The Fitting length of finite soluble groups II

Fixed-point-free automorphisms

Enrico Jabara

Abstract. *Let G be a finite soluble group, and let $h(G)$ be the Fitting length of G. If φ is a fixed-point-free automorphism of G, that is $C_G(\varphi) = \{1\}$, we denote by $W(\varphi)$ the composition length of $\langle\varphi\rangle$. A long-standing conjecture is that $h(G) \leq W(\varphi)$, and it is known that this bound is always true if the order of G is coprime to the order of φ. In this paper we find some bounds to $h(G)$ in function of $W(\varphi)$ without assuming that $(|G|, |\varphi|) = 1$. In particular we prove the validity of the "universal" bound $h(G) < 7W(\varphi)^2$. This improves the exponential bound known earlier from a special case of a theorem of Dade.*

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§1. Introduction

In this paper we apply some results obtained in [2] to the study of finite soluble groups with a fixed-point-free automorphism. We only deal with finite soluble groups and so for us group will always mean "finite soluble group". If G is a group and $\varphi \in \mathrm{Aut}(G)$, then φ is called fixed-point-free if the centralizer

$$C_G(\varphi) = \{g \in G \mid g^\varphi = g\}$$

is the trivial subgroup of G. We shall denote by $h(G)$ the Fitting length of G, by $\pi(G)$ (resp. $\pi(\varphi)$) the set of prime divisors of $|G|$ (of $|\langle\varphi\rangle|$) and by $w(G)$ (resp. $w(\varphi)$) the cardinality of $\pi(G)$ (of $\pi(\varphi)$). Also, we shall write $W(\varphi)$ for the composition length of $\langle\varphi\rangle$ (that is the number of prime divisors of $|\langle\varphi\rangle|$ counted with their multiplicities). Sometimes we will write π, h, w and W instead of $\pi(G)$, $h(G)$, $w(\varphi)$ and $W(\varphi)$ respectively, when there is no possible ambiguity.

If the order of φ is coprime to $|G|$, it was then proved, through a long series of papers (see, in particular, [10], [12] and [13]), that

$$h(G) \leq W(\varphi).$$

Moreover if A is a solvable group of automorphisms of G and $(|A|, |G|) = 1$, then

$$h(G) \leq 2W(A) + h(C_G(A)),$$

by a result of Turull ([18]). So, if $C_G(A) = 1$ (that is A is fixed-point-free), then $h(G) \leq 2W(A)$ and in many cases $h(G) \leq W(A)$ (Turull, [17] and [19]).

Here we turn our attention to the so called *noncoprime case*, in which the hypothesis $(|G|, |\varphi|) = 1$ is omitted. If $w(\varphi) = 1$, then $|\varphi| = p^{W(\varphi)}$ (p a prime number) and an easy argument shows that G is a p'-group. Hence we suppose $w(\varphi) \geq 2$ and this hypothesis will be often implicitly assumed. In this case, from Theorem 8.4 of [4], we can deduce the exponential bound $h(G) \leq 5(2^W - 1)$. Our main result is.

THEOREM 1.1 *Let G be a group and let φ be a fixed-point-free automorphism of G. If $w(\varphi) \geq 2$, then*

$$h(G) < \big(7w - 9\big)W.$$

The inequality proved in Theorem 1.1 is particularly satisfactory if $w(\varphi) = 2$, as it provides the bound

$$h(G) < 5W$$

when the order of φ is divisible by only two primes.

Since $w(\varphi) \leq W(\varphi)$, Theorem 1.1 easily implies the following

COROLLARY 1.2. *Let G be a group and let φ be a fixed-point-free automorphism of G, then $h(G) < 7W^2$.*

Furthermore, in some cases, the previous inequality may be improved, as, for example, in the following two propositions.

PROPOSITION 1.3. *Let G be a group and let φ be a fixed-point-free automorphism of G. If $|\varphi| = p^\alpha q$ with p and q distinct primes, then $h(G) \leq 3W + 1$.*

PROPOSITION 1.4. *Let G be a group and let φ be a fixed-point-free automorphism of G. If the order of φ is square-free and $W(\varphi) \geq 3$, then*

$$h(G) < \frac{1}{2}\big(3W^2 - 7W\big).$$

REMARK 1.5. If the order of $\varphi \in \mathrm{Aut}(G)$ is square-free and $W(\varphi) \leq 3$, then the best possible bound

$$h(G) \leq W$$

was proven. If $W = 1$, then φ has prime order and it is well known that G is nilpotent. If $W = 2$, then $|\varphi| = pq$ (p, q primes, $p \neq q$) and in this case $h(G) \leq 2$ by [3]. If $W = 3$, then $|\varphi| = pqr$ (p, q, r primes, $p \neq q \neq r \neq p$) and $h(G) \leq 3$ follows from [5].

