Boolean-like algebras

ANTONINO SALIBRA, ANTONIO LEDDA, FRANCESCO PAOLI, AND TOMASZ KOWALSKI

ABSTRACT. Using Vaggione's concept of central element in a double-pointed algebra, we introduce the notion of *Boolean-like variety* as a generalisation of Boolean algebras to an arbitrary similarity type. Appropriately relaxing the requirement that every element be central in any member of the variety, we obtain the more general class of *semi-Boolean-like varieties*, which still retain many of the pleasing properties of Boolean algebras. We prove that a double-pointed variety is discriminator iff it is semi-Boolean-like, idempotent, and 0-regular. This theorem yields a new Maltsev-style characterisation of double-pointed discriminator varieties.

1. Introduction

Boolean algebras have an exceptionally rich and smooth structure theory, of which Stone's representation theorem is a prominent example. What is so special about Boolean algebras that is responsible for this nice behaviour? Given a similarity type ν, can we always find a class of algebras of type ν that displays Boolean-like features? And what does it mean, for an algebra of a given type ν that may not exhibit such desirable properties, to have at least a subset of Boolean elements that behave well? To address these questions, we use the concept, due to Vaggione [39], of a *central element* in a double-pointed algebra, meaning an element which induces therein, in a specified sense, a pair of complementary factor congruences. Roughly speaking, given a similarity type ν including at least two constants but otherwise fully arbitrary, we associate the presence of a "well-behaved Boolean core" in a ν-algebra with the presence of a retract of central elements, and we identify Boolean ν-algebras with ν -algebras where every element is central. In order to fully appreciate what properties of Boolean algebras are responsible for the most important results concerning this variety, however, the issue is best addressed in a step-by-step fashion. Therefore, following [31], we will decompose the property of centrality into several equational properties, trying to investigate what happens when some of them are satisfied but other ones are dropped. This approach will give rise to a few successive approximations to a full-fledged notion of "Boolean algebra of arbitrary similarity type".

Presented by ...
Received ...; accepted in final form ...
1991 *Mathematics Subject Classification*: Primary: 08B25; Secondary: 08B05, 08A70.
Key words and phrases: Boolean-like algebra, double-pointed variety, discriminator variety.

Our work ties nicely with at least three research streams that have received considerable attention in universal algebra and in the investigation into the mathematical foundations of computer science:

- *(Weak) Boolean product representations.* It has been known for a long time that Stone's representation theorem, perhaps the most distinctive result characterising Boolean algebras (or Boolean rings), can be generalised to a much larger class of algebras. The appropriate tool to attain this goal is the technique of Boolean products, which can be loosened to the notion of *weak Boolean product* to take care of somewhat less manageable cases (see e.g. [24]). Pierce [36] proved that every commutative ring with unit is representable as a weak Boolean product of directly indecomposable rings; Stone's representation theorem follows as a corollary by observing that the 2-element ring of truth values is the unique directly indecomposable Boolean ring. The technique of Boolean products underwent remarkable developments over the subsequent years (see e.g. [13, Ch. 4.8]), giving rise to further generalisations of Stone's theorem by Comer (covering the case of algebras with Boolean factor congruences [16]) and Vaggione [39].
- *Discriminator varieties and noncommutative lattice theory.* Discriminator varieties [41] are referred to by Burris and Sankappanavar as

  > the most successful generalization of Boolean algebras to date, successful because we obtain Boolean product representations (which can be used to provide a deep insight into algebraic and logical properties) [13, p. 186].

