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Testing for boundary conditions in case of fractionally integrated processes

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Abstract

Bounded integrated time series are a recent development of the time series literature. In this paper, we work on testing the presence of unknown boundaries with particular attention to the class of fractionally integrated time series. We firstly show, via a preliminary Monte Carlo experiment, the effects of neglected boundaries conditions on the most commonly used estimators of the long memory parameter. Then, we develop a sieve bootstrap test to distinguish between unbounded and bounded fractionally integrated time series. We assess the finite sample performance of our test with a Monte Carlo experiment and apply it to the data set of the time series of the Danish Krone/Euro exchange rate.

Keywords Bounded fractionally integrated processes · Range statistics · Sieve bootstrap

JEL Classifications C1: Econometric and Statistical Methods: general

1 Introduction

In recent years, interest has been growing in bounded time series, i.e. series whose evolution is limited either below or above or both (Granger 2010). From an economic point of view, bounded time series can be observed when agents regulate the dynamic of an economic variable only in case its value overtakes a certain threshold (bound). Time series can be also bounded by construction: expenditure shares, unemployment rate, exchange rates are only some of the possible examples of real data time series that can correspond to this definition.

Until recently, Cavaliere (2002, 2005a) and Cavaliere and Xu (2014) were the only efforts to develop a theory for bounded integrated processes. In particular, Cavaliere

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(2005a) works on asymptotic distributions for well known unit root test statistics when the driving series is a bounded $I(1)$ process. These results were further studied in Cavaliere and Xu (2014) by broadening the framework to allow for more general innovation structures.

As for fractionally integrated processes, $I(d)$ (for some $d \in R$), to the best of our knowledge the only proposal is by Trokic (2013), who, building on Cavaliere (2005a), introduces a model for bounded fractionally integrated processes. Specifically, he extends Cavaliere 2005a's framework and develops the limiting distribution in case of bounded $I(d)$ processes under general innovation structures and shows that, effectively, bounded $I(d)$ processes tend in distribution to a regulated fractional Brownian motion.

The theoretical papers mentioned above are based on the hypothesis that the boundaries that constrain the series to move within a closed interval are known. Undoubtedly, a priori knowledge about the existence of boundary conditions is often not realistic and in this vein Cavaliere (2002) proposes a class of tests to distinguish between bounded and unbounded $I(1)$ processes, based on the rescaled range of the process.

To our knowledge, there is nothing to test for boundary conditions in case of fractionally integrated processes. Here we intend to fill this gap by extending the class of tests introduced by Cavaliere (2002) to the fractional case. In fact, being aware of the existence of boundary conditions is in the fractional case even more important not only from the interpretation point of view, but also from the statistical properties point of view. As shown in the Monte Carlo experiment in the second section of this paper, neglecting the bounded nature of the process can affect the estimates of the long memory parameter leading to values that tend to be lower than what they should be.

With this in mind we adopt the rescaled range statistics in Cavaliere (2002) to test for boundary conditions in fractionally integrated time series. Due the unavoidable dependence on the long memory parameter d, the critical values of the test are here obtained via sieve bootstrap (Bühlmann 1997; Poskitt 2008). The finite sample performance of the test is assessed through a Monte Carlo experiment.

The structure of the paper is as follows. In the second section we provide a description of the bounded fractionally integrated processes and we also present the results of an introductory Monte Carlo experiment to investigate the effects of existing bounds on the most commonly used estimators of the long memory parameter. In the third section we present our proposal to tests for no boundary conditions in fractionally integrated processes. In the fourth section we show the details of the Monte Carlo experiment to ascertain the performance of the test. In the fifth section there is an empirical illustration of our proposal to time series of the exchange rate Danish Krone/Euro over the last two decades. The sixth section concludes.

2 Bounded fractionally time series

2.1 Overview

In general, let a bounded time series X_t, $t = 1, \ldots, T$ with fixed bounds $\underline{b}$ and $\overline{b}$ ($\underline{b} < \overline{b}$), be the finite realization of a stochastic process such that $X_t \in \left[\underline{b}, \overline{b}\right]$

almost surely for all t (Cavaliere 2002). In this framework, Cavaliere (2005a) gives the definition of the bounded integrated process, $BI(1)$, as follows:

$$\begin{aligned} X_t &= \theta + Y_t \\ Y_t &= \phi Y_{t-1} + u_t, \ \phi = 1 \\ u_t &= \epsilon_t + \underline{\xi}_t - \overline{\xi}_t \end{aligned} \tag{1}$$

where ϵ_t is a stationary unbounded process process with zero mean and $\underline{\xi}_t$ and $\overline{\xi}_t$ are non-negative processes such that $\underline{\xi}_t > 0$ if and only if $Y_{t-1} + \epsilon_t < \underline{b} - \theta$ and $\overline{\xi}_t > 0$ if and only if $Y_{t-1} + \epsilon_t > \overline{b} - \theta$. The terms $\underline{\xi}_t$ and $\overline{\xi}_t$, also called regulators, force the process between $\underline{b}$ and $\overline{b}$. As clarified in Cavaliere (2005a), the relation between $(\underline{b}, \overline{b})$ and the sample size T is stated as $\underline{b} = \underline{c}\lambda T^{1/2}$ and $\overline{b} = \overline{c}\lambda T^{1/2}$. The nuisance parameters $(\underline{c}, \overline{c})$ provide a way to measure the influence of the bounds, while λ^2 is the long run variance of ϵ_t.