We want to recall that a result of Ercan and Güloğlu (Theorem A of [6]) asserts that if G has odd order, A is abelian of squarefree exponent coprime to 6 and $C_G(A) = 1$, then $h(G) \leq W(A)$.

Using the above-mentioned result of Turull, we can generalize our Theorem 1.1 to

THEOREM 1.6. *Let G be a group and let φ be an automorphism of G. Suppose that $(|C_G(\varphi)|, |\langle\varphi\rangle|) = 1$, $w(\varphi) \geq 2$ and $h(C_G(\varphi)) = h_0$, then*

$$h(G) < \big(8w - 10\big)W + \frac{3}{2}\big(w - 1\big)wh_0.$$

We wish to emphasize that we have not wanted to optimize our bounds, but only indicate a new method to obtain general results.

REMARK 1.7. Let G be a group, let A a fixed-point-free nilpotent group of automorphisms of G and let $W = W(A)$. In his seminal paper [4] (Theorem 8.4), Dade proved that

$$h(G) \leq 5\big(2^W - 1\big),$$

hence there is always a function Γ such that $h(G) \leq \Gamma(W)$. Moreover Dade (in Conjecture 2.9 of [4]) suggests that Γ can be chosen so that $\Gamma(W) = O(W)$ as $W \to \infty$.

Our Theorem 1.1 (and Corollary 1.2) shows that, in the particular case where A is cyclic, $\Gamma(W)$ is at most quadratic in W (compare this result with the main theorem of [16]). Furthermore we have the linear bound

$$\sup \left\{ \frac{h(G)}{W(\langle\varphi\rangle)} \;\middle|\; \begin{array}{c} \varphi \in \mathrm{Aut}(G) \text{ is} \\ \text{fixed-point-free and} \\ w(\langle\varphi\rangle) = 2 \end{array} \right\} \leq 5$$

thanks to the observation made after Theorem 1.1 (see Proposition 3.1).

REMARK 1.8. We point out that if A is a fixed-point-free group of automorphisms of the group G at least one of the two hypotheses (1) A is nilpotent (2) $(|G|, |A|) = 1$ is needed to bound $h(G)$ by a function of $W(A)$. Indeed in [1] it is proved that if A is any finite non nilpotent group and H is any finite group, then there exists a finite group G on which A acts fixed-point-freely, such that H is a homomorphic image of G. Further, if H is soluble, so is G.

§2. Notation and preliminary results

In this paper we use the same notations employed in [2]. In particular if G is a group, with $\{G_p\}_{p\in\pi}$ we denote a Sylow system of G, namely a set of Sylow subgroups of G, one for any $p \in \pi$, such that $G_pG_q = G_qG_p$ for every $p, q \in \pi$. If σ is a subset of π, by σ-Hall subgroup of G we mean $G_\sigma = \prod_{p\in\sigma} G_p$ and by $G_{p'}$ we denote a $\pi \setminus \{p\}$-Hall subgroup of G.

The symbols π, w, W have already been defined. If σ is a set of primes, we denote by $\ell_\sigma(G)$ (or by ℓ_σ) the σ-length of G and by $\ell_p(G) = \ell_{\{p\}}(G)$ (or by ℓ_p) the p-length of G ([14], 9.1.4).

A substantial tool for the proofs in this paper is the following result.

THEOREM 2.1. (Theorem 1.1 of [2]) *Let G be a group and let σ, τ, υ be three subsets of $\pi(G)$ such that $\sigma \cup \tau = \tau \cup \upsilon = \upsilon \cup \sigma = \pi$. Then*

$$h(G) \leq h(G_\sigma) + h(G_\tau) + h(G_\upsilon) - 2.$$

In particular, if $p, q \in \pi$ and $p \neq q$, then

$$h(G) \leq h(G_{p'}) + h(G_{q'}) + h(G_{\{p,q\}}) - 2.$$

Theorem 2.1 is consequence of Theorem 2.3, a more general and technical result, for which the following definition is needed.