  One of the most elegant characterisations of discriminator varieties in the pointed case was obtained by Bignall and Leech [7], who linked them to a noncommutative generalisation of Boolean algebras called *left handed skew Boolean $\cap$-algebras*. More precisely, Bignall and Leech proved that: (i) the variety of type $(3, 0)$ generated by the class of all pointed discriminator algebras $\mathbf{A} = (A; t, 0)$, where t is the discriminator function on A and 0 is a constant, is term equivalent to the variety of left handed skew Boolean $\cap$-algebras; (ii) every pointed discriminator variety is term equivalent to a variety of left handed skew Boolean $\cap$-algebras with additional compatible operations. This result can be easily adapted to the double-pointed case[1], which is particularly significant in that the variety of Boolean algebras is double-pointed [10]. Some more steps in this direction are taken in what follows.
- *Algebraic investigation of the if-then-else construct.* There is a thriving literature on abstract treatments of the fundamental if-then-else construct of computer science, starting with McCarthy's seminal investigations [34]. On the algebraic side, one of the most influential approaches originated

[1]Following [10], we say that a class of similar algebras is *double-pointed* if its type has at least two constants that realise distinct elements in any nontrivial member of the class.

with a paper by Bergman [6]. Bergman modelled the if-then-else by considering Boolean algebras acting on sets: if the Boolean algebra of actions is the 2-element algebra, one simply sets $1(a, b) = a$ and $0(a, b) = b$ (see e.g. [22] for details). Alternative perspectives make recourse to dynamic algebras [37] or Kleene algebras with tests [27]. The approach followed in this paper originates with Dicker's axiomatisation of Boolean algebras in the type $(3, 0, 0)$ [18], and differs from Bergman's in that the if-then-else is treated as a proper algebraic ternary operation q on a double-pointed algebra $\mathbf{A}$, having the property that for every $a, b \in A$, $q(1, a, b) = a$ and $q(0, a, b) = b$. The resulting variety of *Church algebras* is investigated in [30, 31, 32, 33] and is one of the fundamental notions in the present work as well.

In greater detail, our paper is structured as follows. In Section 2 we will dispatch some necessary preliminaries. In Section 3 we recall from [31] the definition of Church algebra, introducing the concept of central element and providing an equational characterisation thereof. In a generic Church algebra, of course, there is no need for the set of central elements to comprise all of the algebra. Church algebras where this is the case are (unimaginatively) called *Boolean-like*, while the ones that miss the mark by possibly failing one designated equation (in other words, where every element is *semi-central*) are termed *semi-Boolean-like*. Both notions are the focus of Section 4. We provide a characterisation of Boolean-like varieties as discriminator varieties in which the directly indecomposable members are two-element algebras. Moreover, we prove several properties of the pure semi-Boolean-like variety, e.g. that it has no congruence identities. In Subsection 4.2, we give a purely algebraic characterisation of semi-Boolean-like varieties along the lines of the one provided in [10] for discriminator varieties. In Section 5 we use the previous concepts and results to provide several descriptions of double-pointed discriminator varieties. With such an aim in mind, in Subsection 5.1 we consider semi-Boolean-like algebras where a term definable meet-like binary operation is idempotent. We prove that a double-pointed variety is discriminator iff it is semi-Boolean-like, idempotent, and 0-regular. This theorem yields a new Maltsev-style characterisation of double-pointed discriminator varieties.

Our notational and terminological conventions are the usual ones in universal algebra. Deviations from standard usage will be explicitly noted in what follows.

2. Preliminaries

The present section reviews concepts and definitions needed to make the present paper as self-contained as possible. Nonetheless, standard elementary material, e.g. on Boolean algebras or discriminator varieties, is not covered here. The interested reader may consult [20] or [13, Ch. 4].

2.1. Weak Boolean products. Burris and Werner [14, 15] obtained Boolean product representations for large classes of algebras, based on properties of their congruence lattices; as mentioned in the introduction, important results on Boolean products were proved by Comer [16] and Vaggione [39]. A good recent account of (weak) Boolean products of lattice-ordered algebras is included in [23]. The relevant definitions follow hereafter.

