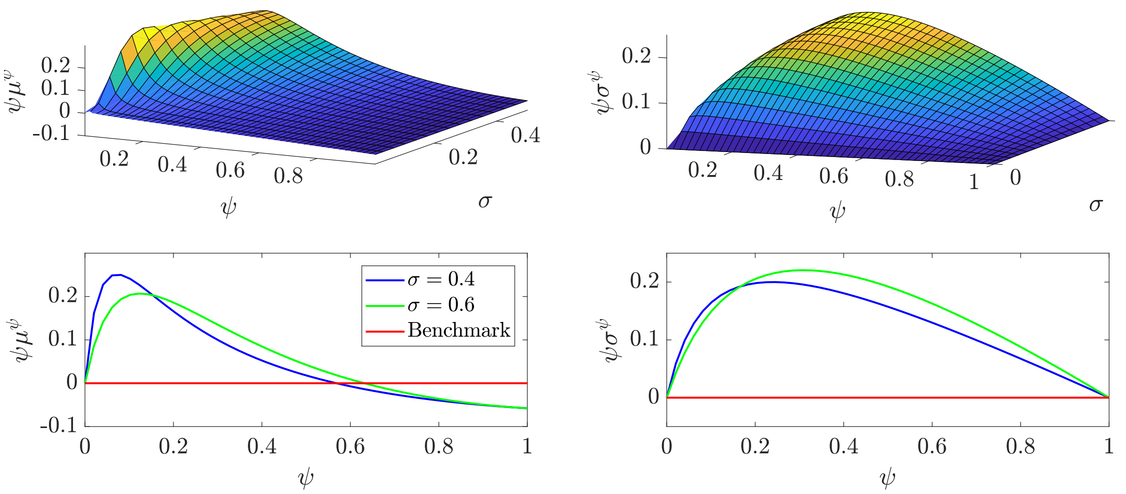

Figure 18: Top: Drift (left) and diffusion (right) of the process $d\psi_t$ for different values of systematic volatility σ. Bottom: Drift and diffusion for high (green) and low (blue) values of σ. In red, the benchmark case of the *full risk pooling* economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 1.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

the lower the risk, the lower both state drift and diffusion. When instead ψ approaches the left side boundary $\underline{\psi} = 0$, a lower σ is associated to higher leverage and reduced risky asset in h/entrepreneurs' portfolio (Figure 19). Indeed, higher leverage is associated to a sharper drift μ^ψ. This phenomenon is associated to the so-called *volatility paradox* (Adrian and Boyarchenko, 2012; Brunnermeier and Sannikov, 2014; Phelan, 2016).

Figure 20 displays a similar exercise by plotting equilibrium portfolio choices over $\psi \in (0, 1)$ with respect to different values of idiosyncratic diffusion $\tilde{\sigma}$. In the bottom graphs, we consider two sections of the upper ones for increasing levels of $\tilde{\sigma} = 0.8$ (blue) and 1.2 (green). In red, we plot the benchmark case of the *full risk pooling* economy. What stands out is that the lower the idiosyncratic risk the lower the equilibrium leverage of the financial sector. This pattern is the consequence of a reduced advantage of the financial sector due to pooling: when idiosyncratic risk is relatively lower, the demand for mitigation is also reduced, h/entrepreneurs keep a wider fraction of their wealth allocated in risky claims, and equilibrium risk-free rate is higher.

Finally, in Figure 21 (top) we repeat the same analysis for different values of intermediation costs η. In the bottom graphs, we consider two sections of the upper ones for null (blue) and positive (green) intermediation costs η. In red, we