DEFINITION 2.2. Let G be a group and let $t \geq 3$ be an integer. The set

$$\mathcal{R} = \left\{ \varrho_1, \varrho_2, \ldots, \varrho_t \;\middle|\; \varrho_i \subseteq \pi \right\}$$

is called a *t-cover* if $\varrho_i \cup \varrho_j = \pi$ for every $i, j \in \{1, 2, \ldots, t\}$, $i \neq j$. The *weight* of a t-cover $\mathcal{R}$ is the number

$$\Theta(\mathcal{R}) = \sum_{i=1}^{t} h(G_{\varrho_i}).$$

THEOREM 2.3. (Proposition 3.1 of [2]) *Let G be a group and let $\mathcal{R}$ be a t-cover of $\pi(G)$ of weight Θ, then*

$$h(G) \leq \frac{\Theta - 2}{t - 2}.$$

We now turn our attention to the structure of groups that admit particular types of automorphisms.

THEOREM 2.4. *Let φ be a fixed-point-free automorphism of the group G, then φ leaves invariant a unique p-Sylow subgroup P of G for each $p \in \pi(G)$. Furthermore, P contains every φ-invariant p-subgroup of G.*

PROOF. See Theorem 10.1.2 of [9]. □

From Theorem 2.4 we can easily deduce that if G is soluble, then G admits a (unique) φ-invariant Sylow system. We remark that, using the classification of finite simple groups, Rowley ([15]) proved that any group admitting a fixed-point-free automorphism is soluble.

THEOREM 2.5. *Let G be a group with a fixed-point-free automorphism of order p^α, p a prime. Then $h(G) \leq \alpha$.*

PROOF. This result is proved in [12] and [10] in the case where p is odd and in [13] if $p = 2$. □

LEMMA 2.6. *Let G be a group and let φ be a fixed-point-free automorphism of G of order $p^\alpha k$, where p is a prime number and $k \in \mathbb{N}$ with $(p,k)=1$. If P is a φ-invariant p-subgroup of G, then $C_P(\varphi^{p^\alpha})=1$.*

PROOF. Suppose, arguing by contradiction, that $C_P(\varphi^{p^\alpha}) \neq 1$. Then φ induces on $P_0 = C_P(\varphi^{p^\alpha})$ an automorphism of order dividing p^α and we have $C_{P_0}(\varphi) \neq 1$. □

A result proved by Espuelas is essential in order to obtain our results.

THEOREM 2.7. (Theorem 2.1 of [8]) *Let G be a group admitting an automorphism φ of order p^α acting fixed-point-freely on every φ-invariant p'-section of G, where p is an odd prime. Then $\ell_p(G) \leq \alpha+1$ and $h(G) \leq 2\alpha+1$. These bounds are best possible.*

Theorem 2.7 is a sharp generalization of the following result, proved by Hartley and Rae, valid also in the case $p=2$.

THEOREM 2.8. (Theorem 2 of [11]) *Let G be a group admitting an automorphism φ of order p^α acting fixed-point-freely on every φ-invariant p'-section of G. Then $\ell_p(G) \leq 2\alpha$.*

The following fundamental result is due to Turull (see §1).

THEOREM 2.9. (Corollary 3.2 of [18]) *Let G be a group and let A be a soluble subgroup of $\mathrm{Aut}(G)$ with $(|G|,|A|)=1$. Then*

$$h(G) \leq 2W(A) + h(C_G(A)).$$

We conclude this section with some more technical results.

LEMMA 2.10. *Let G be a $\{p,q\}$-group with p and q distinct primes. Let φ be a fixed-point-free automorphism of G of order $p^\alpha q^\beta$, then*

$$h(G) < 2W.$$

Moreover if p and q are odd, then

$$h(G) \leq W+1.$$

PROOF. By Theorem 2.4, in G there is a φ-invariant Sylow p-subgroup P and a φ-invariant Sylow q-subgroup Q which are φ-invariant; we have $G=PQ$ because $\pi(G)=\{p,q\}$. Since $p \neq q$ we can suppose, without loss of generality, that q is odd. By Theorem 2.7 we deduce

$$h(G) \leq 2\beta+1 \leq 2 \cdot \max\{\alpha,\beta\}+1 < 2W.$$

If also p is odd, then $h(G) \leq 2\alpha+1$, so

$$h(G) \leq \min\{2\alpha+1, 2\beta+1\} \leq \alpha+\beta+1 = W+1,$$

and the result follows. $\square$

It is well known that a group with a fixed-point-free automorphism of order 2 is abelian. From this fact we can derive