Definition 2.1. A *weak Boolean product* of a family $(\mathbf{A}_i)_{i\in I}$ of algebras is a subdirect product $\mathbf{A} \leq \prod_{i\in I} \mathbf{A}_i$, where I can be endowed with a Boolean space topology such that: (i) the set $\{i \in I : a_i = b_i\}$ is open for all $a, b \in A$, and (ii) if $a, b \in A$ and $N \subseteq I$ is clopen, then the element c, defined by $c_i = a_i$ for $i \in N$ and $c_i = b_i$ for $i \in I - N$, belongs to A.

Definition 2.2. A *Boolean product* of a family $(\mathbf{A}_i)_{i\in I}$ of algebras is a weak Boolean product of such, with the property that the set $\{i \in I : a_i = b_i\}$ is clopen for all $a, b \in A$.

2.2. Subtractive and quasi-subtractive varieties. Subtractive varieties were introduced by Ursini [38] to enucleate the common features of pointed varieties with a good ideal theory, namely varieties of algebras — like groups, rings or Boolean algebras — whose congruences can be replaced to all intents and purposes by ideals of sorts. They were further investigated in [1, 2, 3].

Definition 2.3. A variety $\mathcal{V}$ whose type ν includes a term definable constant 1 is called 1-*subtractive* if there exists a binary term $\rightarrow$ of type ν (hereafter written in infix notation) such that $\mathcal{V}$ satisfies the identities $x \rightarrow x \approx 1$ and $1 \rightarrow x \approx x$. A variety of type ν which is 1-subtractive w.r.t. at least one constant 1 of type ν is called *subtractive* tout court.

It is not hard to see that subtractivity is a congruence property: in fact, a variety $\mathcal{V}$ is 1-subtractive just in case in each $\mathbf{A} \in \mathcal{V}$ congruences permute at 1 (meaning that for all θ, φ in $\mathrm{Con}(\mathbf{A})$, $1/\theta \circ \varphi = 1/\varphi \circ \theta$).

To show that subtractive varieties have a good ideal theory we need a workable general notion of ideal encompassing all the intended examples mentioned above (normal subgroups of groups, two-sided ideals of rings, ideals or filters of Boolean algebras). Ursini's candidate for playing this role is defined below.

Definition 2.4. If $\mathcal{K}$ is a class of similar algebras whose type ν is as in Definition 2.3, a term $p(\overline{x}, \overline{y})$ of type ν is a $\mathcal{K}$-*ideal term* in $\overline{x}$ if $\mathcal{K} \vDash p(1, \dots, 1, \overline{y}) \approx 1$. A nonempty subset J of the universe of $\mathbf{A} \in \mathcal{K}$ is a $\mathcal{K}$-*ideal* of $\mathbf{A}$ (w.r.t. 1) if for any $\mathcal{K}$-ideal term in $\overline{x}$ $p(\overline{x}, \overline{y})$ we have that $p^{\mathbf{A}}\left(\overline{a}, \overline{b}\right) \in J$ whenever $\overline{a} \in J, \overline{b} \in A$.

Under the additional assumption of point regularity, one can show that ideals can indeed replace congruences in members of subtractive varieties:

Theorem 2.5. *If $\mathcal{V}$ is a 1-subtractive and 1-regular variety, then in every $\mathbf{A} \in \mathcal{V}$, $\mathrm{Con}(\mathbf{A})$ is isomorphic to the lattice of $\mathcal{V}$-ideals of $\mathbf{A}$.*

Powerful and general as it may be, Theorem 2.5 does not subsume all the known ideal-congruence isomorphism theorems known in the literature; in particular, there are varieties (e.g. pseudointerior algebras [11], or quasi-MV algebras [28]) which fail to be subtractive but seem to have a good ideal theory notwithstanding. In [26] the following generalisation of subtractive varieties has been suggested:

Definition 2.6. A variety $\mathcal{V}$ whose type ν includes a term definable constant 1 and a term definable unary term $\Box$ is called *quasi-subtractive* w.r.t. 1 and $\Box$ iff there is a binary term $\rightarrow$ of type ν (hereafter written in infix notation) such that $\mathcal{V}$ satisfies the equations

(**Q1**)	$\Box x \rightarrow x \approx 1$	(**Q2**)	$1 \rightarrow x \approx \Box x$
(**Q3**)	$\Box (x \rightarrow y) \approx x \rightarrow y$	(**Q4**)	$\Box (x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$

Clearly, subtractive varieties are in particular quasi-subtractive (just trivialise $\Box$ in the above definition). It is not known if quasi-subtractivity is a congruence property; quasi-subtractive varieties are τ*-permutable* in the sense of Blok and Raftery [12] (for $\tau = \{\Box x \approx 1\}$), but the converse need not hold.