Building on the definition of $BI(1)$ processes depicted above, Trokic (2013) introduces a more general class of bounded integrated models, where the integration order is no more integer. To describe his proposal, we begin by recalling that a general fractionally integrated process Z_t of order d' is defined as

$$(1-B)^{d'} Z_t = u_t, \qquad u_t = \sum_{j=0}^{\infty} \Psi_j \epsilon_{t-j}, \quad t = 1, 2, \ldots$$

where $d' > -1/2$ (stationarity holds for $-1/2 < d' < 1/2$), ϵ_t are zero mean, finite variance, i.i.d. random variables, B is the backshift operator and $(1-B)^{d'}$ is defined by the MacLaurin series:

$$(1-B)^{d'} = \sum_{j=0}^{\infty} \frac{\Gamma(-d'+j)}{\Gamma(-d')\Gamma(j+1)} B^j$$

In the setting by Trokic (2013), X_t defined as

$$\begin{aligned} X_t &= \gamma + Y_t \\ Y_t &= \phi Y_{t-1} + Z_t, \ \phi = 1 \\ Z_t &= \Delta_+^{-d} u_t \\ u_t &= v_t + \underline{\xi}_{d',t} - \overline{\xi}_{d',t} \end{aligned} \tag{2}$$

is a bounded[1] fractionally integrated process of order d, $BFI(d)$, where $d = d' + 1$ and d', $|d'| < 1/2$, is the fractional integration order of the first difference. The error term v_t is assumed to be a general linear model. In addition, $\underline{\xi}_{d',t} \equiv \Delta_+^{d'} \underline{\xi}_t$ and

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[1] Trokic (2013) adopts the terms regulated and bounded as synonyms. We will follow Cavaliere's terminology (Cavaliere 2002, 2005a; Cavaliere and Xu 2014) and adopt only the term bounded process.

$\overline{\xi}_{d',t} \equiv \Delta_{+}^{d'}\overline{\xi}_t$. As in Cavaliere (2005a), the terms $\underline{\xi}_t$ and $\overline{\xi}_t$ ensure that the process x_t moves in the interval $\left[\underline{b}, \overline{b}\right]$. They are both non-negative and satisfy the following conditions

$$\underline{\xi}_t > 0 \quad if \quad Y_{t-1} + Z_t < \underline{b} - \gamma$$
$$\overline{\xi}_t > 0 \quad if \quad Y_{t-1} + Z_t > \overline{b} - \gamma$$

where, in the same fashion as in $BI(1)$ processes, the parameters $\left[\underline{b}, \overline{b}\right]$ are defined as follows (Trokic 2013)

$$\underline{b} = \underline{c}\left[\frac{\lambda^2}{\Gamma^2(d'+1)}T^{2(d'+1/2)}\right]^{1/2} \qquad \overline{b} = \overline{c}\left[\frac{\lambda^2}{\Gamma^2(d'+1)}T^{2(d'+1/2)}\right]^{1/2}$$

2.2 The effects of boundary conditions: Monte Carlo results

While presenting the bounded fractionally integrated processes, Trokic (2013) does not address the issue of the effects of the bounds on the estimates of the long memory parameter d. To shed some light on this, we conduct a Monte Carlo experiment where we generate (nonstationary) $BFI(d)$ time series and analyze the performance of the estimation methods of the long memory parameter[2] that, to our knowledge, are still among the most used by practitioners (and whose performance is fairly good also when the long memory parameter is in the nonstationarity interval $(1/2, 1)$. Specifically, we consider R/S (Lo 1991), GPH (Geweke and Porter-Hudack 1983), GPH modified by Robinson, GPH-m (Robinson 1995a), Whittle, W (Fox and Taqqu 1986; Dahlhaus 1989) and local Whittle, LW (Robinson 1995b).[3]

The details of the experiment are as follows.[4] Nonstationary $BFI(d)$ time series (2) have been generated according to the algorithms by Trokic (2013), both the reflection and censoring one. The values of the (overall) long memory parameter are $d = 0.6, 0.7, 0.8, 0.9$, for sample sizes $T = 250, 500, 1000$. For all models, innovations are distributed as $N(0, 1)$. The parameters $\underline{c}$ and $\overline{c}$ are set equal to, respectively, $\pm\, 0.4, 0.6, 0.8$ as in Cavaliere and Xu (2014), corresponding to increasingly wider symmetric bounds. Also the case of one single (positive) bound has been studied, *i.e.* $\overline{c} = 0.4, 0.6, 0.8$. For each combination of the c, d parameters we generated 2000 time series and computed the estimates of the long memory parameters with the 5 methods cited above. As a benchmark, the corresponding unbounded version of the time series is also considered. The performance is expressed in terms of average bias and standard error of the estimates across Monte Carlo replications. For each series,

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[2] For clarity's sake, we stress once more that we refer here to the overall long memory of the process X_t, denoted by $d = d' + 1$, where d' is the long memory parameter of $\Delta X_t = Z_t$

[3] We are well aware of the existence of many other estimation methods, but we believe that it is of interest to the reader the performance of the methods that are effectively the best-known and, also, implemented in the most common packages (e.g. R, Matlab,…).

[4] All the codes (throughout the paper) are written in R language (R Core Team 2015) and are available upon request by the authors.

the estimates are repeated also on the first difference of the series, as in Hurvich and Ray (1995).