LEMMA 2.11. *Let q be an odd prime and let G be a $\{2,q\}$-group. If G admits a fixed-point-free automorphism φ of order $2q^\alpha$, then $h(G) \leq 3$.*

PROOF. By Theorem 2.4, we can choose a φ-invariant Sylow q-subgroup Q of G. By Lemma 2.6, φ^{q^α} is a fixed-point-free automorphism of order 2 of Q, and hence Q is abelian. From 9.3.7 of [14] we deduce $\ell_q(G) \leq 1$, and hence we can conclude that $h(G) \leq 3$. $\square$

The following lemma is needed in the proof of Theorem 1.6.

LEMMA 2.12. *Let G be a group, φ an automorphism of G and suppose that $(|C_G(\varphi)|, |\langle\varphi\rangle|) = 1$. Then*

(a) if N is a normal φ-invariant subgroup of G, then $(|C_{G/N}(\varphi)|, |\langle\varphi\rangle|) = 1$;

(b) for every $p \in \pi(G)$ there is a φ-invariant Sylow p-subgroup of G.

PROOF. We prove (a) arguing by induction on $|G| + |N|$. If $N = 1$, then (a) is trivially verified, in particular the induction basis is proved and we can suppose $N \neq 1$.

Let $L \neq 1$ be a normal minimal φ-invariant subgroup of G contained in N. If $L < N$, then, by induction hypothesis $(|C_{G/L}(\varphi)|, \langle\varphi\rangle) = 1$. Since $|G/L| + |N/L| < |G| + |N|$, the induction hypothesis yields the conclusion.

Hence N is a (non trivial) minimal normal φ-invariant subgroup of G, in particular N is an elementary abelian p-group for some $p \in \pi(G)$.

Suppose, arguing by contradiction, that $(|C_{G/N}(\varphi)|, |\langle\varphi\rangle|) \neq 1$. Hence there is a prime $q \in \pi(\langle\varphi\rangle)$ and an element $y \in G$ such that $y \notin N$, $y^q \in N$ and $yy^{-\varphi} \in N$. If $N\langle y\rangle < G$ then, applying the induction hypothesis to $N\langle y\rangle$ we obtain a contradiction, so $G = N\langle y\rangle$. Furthermore $C_G(\varphi)$ is a p-group, since $\pi(G) = \{p, q\}$ and $q \in \pi(\langle\varphi\rangle)$. We now distinguish two cases.

• $C_N(\varphi) \neq 1$. Let $x \in C_G(N)$, since N is abelian and $yy^{-\varphi} \in N$, we have $yy^{-\varphi} = (yy^{-\varphi})^x = y^x(y^x)^{-\varphi}$ and $[x,y] = (y^{-1})^x y = (y^x)^{-\varphi} y^\varphi = [x,y]^\varphi$. This shows that y normalizes $C_N(\varphi)$ and, since $G = N\langle y\rangle$ and N is abelian, we can conclude that $C_N(\varphi) \trianglelefteq G$. From the hypothesis that $C_N(\varphi) \neq 1$ and from the minimality of N we obtain $N = C_N(\varphi)$. Since $yy^{-\varphi} \in N$ we can write $yy^{-\varphi} = x^{-1}$ for some $x \in C_N(\varphi)$; if n is the order of φ, then, as $y^\varphi = xy$, applying n times φ we obtain $y = y^{\varphi^n} = x^n y$, that is $x^n = 1$. Since $x \in C_G(\varphi)$ and $(|C_G(\varphi)|, n) = 1$ we have $x = 1$ and $y \in C_G(\varphi)$. Then $G = C_G(\varphi)$ and $q \in \pi(G) \cap \pi(\langle\varphi\rangle)$, a contradiction.