The rôle played by ideals in the theory of subtractive varieties is played by *open filters* in the suggested generalisation:

Definition 2.7. Let $\mathcal{V}$ be a variety whose type ν is as in Definition 2.6. A $\mathcal{V}$*-open filter term* in the variables $\overline{x}$ is an $n+m$-ary term $p(\overline{x}, \overline{y})$ of type ν such that

$$\{\Box x_i \approx 1 : i \leq n\} \vDash_{\mathcal{V}} \Box p(\overline{x}, \overline{y}) \approx 1.$$

A $\mathcal{V}$*-open filter* of $\mathbf{A} \in \mathcal{V}$ is a subset $F \subseteq A$ with the following properties: (i) it is closed w.r.t. all $\mathcal{V}$-open filter terms p: whenever $a_1, \ldots, a_n \in F, b_1, \ldots, b_m \in A$, $p\left(\overline{a}, \overline{b}\right) \in F$; (ii) for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.

Remarkably enough, in every member of a quasi-subtractive variety the lattice of open filters is modular.

Recall that τ-regular varieties [9] are a generalisation of point regular varieties to the case of an arbitrary translation τ: a variety $\mathcal{V}$ is *weakly* τ*-regular* iff its τ-assertional logic[2] is strongly and finitely algebraisable, and τ*-regular* (tout court) if, in addition, $\mathcal{V}$ is its equivalent variety semantics.

Under the additional assumption of weak $\{\Box x \approx 1\}$-regularity, one can show that open filters can indeed replace congruences with the appropriate quotients in members of quasi-subtractive varieties.

Theorem 2.8. *If* $\mathcal{V}$ *is quasi-subtractive and weakly* $\{\Box x \approx 1\}$*-regular, and* $\mathcal{V}'$ *is the equivalent variety semantics of the* $\{\Box x \approx 1\}$*-assertional logic of* $\mathcal{V}$*,*

[2]If $\mathcal{K}$ is a class of algebras of type ν and $\tau(x) = \{t_i(x) \approx s_i(x)\}_{i \in I}$ a translation (a mapping from ν-terms to sets of ν-equations in a single variable), the τ*-assertional logic of* $\mathcal{K}$ is the logic $\mathcal{S}(\mathcal{K}) = (\mathbf{Tm}_\nu, \vdash)$, where

$$\Gamma \vdash t \text{ iff } \{\tau(s) : s \in \Gamma\} \vDash_{\mathcal{V}} \tau(t).$$

then in any $\mathbf{A} \in \mathcal{V}$*, there is a lattice isomorphism between the lattice of all congruences* θ *on* $\mathbf{A}$ *such that* $\mathbf{A}/\theta \in \mathcal{V}'$ *and the lattice of* $\mathcal{V}$*-open filters on* $\mathbf{A}$*.*

Clearly, this result specialises to a full isomorphism theorem between the lattices of congruences and of open filters in case $\mathcal{V}$ is quasi-subtractive and $\{\Box x \approx 1\}$-regular.

3. Church algebras

The key observation motivating the introduction of *Church algebras* [30, 31, 32, 33] is that many algebras arising in completely different fields of mathematics — including Heyting algebras, rings with unit, or combinatory algebras — have a term operation q satisfying the fundamental properties of the if-then-else connective : $q(1, x, y) \approx x$ and $q(0, x, y) \approx y$. As simple as they may appear, these properties are enough to yield rather strong results.