The results of the Monte Carlo experiment are presented in Tables 1, 2, 3 and refer specifically to series generated with the reflection algorithm (results relative to the censoring algorithm are substantially identical and are available upon request by the authors). As we can see from the tables, in particular in comparison with the performance of the estimators in case of no bounds, the effect of the bounds is not negligible. Unsurprisingly, the effect changes with the sample size, with the wideness of the bounds and the estimation methods. The narrower are the bounds, the stronger is the effect on the estimates of the long memory parameter; *ceteris paribus*, this is more visible for lower sample sizes. Although there are estimation methods that in general are somewhat robust, as the local Whittle whose performance is not too bad in case of boundary conditions, still, when the bounds are narrow, also these methods have difficulty in capturing correctly long memory.

Interestingly, the strategy of preliminarily differentiating the time series (as suggested by Hurvich and Ray (1995) in case of nonstationary time series) does not appear to be a solution as the effects of the bounds on the estimates are still quite perceivable.

The results of this Monte Carlo experiment show that it is important to be aware of the bounded behaviour as it affects the estimates of the long memory parameter. For some estimation methods (W and LW) these effects are less severe, yet not negligible; for other methods (GPH, GPH-m and R/S) the effects are very severe. Hence, although these estimation techniques are widely used and implemented in several packages (especially GPH), their use should be judicious in case of bounded time series as it could lead to wrong inferential conclusions.

3 Testing for boundary conditions

Given the importance of being aware of the existence of boundary conditions, we now present a test to find out whether a long memory time series is bounded or not. In principle, we extend to nonstationary long memory time series the test proposed by Cavaliere (2002) to investigate the presence of unknown boundaries for a $I(1)$ process. Briefly, for a time series X_t, $t = 1, \ldots, T$, Cavaliere (2002) tests $H_0 : I(1)$ versus $H_1 : BI(1)$ by means of the following range statistic:

$$\hat{r}_\mu = \frac{max_t \{X_t\} - min_t \{X_t\}}{\tilde{\lambda}_T T^{1/2}} \tag{3}$$

where $\tilde{\lambda}_T^2$ is the estimator of the long run variance of the difference process ΔX_t.

An interesting feature of (3) is that, as illustrated by Cavaliere (2002), it represents a general framework of which the rescaled range (R/S) statistic by Hurst (1951) and Mandelbrot (1972, 1975) is a special case. To see this, let us recall that for a stationary stochastic process u_t, the R/S statistic is defined (up to a normalization factor) as

Table 1 Average bias across Monte Carlo replications-$T = 250$

Two-bounded time series

c	*d*	Original series				Differenced series			
		R/S	GPH	W	LW	R/S	GPH	W	LW
0.4	0.7	− 0.214	− 0.081	− 0.093	−0.052	0.122	− 0.069	− 0.089	0.110
0.4	0.8	− 0.295	− 0.139	− 0.085	−0.157	0.037	− 0.117	− 0.096	0.041
0.4	0.9	− 0.327	− 0.112	− 0.057	−0.169	−0.002	− 0.101	− 0.070	0.014
0.6	0.7	− 0.260	− 0.153	− 0.120	−0.144	0.085	− 0.121	− 0.105	0.062
0.6	0.8	− 0.283	− 0.112	− 0.079	−0.138	0.023	− 0.099	− 0.086	0.037
0.6	0.9	− 0.303	− 0.085	− 0.048	−0.135	0.012	− 0.072	− 0.061	0.042
0.8	0.7	− 0.266	− 0.171	− 0.118	−0.157	0.059	− 0.122	− 0.109	0.070
0.8	0.8	− 0.263	− 0.089	− 0.068	−0.108	0.032	− 0.076	− 0.079	0.054
0.8	0.9	− 0.297	− 0.067	− 0.043	−0.124	0.016	− 0.071	− 0.057	0.038

Unbounded time series

-	*d*	Original series				Differenced series			
		R/S	GPH	W	LW	R/S	GPH	W	LW
-	0.7	− 0.148	0.025	0.000	0.052	0.154	0.012	− 0.018	0.183
-	0.8	− 0.192	0.035	0.005	0.019	0.110	0.010	− 0.019	0.141
-	0.9	− 0.238	0.027	0.003	−0.032	0.080	0.007	− 0.018	0.145

Table 2 Average bias across Monte Carlo replications-$T = 500$

Two-bounded time series

c	*d*	Original series				Differenced series			
		R/S	GPH	W	LW	R/S	GPH	W	LW
0.4	0.7	− 0.188	−0.054	− 0.079	0.000	0.105	− 0.054	− 0.083	0.054
0.4	0.8	− 0.280	−0.122	− 0.066	− 0.090	0.019	− 0.107	− 0.076	−0.011
0.4	0.9	− 0.318	−0.102	− 0.040	− 0.095	−0.021	− 0.098	− 0.052	−0.029
0.6	0.7	− 0.262	−0.156	− 0.102	− 0.100	0.064	− 0.123	− 0.097	0.012
0.6	0.8	− 0.281	−0.121	− 0.064	− 0.079	−0.002	− 0.103	− 0.071	−0.014
0.6	0.9	− 0.297	−0.078	− 0.032	− 0.056	−0.009	− 0.079	− 0.043	0.001
0.8	0.7	− 0.263	−0.158	− 0.101	− 0.095	0.035	− 0.123	− 0.100	0.004
0.8	0.8	− 0.260	−0.094	− 0.051	− 0.049	0.011	− 0.082	− 0.059	0.007
0.8	0.9	− 0.292	−0.062	− 0.027	− 0.050	0.001	− 0.062	− 0.039	0.009