• $C_N(\varphi) = 1$. By Lemma 10.1.1 of [9], $N = \{x^{-1}x^\varphi \mid x \in N\}$ and we can write $yy^{-\varphi} = x^{-1}x^\varphi$ for some $x \in N$. Since $xy \in C_G(\varphi)$, xy is a p-element and $(xy)^{p^k} = 1$ for some $k \in \mathbb{N}$. If $p \neq q$ then $y^{p^k} \in N$ and $y^q \in N$ implies $y \in N$, a contradiction. If $p = q$, then G is a q-group and $q \notin \pi(C_G(\varphi))$ implies that

$C_G(\varphi) = 1$. By Lemma 10.1.3 of [9], $C_{G/N}(\varphi) = 1$, we have thus obtained the contradiction $yN \in C_{G/N}(\varphi) = N$ and so we have proved (a).

In order to prove (b) we begin by observing that, if φ has prime power order q^k (q a prime, $k \in \mathbb{N}$), then $(|G|, q) = 1$. We argue by induction on the order of G (the basis is trivial). If G is an elementary abelian p-group for some prime p, and if $p = q$, then $C_G(\varphi) \neq 1$, against the hypothesis. Let N be a normal elementary abelian φ-invariant p-subgroup of G, then $p \neq q$. By (a) in G/N we have $(|C_{G/N}(\varphi)|, q) = 1$ and, by the induction hypothesis, $(|G/N|, q) = 1$, so $(|G|, q) = 1$.

We now prove (b) arguing by induction on $|G| + |\pi(\langle\varphi\rangle)|$. If $|\pi(\langle\varphi\rangle)| = 1$, then the order of G is coprime to $|\langle\varphi\rangle|$ and (b) follows by 6.2.2 of [9], in particular the induction basis is proved.

Fixed $p \in \pi(G)$, our aim is to prove that there is a Sylow p-subgroup of G which is φ-invariant. If $O_p(G) \neq 1$ then, by (a), we can consider $G/O_p(G)$ and we can easily conclude by induction hypothesis. Let N be a non trivial minimal φ-invariant normal subgroup of G, then N is an elementary abelian q-group for some $q \in \pi(G)$ and $q \neq p$. By induction hypothesis in G/N there is a φ-invariant Sylow p-subgroup and hence we can suppose $G = NP$, with P a Sylow p-subgroup of G. If $C_N(\varphi) \neq 1$, then $q \notin \pi(\langle\varphi\rangle)$, and hence $p \in \pi(\langle\varphi\rangle)$, since otherwise $(|G|, |\langle\varphi\rangle|) = 1$ and the conclusion follows by 6.2.2 of [9]. Write $|\langle\varphi\rangle| = p^k m$ with $(p, m) = 1$ and let $\psi = \varphi^m$. Let $\Gamma = G\langle\psi\rangle$ be the semidirect product of G by $\langle\psi\rangle$, then in Γ there is a Sylow p-subgroup Π such that $\psi \in \Pi$. The subgroup $G \cap \Pi$ is normal in Π and hence $C_{G\cap\Pi}(\psi) \neq 1$, in particular $C_G(\psi) \neq 1$. Moreover $C_G(\psi) \neq G$, as otherwise φ would have order m, coprime to p, hence, by induction hypothesis, $C_G(\psi)$ contains a non trivial φ-invariant Sylow p-subgroup P_0. Let M be a φ-invariant p-subgroup of G of maximal order and let P be a Sylow p-subgroup of G containing M. If $M < P$, then $N_G(M) \geq N_P(M) > M$. The two conditions $O_p(G) = 1$ and $M \neq 1$ imply $N_G(M) < G$ and, by the induction hypothesis, $N_G(M)$ contains a φ-invariant subgroup of order greater than $|M|$. This forces $M = P$ and (b) is proved. □

Remark 2.13. Lemma 2.12 allows us to state that, if $(|C_G(\varphi)|, |\langle\varphi\rangle|) = 1$, then G admits a φ-invariant Sylow system. In particular for every $\sigma \subseteq \pi(G)$, in G there is a φ-invariant Hall σ-subgroup.

Remark 2.14. Without the hypothesis $(|C_G(\varphi)|, |\langle\varphi\rangle|) = 1$, Lemma 5.9 is no longer true. As a simple counterexample we can consider $G \simeq S_3$ and φ the inner automorphism of order 3.