Definition 3.1. An algebra $\mathbf{A}$ of type ν is a *Church algebra* if there are term definable elements $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a ternary term operation $q^{\mathbf{A}}$ such that, for all $a, b \in A$, $q^{\mathbf{A}}(1^{\mathbf{A}}, a, b) = a$ and $q^{\mathbf{A}}(0^{\mathbf{A}}, a, b) = b$. A variety $\mathcal{V}$ of type ν is a *Church variety* if every member of $\mathcal{V}$ is a Church algebra with respect to the same term $q(x, y, z)$ and the same constants $0, 1$.

If $\mathbf{A}$ is a Church algebra, then $\mathbf{A}_0 = (A; q^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}})$ is the *pure reduct* of $\mathbf{A}$. Henceforth, the superscript in $q^{\mathbf{A}}$ will be dropped whenever the difference between the operation and the operation symbol is clear from the context, and the same policy will be followed in similar cases throughout the paper. The following proposition, whose proof is left to the reader, provides some examples of Church varieties.

Proposition 3.2. *Let* $\mathcal{V}$ *be a double-pointed variety.*

(1) *If* $\mathcal{V}$ *is* 1*-subtractive with witness term* $\to$*, and every* $\mathbf{A} \in \mathcal{V}$ *has a term reduct* $(A; \cdot, 1, 0)$ *of type* $(2, 0, 0)$ *satisfying*

$$1 \cdot x \approx x \approx x \cdot 1; \quad 0 \cdot x \approx 0,$$

then $\mathcal{V}$ *is a Church variety w.r.t.* $q(x, y, z) = ((x \to (x \cdot z)) \to z) \cdot (x \to (x \cdot y))$*.*

(2) *If* $\mathcal{V}$ *has a term reduct* $(A; +, \cdot, ', 1, 0)$ *of type* $(2, 2, 1, 0, 0)$ *satisfying*

$$1 \cdot x \approx x; \quad 0 \cdot x \approx 0; \quad x + 0 \approx x \approx 0 + x; \quad 0' \approx 1; \quad 1' \approx 0,$$

then $\mathcal{V}$ *is a Church variety w.r.t.* $q(x, y, z) = (x \cdot y) + (x' \cdot z)$*.*

Moreover, the following are easily checked to be Church algebras:

Example 3.3. (1) Rings with unit: $q(x, y, z) = xy + (1 - x)z$;
(2) FL_{ew}*-algebras* [19]: $q(x, y, z) = (x \vee z) \wedge ((x \to 0) \vee y)$;
(3) *Ortholattices* [13, p. 29]: $q(x, y, z) = (x \vee z) \wedge (x' \vee y)$;

(4) *Combinatory algebras* [5]: $q(x, y, z) = (x \cdot y) \cdot z$, $1 = \mathbf{k}$ *and* $0 = \mathbf{s} \cdot \mathbf{k}$, where **k** and **s** are the basic combinators.

Hereafter, we consider the following terms:

$x \wedge y = q(x, y, 0)$;	$x \vee y = q(x, 1, y)$;
$x' = q(x, 0, 1)$;	$c(x) = q(x, 1, 0)$;
$x - y = y' \wedge x$.	

Exploiting an idea by Vaggione [39], we also define:

Definition 3.4. An element e of a Church algebra $\mathbf{A}$ is called *central* if the pair $(\theta(e, 0), \theta(e, 1))$ is a pair of complementary factor congruences of $\mathbf{A}$. A central element e is *nontrivial* if $e \notin \{0, 1\}$. By $\mathrm{Ce}(A)$ we denote the set of central elements of the algebra $\mathbf{A}$.

With reference to Example 3.3, it is known that central elements coincide with central idempotents in rings with unit, with complemented elements in FL_{ew}-algebras, and with members of the centre in ortholattices. The next characterisation of central elements in a Church algebra is extremely useful; the proofs of Proposition 3.6, and Theorem 3.7 can be found in [32] for the particular case of combinatory algebras. We start with a lemma.