Unbounded time series

-	*d*	Original series				Differenced series			
		R/S	GPH	W	LW	R/S	GPH	W	LW
-	0.7	− 0.135	0.029	0.005	0.094	0.129	0.015	−0.008	0.119
-	0.8	− 0.190	0.033	0.011	0.065	0.092	0.001	− 0.009	0.099
-	0.9	− 0.226	0.027	0.010	0.022	0.064	0.001	− 0.010	0.092

Table 3 Average bias across Monte Carlo replications-$T = 1000$

Two-bounded time series									
c	*d*	Original series				Differenced series			
		R/S	GPH	W	*LW*	*R/S*	*GPH*	*W*	*LW*
0.4	0.7	– 0.188	– 0.067	– 0.073	−0.001	0.086	−0.069	−0.077	0.015
0.4	0.8	– 0.279	– 0.124	– 0.053	−0.053	0.001	−0.118	−0.062	−0.034
0.4	0.9	– 0.305	– 0.092	– 0.029	−0.037	−0.037	−0.099	−0.039	−0.016
0.6	0.7	– 0.250	– 0.145	– 0.084	−0.076	0.051	−0.110	−0.085	−0.020
0.6	0.8	– 0.258	– 0.107	– 0.047	−0.033	−0.018	−0.101	−0.054	−0.017
0.6	0.9	– 0.281	– 0.061	– 0.021	−0.007	−0.020	−0.066	−0.031	0.018
0.8	0.7	– 0.249	– 0.141	– 0.078	−0.067	0.023	−0.108	−0.079	−0.022
0.8	0.8	– 0.248	– 0.087	– 0.042	−0.009	−0.009	−0.081	−0.049	0.003
0.8	0.9	– 0.275	– 0.045	– 0.018	0.006	−0.006	−0.051	−0.028	0.028
Unbounded time series									
-	*d*	Original series				Differenced series			
		R/S	GPH	W	*LW*	*R/S*	*GPH*	*W*	*LW*
-	0.7	– 0.134	0.024	0.007	0.096	0.152	0.067	0.056	0.143
-	0.8	– 0.171	0.029	0.012	0.088	0.078	0.006	−0.005	0.083
-	0.9	– 0.213	0.027	0.014	0.051	0.007	−0.057	−0.065	0.023

$$R/S = max_{t=1,\ldots,T}\left\{\sum_{j=1}^{t}(u_t - \bar{u})\right\} - min_{t=1,\ldots,T}\left\{\sum_{j=1}^{t}(u_t - \bar{u})\right\} \tag{4}$$

Lo (1991) standardizes properly the R/S statistic and obtains a test for long memory (also presented in the Monte Carlo experiment reported in the previous section), whose underlying logic is that the higher the degree of persistence of the process, the wider the cumulate deviations from the sample mean. Bearing in mind that

$$\sum_{j=1}^{t}(u_j - \bar{u}) = \sum_{j=1}^{t}u_j - \frac{t}{T}\sum_{j=1}^{T}u_j \quad and \quad X_t = X_0 + \sum_{j=1}^{t}u_j$$

then the R/S statistic in (4) coincides with the range statistic in (3).

Our idea is to adopt the $\hat{r}_\mu$ statistic to ascertain the existence of boundary conditions also in case of long memory time series. Therefore, following Cavaliere (2002)'s setting, we test the null hypothesis of unbounded long memory versus the alternative of bounded long memory.[5] As in the original idea by Cavaliere (2002), we reject the null hypothesis if the test statistic takes on small values, because that represents the

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[5] We concentrate only on the $\hat{r}_\mu$ statistic proposed by Cavaliere (2002) and do not focus on the other versions (presented in the same article) accounting for a deterministic trend in the data, that is a case into which we are not interested.

case when the numerator is not sufficiently large to be the range of an unbounded time series.

Given that, as recalled before, under the null hypothesis the $\hat{r}_\mu$ statistics coincides with the R/S statistic by Lo (1991), it is possible to say something at least heuristically, about the asymptotics of this test. Provided, in particular, that the long run variance is consistently estimated, proposition 1 and corollary 1 in Cavaliere (2002) and theorem 1 in Trokic (2013) allows us to guess that our BLM test statistic should weakly convergence to the range of fractional brownian motion under H_0 and to the range of a bounded (or regulated) fractional brownian motion under H_1.

Fractional brownian motion depend on the long memory parameter of the process and in practice, the dependence on d of the asymptotic distribution of the test under H_0 creates an issue. Indeed, it is not possible to refer to the asymptotic distribution to derive the critical values, unless a consistent estimate of the long memory parameter is plugged-in and the critical values are obtained via response surface approach as in Cavaliere (2002). However, this could be quite risky in the current case, since the estimation process of d can be seriously affected by the existence of boundary conditions leading to a vicious circle. If no asymptotically pivotal statistics are available, like in this case, and there is no guarantee that a consistent estimate of the nuisance parameter is available, the bootstrap can be an advantageous tool as it allows for inference by automatically taking the nuisance parameters into account by somewhat replicating them into the bootstrap distribution. In the same vein, Cavaliere and Xu (2014) adopt (and prove validity of) a simulation based method to approximate quantiles from the non pivotal limiting distribution of the standard ADF test (Dickey and Fuller 1979) and M test (Perron and Ng 1996) for unit root in case of boundary conditions.

Consequently, we decide to adopt bootstrap methods to derive critical values. Bootstrapping time series poses serious issues due to the lack of iid requirement on which Efron's (1979) original idea is built and the literature in the last decades has presented a number of proposals to handle such forms of dependence in the data (see Lahiri (2003) for a very good review).