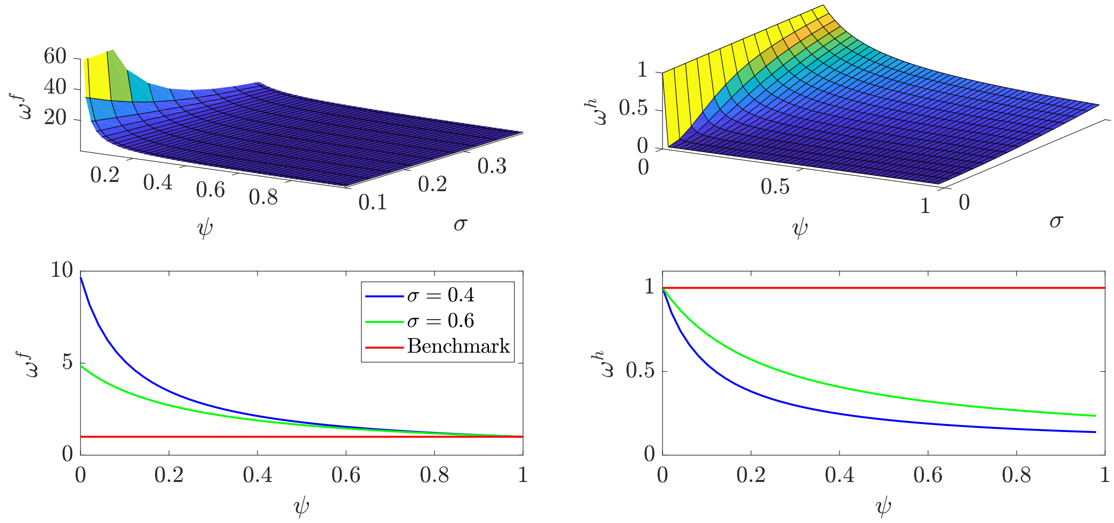

Figure 19: Top: Equilibrium portfolio shares ω^f (left) and ω^h (right) for different values of systematic diffusion σ. Bottom: Portfolio shares for high (green) and low (blue) values of σ. In red, the benchmark case of the *full risk pooling* economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 1.2$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

plot the benchmark case of complete markets. From Figure 21 we notice that, when there are no intermediation costs, the drift μ_t^ψ is positive for each ψ. In the long-run the financial sector dominates and thus its drains the whole wealth in the economy.[46] Moreover, positive intermediation costs (green) mainly affect the right-hand side of the state space, when ψ approaches $\bar{\psi} = 1$. Higher costs progressively sharpen the negative drift, when the financial sector is relatively well capitalised, making faster the recovery of h/entrepreneurs relative wealth.

Comparative Ergodic Figure 22 (top) shows the density $\pi(\psi)$ for different values of σ (left). We repeat the same exercise for different values of $\tilde{\sigma}$ (right). In the bottom panels, we graph two sections of the upper ones for high (green) and low (blue) values of σ and $\tilde{\sigma}$ respectively.

The most relevant feature of Figure 22 (right) is that when σ increases, so it does the average financial sector wealth share, whereas when the systematic risk crosses a certain threshold (the magnitude of the idiosyncratic risk $\tilde{\sigma}$), the ergodic wealth share shifts back towards a lower financial sector wealth share.

In this regards, whatever the sources of risk which is increased, the pooling power of the financial sector gets more valuable. This pattern persists as long

[46] This case is equivalent to the equilibrium where markets are complete for both classes agents.

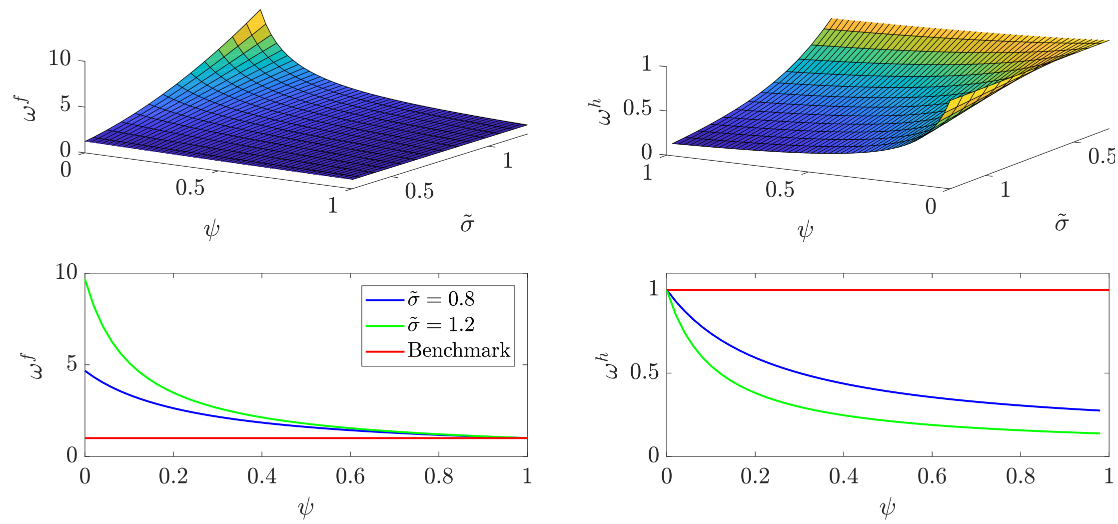

Figure 20: Top: Equilibrium portfolio shares ω^f (left) and ω^h (right) for different values of idiosyncratic diffusion $\tilde{\sigma}$. Bottom: Portfolio shares for high (green) and low (blue) values of $\tilde{\sigma}$. In red, the benchmark case of the *full risk pooling* economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\sigma = 0.4$, $\eta = 0.1$, $\theta = 2$, and $\rho = 0.05$.