Lemma 3.5. *Let* $\mathbf{A}$ *be a Church algebra and* $e \in A$. *Then we have, for all* $a, b \in A$,

$$a \ \theta(e, 1) \ q(e, a, b) \ \theta(e, 0) \ b.$$

Proof. From $1 \ \theta(e, 1) \ e \ \theta(e, 0) \ 0$ it follows that

$$q(1, a, b) \ \theta(e, 1) \ q(e, a, b) \ \theta(e, 0) \ q(0, a, b).$$

By applying the identities characterizing Church algebras we get the conclusion. □

Factor congruences can be characterised in terms of certain algebra homomorphisms called decomposition operations (see e.g. [35, Def. 4.32] for more details).

Proposition 3.6. *If* $\mathbf{A}$ *is a Church algebra of type* ν *and* $e \in A$, *the following conditions are equivalent:*

(1) *e is central;*
(2) $\theta(e, 0) \cap \theta(e, 1) = \Delta^{\mathbf{A}}$;
(3) *for all* $a, b \in A$, *the element* $q(e, a, b)$ *is the unique* $c \in A$ *such that* $a \ \theta(e, 1) \ c \ \theta(e, 0) \ b$;
(4) *For all* $a, b, \overline{a}, \overline{b} \in A$:

1. $q(e, a, a) = a$
2. $q(e, q(e, a, b), c) = q(e, a, c) = q(e, a, q(e, b, c))$
3. $q(e, f(\overline{a}), f(\overline{b})) = f(q(e, a_1, b_1), \ldots, q(e, a_n, b_n))$ *(for every* $f \in \nu$*)*
4. $q(e, 1, 0) = e$

(5) *The function* $f_e(a,b) = q(e,a,b)$ *is a decomposition operation on* $\mathbf{A}$ *such that* $f_e(1,0) = e$.

Proof. (1) $\Rightarrow$ (3) $(\theta(e,1), \theta(e,0))$ is a pair of complementary factor congruences; then by Lemma 3.5 $c = q(e,a,b)$ is the unique element satisfying $a\ \theta(e,1)\ c\ \theta(e,0)\ b$.

(3) $\Rightarrow$ (2) By $a\,\theta(e,1)\,a\,\theta(e,0)\,a$ and by Lemma 3.5 we have that $q(e,a,a) = a$. If $a(\theta(e,0) \cap \theta(e,1))b$ then $a\ \theta(e,1)\ b\ \theta(e,0)\ a$, so that $b = q(e,a,a) = a$.

(2) $\Rightarrow$ (1) From Lemma 3.5 it follows that $\theta(e,1) \circ \theta(e,0) = \nabla^{\mathbf{A}}$.

(4) $\Leftrightarrow$ (5) The identities of item (4) express that f_e is a decomposition operator with the required property.

(1) $\Rightarrow$ (5) f_e is a decomposition operator because $(\theta(e,1), \theta(e,0))$ is a pair of complementary factor congruences and $q(e,a,b)$ is the unique element satisfying $a\ \theta(e,1)\ q(e,a,b)\ \theta(e,0)\ b$. Moreover, $f_e(1,0) = q(e,1,0) = e$ follows from $1\ \theta(e,1)\ e\ \theta(e,0)\ 0$.

(5) $\Rightarrow$ (1) Let $(\phi, \overline{\phi})$ be the pair of complementary factor congruences associated with f_e. From $f_e(1,0) = q(e,1,0) = e$ it follows that $1\,\phi\,e\,\overline{\phi}\,0$, so that $\theta(e,1) \subseteq \phi$ and $\theta(e,0) \subseteq \overline{\phi}$. For the opposite direction, let $a\phi b$, which is equivalent to $q(e,a,b) = b$ by definition of decomposition operator. Then by $1\theta(e,1)e$ we derive $a = q(1,a,b)\theta(e,1)q(e,a,b) = b$, that implies $\phi \subseteq \theta(e,1)$. Similarly for $\overline{\phi}$. □

Observe that Church varieties are *Pierce varieties*, in the sense of [39]. Hence, as a consequence of [39, Theorem 5], every Church algebra has factorable congruences and then by [8, Corollary 1.4] it has Boolean factor congruences.