In the current framework, *i.e.* long memory time series, the issue is even more challenging due to additional difficulty in capturing the essential feature of such a data generating process and for this reason we opted for the sieve bootstrap (Bühlmann 1997), that is known to have good performance in case of long memory (Poskitt 2008) and does not require any form of *a priori* knowledge on d. More in details, Bühlmann's sieve bootstrap approximates a general linear invertible process by a finite autoregressive model of order $p = p(T)$, where $p(T) \to \infty$, then it resamples from the approximated autoregression. This method takes up the older idea of fitting parametric models first and then resampling from the residuals, but instead of considering a fixed finite-dimensional model, an infinite-dimensional, non-parametric model is approximated by a sequence of finite-dimensional models. This method can be considered non parametric because it is model-free in the class of the linear invertible processes.

The properties of the sieve bootstrap have been rigorously investigated, among other, by Kreiss (1992), Paparoditis (1996), Bühlmann (1997), Bickel and Bühlmann (1999) who established its asymptotic validity for several statistics assuming that the data generating process is an infinite order autoregressive. Kapetanios and Psaradakis (2006) and Poskitt (2008) proved that under regularity conditions (satisfied by stationary long memory processes) the sieve bootstrap provides and asymptotically valid approximation to the distributions of several statistics.[6]

Note that, considering that in our case the process under examination, X_t, is essentially nonstationary, we effectively need to preliminarily differentiate the time series and implement the sieve bootstrap algorithm to the stationary long memory $\Delta X_t = Z_t$, $Z_t \sim I(d')$. This is in line with Psaradakis (2001) and Chang and Park (2003) who proved that applying the sieve bootstrap to first difference is a valid bootstrap approach to nonstationary time series.[7]

The following is the algorithm to implement our bootstrap test for a given X_t, $t = 1, \ldots, T$ time series.

1. Fit an $AR(h)$, $h > 0$, model in $Z_t = \Delta X_t$, the first difference of the data. Obtain the residuals of the $AR(h)$ and standardize them, denoted by $\hat{\zeta}_t$.
2. Create a new randomly resampled residuals set, denoted by $\hat{\zeta}_t^*$
3. Generate the bootstrap time series Z_t^* as

$$Z_t^* = \hat{\alpha}_1 Z_{t-1}^* + \cdots + \hat{\alpha}_h Z_{t-h}^* + \hat{\zeta}_t^*$$

   and then obtain

$$X_t^* = X_{t-1}^* + Z_t^*$$

4. Apply the desired test statistic, in this case $\hat{r}_\mu$, on the bootstrap sample X_t^*.
5. By repeating the above procedure a number of times B we obtain a bootstrap approximation to the distribution of the test statistic.

With the empirical distribution obtained by applying the above algorithm it is possible to derive the desired percentiles. As a practical note, we set h adopting the Akaike information criterion (AIC) as Bühlmann (1997) suggests.

4 Finite sample performance

To study the performance of the proposed test we conduct a Monte Carlo experiment aiming at evaluating empirical size and power. To do this we obtain the percentage of rejection of the null hypothesis in case of, respectively, unbounded (H_0) and bounded (H_1) long memory data generating processes (DGPs).

The algorithms used to generate the data are reflexion algorithms, the same as in the previous Monte Carlo experiment. In particular, the parameters $\underline{c}$ and $\overline{c}$ are, as before, equal to, respectively, $\pm 0.4, 0.6, 0.8$ as in Cavaliere and Xu (2014), corresponding

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[6] Poskitt (2008) also showed that the sieve bootstrap performs better than the block bootstrap (Kunsch 1989), that is also a well-known approach to boostrapping dependent data, but more suitable for short range dependence than long range dependence.

[7] Palm et al. (2008) also showed that for some data generating processes (not our case), residuals from a first order autoregression can lead to an even better performance of the sieve bootstrap in terms of asymptotic validity, compared to first difference.

to increasingly wider symmetric bounds. Also the case of one single (positive) bound has been considered, *i.e.* $\overline{c} = 0.4, 0.6, 0.8$.[8]

As in Sect. 2, d is set $d = 0.6, 0.7, 0.8, 0.9$. Innovations are $N(0, 1)$ and $T = 250, 500$ is the sample size. For each type of unbounded time series the number of Monte Carlo replications is 5000, whereas for bounded time series the number of Monte Carlo replications is 2000.

As for the implementation of the test, the number of sieve bootstrap replications is $B = 500$, with the Akaike automatic lag order selection as in Bühlmann (1997). The long run variance is estimated, as in Cavaliere (2002), via the HAC estimator by Andrews (1991), but we did not confine the estimate to the Newey-West weights (Newey and West 1987), we also consider Parzen weights (Parzen 1961), Tukey-Hanning weights (Blackman and Tukey 1958) and quadratic spectral weights, *i.e.* the Bartlett-Priestley kernel (Bartlett 1950; Priestley 1962) with automatic bandwidth selection as in Andrews (1991).

Results are shown in Tables 4, 5 and 6 where we can appreciate the ability of the sieve bootstrap test to deal with bounds and long memory. In particular, Tables 4 and 5 report the empirical power of the test and we observe that for a given value of d, the tighter are the bounds, the higher the power of the test. This improves, as it should be, with the increase of T. In the one-bounded case the power is a little smaller than in the two-bounded case. This is due to the less strong mean reversion induced by the regulation of the dynamics via just one bound. Still, also in this case, the power of the test reaches satisfactory levels and increases with the decrease of the value of parameter c.