§3. Proofs

Proposition 3.1. *Let G be a group and let φ be a fixed-point-free automorphism of G of order $p^\alpha q^\beta$, with distinct primes p and q. Then*

$$h(G) < 5W - 2;$$

moreover if p and q are odd, then

$$h(G) \leq 4W - 1.$$

Proof. Let $J = PQ$, H and K be respectively φ-invariant Hall subgroups of G with $\pi(J) = \{p, q\}$, $\pi(H) = \{p\}'$ and $\pi(K) = \{q\}'$ (see Theorem 2.4 and the remark made after it). By Lemma 2.10 we have that $h(J) < 2W$. If we consider the action of φ on H, we see that φ acts as a fixed-point-free automorphism of order q^β on $C_H(\varphi^{q^\beta})$ and hence $h(C_H(\varphi^{q^\beta})) \leq \beta$. Since $\big(|H|, |\varphi^{q^\beta}|\big) = 1$, by Theorem 2.9 we deduce

$$h(H) \leq 2\alpha + h(C_H(\varphi^{q^\beta})) \leq 2\alpha + \beta$$

and similarly $h(K) \leq \alpha + 2\beta$. By Theorem 2.1 we obtain

$$h(G) \leq h(J) + h(H) + h(K) - 2 < 5W - 2.$$

If p and q are odd, then, by Lemma 2.10, $h(J) \leq W + 1$, and hence we conclude that $h(G) \leq 4W - 1$. □

Proof of Theorem 1.1. We argue by induction on w. Let $|\varphi| = \prod_{i=1}^{w} p_i^{\alpha_i}$ with $\alpha_i \in \mathbb{N}$ and p_i distinct prime numbers. If $w = 2$, then the conclusion follows from Proposition 3.1.

Suppose $w \geq 3$, denote by G_i a φ-invariant Hall p_i'-subgroup of G and write $\langle\varphi\rangle = \langle\varphi_1\rangle \times \langle\varphi_2\rangle \times \ldots \times \langle\varphi_w\rangle$ with φ_i of order $p_i^{\alpha_i}$, for $i \in \{1, 2, \ldots, w\}$. Since $(|G_i|, |\langle\varphi_i\rangle|) = 1$, by the Turull's Theorem 2.9 we have

$$h(G_i) \leq 2W(\varphi_i) + h\big(C_{G_i}(\varphi_i)\big) = 2\alpha_i + h\big(C_{G_i}(\varphi_i)\big).$$

The automorphism ψ_i induced on $C_{G_i}(\varphi_i)$ by φ has order dividing $|\varphi|/p_i^{\alpha_i}$, so we have $W(\psi_i) \leq W(\varphi) - \alpha_i$ and $w(\psi_i) \leq w(\varphi) - 1$. The induction hypothesis leads us to conclude that

$$h\big(C_{G_i}(\varphi_i)\big) < \big(7(w-1) - 9\big)\big(W - \alpha_i\big)$$

and $h(G_i) < 2\alpha_i + \big(7(w-1) - 9\big)\big(W - \alpha_i\big)$.

An easy computation provides

$$\sum_{i=1}^{w} h(G_i) \leq 2W + \big(7(w-1) - 9\big)(w-1)W < (7w - 9)(w-2)W,$$

and applying Theorem 2.3 we obtain

$$h(G) \leq \frac{\Big(\sum_{i=1}^{w} h(G_i)\Big) - 2}{w - 2} < (7w - 9)W,$$

which concludes the proof. □

Remark 3.2. If, in Theorem 1.1, we suppose that $|\varphi|$ is odd, then, thanks to the Proposition 3.1, we can improve the bound for the Fitting length of G to

$$h(G) \leq 2\big(3w-4\big)W.$$

Proof of Proposition 1.3. By hypothesis $|\varphi| = p^\alpha q$, and hence $W = \alpha + 1$. We use the notation of the proof of Proposition 3.1. By Theorem 2.7 (if q is odd) and Lemma 2.11 (if $q = 2$) we obtain $h(J) \leq 3$. Arguing as in the proof of Proposition 3.1, we deduce $h(H) \leq 2\alpha + 1$ and $h(K) \leq \alpha + 2$. So

$$h(G) \leq h(J) + h(H) + h(K) - 2 = 3\alpha + 4 = 3W + 1,$$

and the proof is complete. □

Remark 3.3. In [7] it has been proven that if a group G has a fixed-point-free automorphism of order $p^\alpha q$ with $(pq, 6) = 1$ and if the Sylow 2-subgroups of G are abelian, then $h(G) \leq W(\varphi)$.