Theorem 3.7. *Let* $\mathbf{A}$ *be a Church algebra. Then* $\mathrm{Ce}(\mathbf{A}) = (\mathrm{Ce}(A); \vee, \wedge, ', 0, 1)$ *is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of* $\mathbf{A}$.

Proof. The map $e \mapsto \theta(e,0)$ is a bijective map from the set $\mathrm{Ce}\,(A)$ of central elements onto the Boolean algebra of factor congruences. We show that, for all central elements e and d, the elements e', $e \wedge d$ and $e \vee d$ are central and are respectively associated with the factor congruences $\theta(e,1) = \theta(e',0)$, $\theta(e,0) \cap \theta(d,0)$ and $\theta(e,0) \vee \theta(d,0)$.

We check the details for $e \vee d = q(e,1,d)$. First of all, we show that $q(e,1,d) = q(d,1,e)$. By Proposition 3.6.3 we have that

$$1\ \theta(e,1)\ q(e,1,d)\ \theta(e,0)\ d,$$

while $1\ \theta(e,1)\ q(d,1,e)\ \theta(e,0)\ d$ can be obtained as follows:

$$1 =_{\text{P. 3.6.4}} q(d,1,1)\ \theta(e,1)\ q(d,1,e)\ \theta(e,0)\ q(d,1,0) = d$$

Then, by Proposition 3.6.3 we have the conclusion $q(e,1,d) = q(d,1,e)$. We now show that $q(e,1,d)$ is the central element associated with the factor congruence $\theta(e,0) \vee \theta(d,0)$, that is,

$$1\ (\theta(e,1) \wedge \theta(d,1))\ q(e,1,d)\ (\theta(e,0) \vee \theta(d,0))\ 0.$$

By $q(e,1,d) = q(d,1,e)$ we have $1\ \theta(e,1)\ q(e,1,d)$ and $1\ \theta(d,1)\ q(e,1,d)$, that is, $1\ (\theta(e,1) \wedge \theta(d,1))\ q(e,1,d)$. Finally, by Proposition 3.6 we obtain: $q(e,1,d)\ \theta(e,0)\ d = q(d,1,0)\ \theta(d,0)\ 0$, that is, $q(e,1,d)\ (\theta(e,0) \vee \theta(d,0))\ 0$. A similar reasoning works for $e \wedge d$ and e'. □

We already recalled that discriminator varieties are a successful generalisation of Boolean algebras in that they retain several distinctive properties thereof, including their being amenable to Boolean product representations with simple stalks. A generic Church variety admits a weak Boolean product representation, but falls short of this standard under several respects – for one, in general, stalks need not even be directly indecomposable, as witnessed by the case of rings with unit. The following theorems provide necessary and sufficient conditions for this to be the case, as well as singling out cases where the representation is actually a Boolean product representation.
The next theorem corrects a partly erroneous statement to be found in [32]. Item (1) follows from [16], because Church algebras have Boolean factor congruences. Item (2) follows from [13, Theorem VI.8.12].

Theorem 3.8. *Let* $\mathbf{A}$ *be a Church algebra,* S *be the Boolean space of maximal ideals of* $\mathrm{Ce}(\mathbf{A})$ *and* $f : A \to \Pi_{I\in S} A/\theta_I$ *be the map defined by*

$$f(a) = (a/\theta_I : I \in S),$$

where $\theta_I = \bigcup_{e\in I} \theta(0,e)$. *Then we have:*

(1) f *gives a weak Boolean representation of* $\mathbf{A}$.
(2) f *provides a Boolean representation of* $\mathbf{A}$ *iff, for all* $a \neq b \in A$, *there exists a least central element* e *such that* $q(e,a,b) = a$, *that is,* $(a,b) \in \theta(0,e)$.