As far as the long memory parameter is concerned, the performance of the test reduces the closer is d to the stationary region. This is not surprising as the unbounded time series behaviour is effectively much less recognizable the closer the series is to stationarity.

The performance of the test does not seem to be affected by the chosen approach to obtain the HAC estimator, although Newey-West and Parzen weights appear to reach steadily the highest percentages of correct rejections of the null hypothesis.

As for the empirical size, Table 6, the behaviour of the test is across all cases more than satisfying, being slightly fluctuating just above 0.05. Once more, Newey-West and Parzen weights seem to characterize the best performing approaches.

5 Empirical illustration

In this empirical illustration we show how to apply our sieve bootstrap test to the time series of the exchange rate Danish Krone/Euro. The data set includes monthly data from January 1999 to December 2017 and it mirrors the fixed-exchange-rate Denmark's monetary policy, aiming at keeping the Krone stable against the euro.

As the central bank of Denmark, Danmarks Nationalbank is responsible for conducting monetary policy in Denmark, which it does by setting the monetary-policy

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[8] Results with the censoring algorithm are substantially equivalent and are available upon request by the authors.

Table 4 Power of the test (two-bounded time series), level $\alpha = 0.05$

c	d	$T = 250$				$T = 500$			
		NW87	Parzen	TH	QS	NW87	Parzen	TH	QS
0.4	0.6	0.75	0.77	0.70	0.71	0.83	0.90	0.82	0.75
0.4	0.7	0.77	0.81	0.71	0.72	0.90	0.93	0.85	0.77
0.4	0.8	0.83	0.84	0.74	0.75	0.94	0.95	0.87	0.79
0.4	0.9	0.95	0.92	0.75	0.77	0.98	0.97	0.91	0.82
0.6	0.6	0.78	0.79	0.77	0.72	0.93	0.96	0.90	0.91
0.6	0.7	0.84	0.87	0.78	0.76	0.94	0.98	0.92	0.93
0.6	0.8	0.88	0.90	0.84	0.78	0.97	0.99	0.94	0.95
0.6	0.9	0.96	0.93	0.87	0.85	1	1	0.98	0.98
0.8	0.6	0.81	0.83	0.76	0.84	0.98	0.98	0.97	0.95
0.8	0.7	0.85	0.89	0.78	0.84	1	1	0.98	0.98
0.8	0.8	0.91	0.93	0.84	0.86	1	1	1	1
0.8	0.9	0.98	0.99	0.86	0.91	1	1	1	1

Notes: HAC estimator computed with Newey-West weights (NW87), Parzen weights (Parzen), Tukey-Hanning weights (TH) and quadratic spectral weights (QS)

Table 5 Power of the test (one-bounded time series), level $\alpha = 0.05$

c	d	$T = 250$				$T = 500$			
		NW87	Parzen	TH	QS	NW87	Parzen	TH	QS
0.4	0.6	0.72	0.74	0.67	0.68	0.80	0.87	0.79	0.72
0.4	0.7	0.74	0.78	0.68	0.69	0.87	0.90	0.82	0.74
0.4	0.8	0.80	0.81	0.71	0.72	0.091	0.92	0.84	0.76
0.4	0.9	0.92	0.89	0.72	0.74	0.95	0.94	0.88	0.79
0.6	0.6	0.75	0.76	0.74	0.69	0.90	0.93	0.87	0.88
0.6	0.7	0.81	0.84	0.75	0.73	0.91	0.95	0.89	0.90
0.6	0.8	0.85	0.87	0.80	0.76	0.94	0.96	0.90	0.92
0.6	0.9	0.93	0.90	0.84	0.81	0.98	0.98	0.95	0.96
0.8	0.6	0.79	0.79	0.72	0.79	0.95	0.93	0.94	0.92
0.8	0.7	0.82	0.86	0.74	0.81	0.97	0.98	0.95	0.95
0.8	0.8	0.87	0.90	0.79	0.84	1	1	1	1
0.8	0.9	0.95	0.96	0.81	0.88	1	1	1	1

Notes: HAC estimator computed with Newey-West weights (NW87), Parzen weights (Parzen), Tukey-Hanning weights (TH) and quadratic spectral weights (QS)

Table 6 Size of the test (unbounded time series), level $\alpha = 0.05$

d	$T = 250$				$T = 500$			
	NW87	Parzen	TH	QS	NW87	Parzen	TH	QS
0.6	0.067	0.065	0.071	0.072	0.058	0.055	0.059	0.061
0.7	0.066	0.064	0.070	0.068	0.057	0.054	0.057	0.058
0.8	0.064	0.063	0.069	0.066	0.054	0.053	0.054	0.056
0.9	0.061	0.059	0.064	0.064	0.051	0.052	0.052	0.054

Notes: HAC estimator computed with Newey-West weights (NW87), Parzen weights (Parzen), Tukey-Hanning weights (TH) and quadratic spectral weights (QS)

interest rates. Denmark maintains a fixed-exchange-rate policy vis-à-vis the euro area and participates in the European Exchange Rate Mechanism, ERM 2[9] at a central rate of 746.038 kroner per 100 euro with a fluctuation band of $+/-2.25$ per cent. This exchange-rate regime provides a framework for low and stable inflation in Denmark.

In periods when the foreign-exchange market is calm, Danmarks Nationalbank usually changes its interest rates in step with the monetary-policy interest rates of the ECB. In situations with upward or downward pressure on the Krone, Danmarks Nationalbank unilaterally changes its interest rates in order to stabilise the currency. In the short term, Danmarks Nationalbank may also influence the exchange rate of the Krone by intervening, i.e. buying and selling foreign exchange in the market.