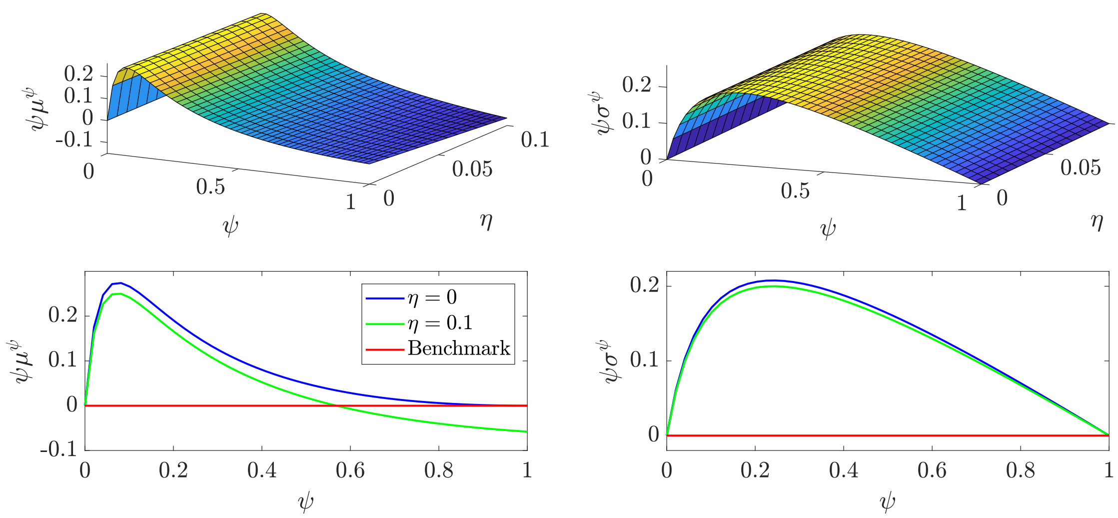

Figure 21: Top: Drift (left) and diffusion (right) of the process $d\psi_t$ for different intermediation costs η. Bottom: Drift and diffusion for high (green) and low (blue) values of η. In red, the benchmark case of the *full risk pooling* economy. Baseline parameters: $A = 0.5$, $\delta = 0.05$, $\tilde{\sigma} = 1.2$, $\sigma = 0.4$, $\theta = 2$, and $\rho = 0.05$.

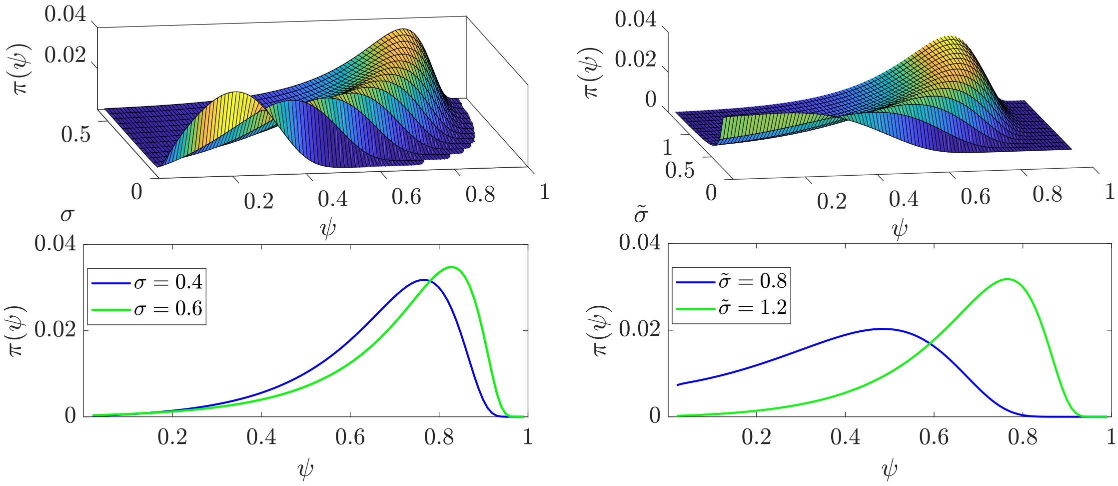

Figure 22: Top: Ergodic density $\pi(\psi)$ of relative wealth share ψ_t for different values of σ (left) $\tilde{\sigma}$ (right). Bottom: Ergodic density for high (green) and low (blue) values of σ (left) and $\tilde{\sigma}$ (right). Baseline parameters: $\delta = 0.05$, $\tilde{\sigma} = 1.2$, $\eta = 0.1$, $\theta = 2$, $\sigma = 0.4$ and $\rho = 0.05$.

as the idiosyncratic diffusion $\tilde{\sigma} > \sigma$. Furthermore, the higher σ, the higher the probability of extremely low capitalization, due to increasing leverage.

Similarly, to decreasing idiosyncratic diffusion is associated greater density mass to states where the financial sector has lower capitalisation (see Figure 22, right). In other words, if the pooling power does not reward the associated cost, the financial sector turns progressively marginal. Conversely, the higher the idiosyncratic risk component with respect to systematic, the higher the ergodic wealth share of the financial sector: since h/entrepreneurs are more exposed to extra risks, the aggregate demand for bonds rises and thus intermediaries have greater pooling power and pay lower interest rates.