Proof of Proposition 1.4. We will proceed as in the proof of Theorem 1.1, using the same notation and adding the conditions

$$\alpha_1 = \alpha_2 = \ldots = \alpha_w = 1.$$

If $w = W = 3$ then, by [5], we know that $h(G) \leq 3$. If we suppose $W \geq 4$, we have

$$h(G_i) \leq 2 + h\big(C_{G_i}(\varphi_i)\big)$$

and, by the induction hypothesis,

$$h(G_i) \leq 2 + \frac{1}{2}\big(3(W-1)^2 - 7(W-1)\big) = \frac{1}{2}\big(3W-7\big)\big(W-2\big),$$

so $\sum_{i=1}^{W} h(G_i) \leq \frac{1}{2}\big(3W-7\big)\big(W-2\big)W$. Now, by Theorem 2.3,

$$h(G) \leq \frac{\Big(\sum_{i=1}^{W} h(G_i)\Big) - 2}{W-2} < \frac{\big(3W-7\big)\big(W-2\big)W}{2(W-2)} = \frac{1}{2}\big(3W^2 - 7W\big),$$

and the theorem is proved. □

The proof of Theorem 1.6 is very similar to that of Theorem 1.1; we report it here for completeness.

Proof of Theorem 1.6. We use induction on w.

By Lemma 2.12 and Remark 2.13 we know that, for every $\sigma \subseteq \pi(G)$, G admits a φ-invariant Hall σ-subgroup.

Suppose $w = 2$, $|\varphi| = p^\alpha q^\beta$ with $p \neq q$ primes. Let $J = PQ$, H and K be respectively φ-invariant Hall subgroups of G with $\pi(J) = \{p, q\}$, $\pi(H) = \{p\}'$ and $\pi(K) = \{q\}'$. By Theorem 2.9 we can write

$$h(J) \leq \min\big\{2\alpha + h(C_J(\varphi^{q^\beta}),\ 2\beta + h(C_J(\varphi^{p^\alpha})\big\} \leq 2\alpha + 2\beta + h_0 = 2W + h_0,$$

$$h(H) \leq 2\alpha + h(C_H(\varphi^{p^\alpha}) \leq 2\alpha + 2\beta + h_0 = 2W + h_0$$

and, as the roles of p and q can be exchanged, $h(K) \leq 2W + h_0$. By Theorem 2.1 we obtain

$$h(G) \leq h(J) + h(H) + h(K) - 2 < 6W + 3h_0,$$

so the induction basis is proved.

Suppose now $w \geq 3$ and let $|\varphi| = \prod_{i=1}^{w} p_i^{\alpha_i}$. Denote by G_i a φ-invariant Hall p_i'-subgroup of G and write $\langle\varphi\rangle = \langle\varphi_1\rangle \times \langle\varphi_2\rangle \times \ldots \times \langle\varphi_w\rangle$ with φ_i of order $p_i^{\alpha_i}$, for $i \in \{1, 2, \ldots, w\}$. Since $(|G_i|, |\langle\varphi_i\rangle|) = 1$, by the Turull's Theorem 2.9 we have

$$h(G_i) \leq 2W(\varphi_i) + h\big(C_{G_i}(\varphi_i)\big) = 2\alpha_i + h\big(C_{G_i}(\varphi_i)\big).$$

By induction hypothesis we can write

$$h\big(C_{G_i}(\varphi_i)\big) < \big(8w - 18\big)\big(W - \alpha_i\big) + \frac{3}{2}\big(w - 2\big)\big(w - 1\big)h_0$$

and an easy computation provides that

$$\sum_{i=1}^{w} h\big(C_{G_i}(\varphi_i)\big) < \big(8w - 18\big)\big(w - 1\big)W + \frac{3}{2}\big(w - 2\big)\big(w - 1\big)wh_0,$$

hence, by Theorem 2.3 the following inequality hold

$$h(G) < \frac{1}{w-2}\left(\big(8w - 18\big)\big(w - 1\big)W + 2W + \frac{3}{2}\big(w - 2\big)\big(w - 1\big)wh_0\right).$$

Since $(8w - 18)(w - 1) + 2 = (8w - 10)(w - 2)$, we have

$$h(G) < (8w - 10)W + \frac{3(w-1)w}{2}h_0$$

and the conclusion. □

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Enrico Jabara
DFBC – Università di Venezia
Dorsoduro 3484/D - 30123 Venezia - ITALY
jabara@unive.it