These stylized facts about Danish monetary policy help us critically look at the evolution over time of the Danish Krone/Euro exchange rate that is suggestive of the possibility that the time series might be affected by some form of boundary conditions. Moreover, from the statistical point of view, this is in line with Cavaliere (2005b) who documents that the use of range-based statistics allows detecting mean reversion in several European exchange rates time series over the period 1987–1997 of the ERM 1[10]. In his paper, Cavaliere clearly explains how crucial is to provide additional information to standard unit root tests, as the Philips-Perron (Phillips and Perron 1988), whose power is known to be low in case of nonlinear dynamics, such as those induced by bounds on exchange rate excursions (Fig. 1) .

Here, we are interested int he possibility of long memory, given some previous results documenting long memory in exchange rate time series (see, for instance, Barkoulas et al. 2016).

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[9] The euro is at the core of ERM 2, and the currencies of participating EU member states have central rates against the euro, but not against each other. The obligation to intervene–that is, to buy or sell currency to support the exchange rate–if a participating currency reaches a fluctuation limit depends only on the central bank of the relevant member state and the ECB. The other participating member states have no obligation to intervene. ERM 2 includes a provision on unlimited intervention credit between the ECB and the participating central banks in connection with intervention at the fluctuation limits. One of the convergence criteria for joining the euro area is to observe the normal fluctuation band within ERM 2 without severe tensions for at least two years. In the same period, the member state in question may not devalue its currency against the euro.

[10] The ERM 1, antecedent to the ERM 2, is one of the arrangements implied by the European Monetary System (EMS), in place between 1979 and May 1998.

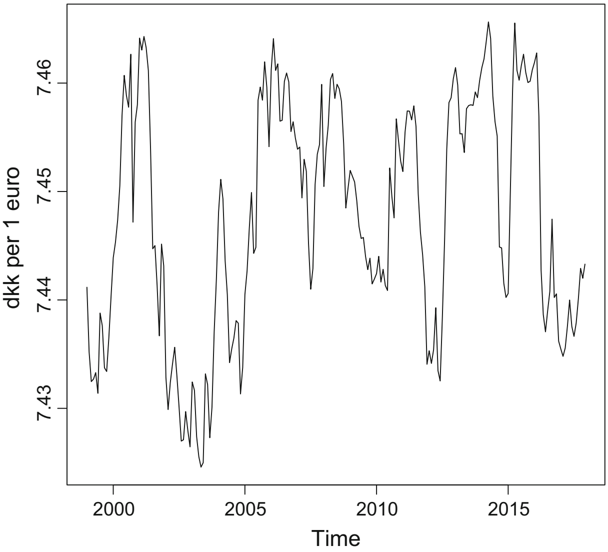

Fig. 1 Dkk/euro exchange rate (Danish krone for one euro) time series from 1999 January to 2017 December

Table 7 Long memory estimates and BLM test results

long memory estimates	GPH	loc Whittle	Whittle	R/S
	0.55	0.87	1.05	0.62
BLM test (p–values)	NW87	Parzen	TH	QS
	0.0013	0.0010	0.0018	0.0022

Notes: HAC estimator computed with Newey-West weights (NW87), Parzen weights (Parzen), Tukey-Hanning weights (TH) and quadratic spectral weights (QS)

In fact, we find evidence of long memory, but with conflicting results. Visibly (Table 7, top panel), the GPH method returns quite a low value, whereas Whittle and local Whittle provide higher values. These not aligned results, coupled with the background information about the evolution over time of the time series, lead us to the suspicion that this might be a case of bounded long memory. Consequently we adopt the sieve bootstrap test, whose results confirm this impression. In particular, we apply the test in combination with all the various versions of long-run variance estimation (as explained in Sect. 4) and steadily the null hypothesis is rejected as shown in the bottom panel of Table 7.

This result, totally in line with what the Danish monetary policy tells us, is very interesting for at least two reasons. First, it sheds some light on the actual possibility of distinguishing bounded from unbounded time series, even when this is more difficult as in case of long memory time series with long memory parameter in $(1/2, 1)$, *i.e.* outside the stationarity region. Second, it gives valuable information about the extent to which a bounds-unaware long memory estimation approach may lead to wrong inferential conclusions.

Finally, from and economic view point this result illustrates that the existence of the fluctuation band is effectively reflected into a specific stochastic process underlying the observed time series, thus confirming, also from the perspective of a different period of time, the analysis by Cavaliere (2005b).

6 Conclusion

In this paper we proposed a sieve bootstrap approach to distinguish between bounded and unbounded fractionally integrated time series. This is quite a serious issue because, as we extensively show via a Monte Carlo experiment, neglecting the existence of boundary conditions can affect the estimates of the long memory parameter leading to values that tend to be downward biased, especially for some wellknown estimation methods, such as GPH. With this in mind, we adopted the rescaled range statistics utilized by Cavaliere (2002) to test for boundary conditions in integrated time series to the case of fractionally integrated ones. Due to the unavoidable dependence on the long memory parameter d, the critical values of the test are here obtained via sieve bootstrap (Bühlmann 1997; Poskitt 2008).

A further Monte Carlo experiment illustrates the good finite sample performance of our procedure, especially for tighter bounds and for all the considered methods to obtain the HAC long run variance estimator.

An empirical illustration shows the actual ability of the method to recognize this feature of the series. This is interesting also from an economic view point as it confirms that the existence of predefined boundaries, such as the fluctuation bands of the Danish Krone/Euro exchange rate, are effectively reflected into a specific stochastic process.

All in all, being aware of the existence of boundary conditions is very important in order to read the results of the long memory parameter estimates in the correct way. In particular the suggestion is being cautious in the use of commonly employed methods, such as the GPH which is very often used, yet seriously affected by the bounds. As a practical implication we recommend to resort more on the answer given by the Whittle and local Whittle method.

Future research lines of the authors are oriented, on the one hand, to the identification of the bounds and, on the other hand, to possible adjustments of the long memory estimators, especially the most commonly used, in order to allow for existing bounds.

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References

Andrews D (1991) Heteroskedasticity and autocorrelation consistent covariance matrix estimation. Econometrica 59:817–858

Bartlett MS (1950) Periodogram analysis and continuous spectra. Biometrika 37:1–16

Barkoulas JT, Barilla AG, Wells W (2016) Long-memory exchange rate dynamics in the euro era. Chaos Solitons Fractals 86:92–100

Bickel PJ, Bühlmann P (1999) A new mixing notion and functional central limit theorems for a sieve bootstrap in time series. Bernoulli 5:413–446

Blackman RB, Tukey JW (1958) The measurement of power spectra from the point of view of communications engineering—part I. Bell Syst Tech J 37:185–282

Bühlmann P (1997) Sieve bootstrap for time series. Bernoulli 3:123–148

Cavaliere G (2002) Bounded integrated processes and unit root tests. Stat Methods Appl 11:41–69

Cavaliere G (2005) Limited time series with a unit root. Econ Theory 21:907–945

Cavaliere G (2005) Testing mean reversion in target-zone exchange rates. Appl Econ 37:2335–2347

Cavaliere G, Xu F (2014) Testing for unit roots in bounded time series. J Econ 178:259–272

Chang Y, Park JY (2003) A sieve bootstrap for the test of a unit root. J Time Ser Anal 24:379–400

Dahlhaus R (1989) Efficient parameter estimation for self-similar processes. Ann Stat 17:1749–1766

Dickey DA, Fuller WA (1979) Distribution of the estimators of an autoregressive time series with a unit root. J Am Stat Assoc 74:427–431

Efron B (1979) Bootstrap methods: another look at the jackknife. Ann Stat 7:1–26

Fox R, Taqqu MS (1986) Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. Ann Stat 14:517–532

Geweke J, Porter-Hudack S (1983) The estimation and application of long-memory time series models. J Time Ser Anal 4:221–237

Granger CWJ (2010) Some thoughts on the development of cointegration. J Econ 158:3–6

Hurst H (1951) Long-term storage capacity of reservoirs. Trans Am Soc Civil Eng 116:770–799

Hurvich C, Ray B (1995) Estimation of the memory parameter for nonstationary or noninvertible fractionally integrated processes. J Time Ser Anal 16:17–41

Kapetanios G, Psaradakis Z (2006) Sieve bootstrap for strongly dependent stationary processes. Working Papers 552, Queen Mary University of London. School of Economics and Finance

Kreiss JP (1992) Bootstrap procedures for AR(∞)-processes. In: Lecture Notes in Economics and Mathematical Systems, vol 376: 107–113 (Proc. Bootstrapping and Related Techniques, Trier)

Kunsch HR (1989) The jacknife and the bootstrap for general stationary observations. Ann Stat 17:1217–1241

Lahiri SN (2003) Resampling Methods for Dependent Data. Springer, New York

Lo A (1991) Long-term memory in stock market prices. Econometrica 59:1279–1313

Mandelbrot B (1972) Statistical methodology for nonperiodic cycles: from the covariance to r/s analysis. Ann Econ Soc Meas 1:259–290

Mandelbrot B (1975) Limit theorems of the self-normalized range for weakly and strongly dependent processes. Z Wahr verw Geb 31:271–285

Newey WK, West KD (1987) A simple positive semi-definite heteroskedasticity and autocorrelation consistent covariance matrix. Econometrica 55:703–708

Palm FC, Smeekes S, Urbain JP (2008) Bootstrap unit-root tests: comparison and extensions. J Time Ser Anal 29:371–400

Paparoditis E (1996) Bootstrapping autoregressive and moving average parameter estimates of infinite order vector autoregressive processes. J Multivar Anal 57:277–296

Parzen E (1961) An approach to time series analysis. Ann Math Stat 32:951–989

Perron P, Ng S (1996) Useful modifications to some unit root tests with dependent errors and their local asymptotic properties. Rev Econ Stud 63:435–463

Phillips PCB, Perron P (1988) Testing for unit root in time series regression. Biometrika 75:335–346

Poskitt DS (2008) Properties of the sieve bootstrap for fractionally integrated and non-invertible processes. J Time Ser Anal 29:224–250

Priestley MB (1962) Basic considerations in the estimation of spectra. Technometrics 4:551–564

Psaradakis Z (2001) Bootstrap tests for an autoregressive unit root in the presence of weakly dependent errors. J Time Ser Anal 22:577–594

R Core Team (2015) R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria

Robinson PM (1995a) Log-periodogram regression of time series with long range dependence. Ann Stat 23:1048–1072

Robinson PM (1995b) Gaussian semiparametric estimation of long range dependence. Ann Stat 23:1630–1661

Trokic M (2013) Regulated fractionally integrated processes. J Time Ser Anal 34:591–601