For the previous representation to be of some interest, we need to be in a position to provide additional information on its stalks. The following theorem is a consequence of [39, Theorem 8], because Church varieties are Pierce varieties in the sense of [39]. Nonetheless, an alternative detailed proof of Theorem 3.9 can be found in the Appendix.

Hereafter we denote by $T_\nu(x)$ the set of ν-terms in one variable x.

Theorem 3.9. *Let* $\mathcal{V}$ *be a Church variety of type* ν. *Then, the following conditions are equivalent:*

(1) *For all* $\mathbf{A} \in \mathcal{V}$, *the stalks* $\mathbf{A}/\theta_I$ *(*$I \in S$ *maximal ideal) are directly indecomposable.*
(2) *The class* $\mathcal{V}_{DI}$ *of directly indecomposable members of* $\mathcal{V}$ *is a universal class.*

(3) *There exists a finite subset* ν_0 *of* ν *and a finite subset* Y *of* $T_{\nu_0}(x)$ *such that, for every* $\mathbf{A} \in \mathcal{V}$ *and* $e \in A$*,* e *is central in* $\mathbf{A}$ *iff the following conditions hold, for all unary* ν_0*-terms* $t, t_1, t_2, \overline{u}, \overline{v} \in Y$*:*
   (a) $q(e, t(e), t(e)) = t(e); \qquad q(e, 1, 0) = e$
   (b) $q(e, q(e, t(e), t_1(e)), t_2(e)) = q(e, t(e), t_2(e))$
   $= q(e, t(e), q(e, t_1(e), t_2(e)))$
   (c) $q(e, f(\overline{u}(e)), f(\overline{v}(e))) = f(q(e, u_1(e), v_1(e)), \ldots, q(e, u_n(e), v_n(e)))$ *for all* $f \in \nu_0$*.*

Observe that, in general, you cannot do any better than this: varieties with factorable congruences where every member has a Boolean product representation are necessarily discriminator varieties [40], while Church algebras, which have factorable congruences [32], need not be discriminator.

4. Semi-Boolean-like algebras

In a generic Church algebra, of course, there is no need for the set of central elements to comprise all of the algebra — not any more than an arbitrary ortholattice needs to be a Boolean algebra, or a ring with unit a Boolean ring. In this section, we define under the name of *Boolean-like algebras* those Church algebras where this actually happens, and we define and investigate under the name of *semi-Boolean-like algebras* those Church algebras that miss the mark, so to speak, by a hair's breadth: in other words, Church algebras where every element satisfies all the conditions characterising central elements except, possibly, $e = q(e, 1, 0)$.

Definition 4.1. We say that a Church algebra $\mathbf{A}$ of type ν is a *semi-Boolean-like algebra* (or a SBlA, for short) if it satisfies the following axioms, for all $e, a, a_1, a_2, \overline{b}, \overline{c} \in A$:

Ax_1. $q(e, a, a) = a$
Ax_2. $q(e, q(e, a_1, a_2), a) = q(e, a_1, a) = q(e, a_1, q(e, a_2, a))$
Ax_3. $q(e, g(\overline{b}), g(\overline{c})) = g(q(e, b_1, c_1), \ldots, q(e, b_n, c_n))$, for every $g \in \nu$.

If every element of $\mathbf{A}$ is central, that is if $\mathbf{A}$ satisfies Ax_1-Ax_3 plus

Ax_4. $q(e, 1, 0) = e$

then we say that $\mathbf{A}$ is a *Boolean-like algebra* (or a BlA, for short).

The elements of a semi-Boolean-like algebra will be called *semi-central.* The same terminology will be used in the more general context of Church algebras to express the fact that an element e of a Church algebra $\mathbf{A}$ satisfies the identities Ax_1-Ax_3 for all $a, a_1, a_2, \overline{b}, \overline{c} \in A$ and all $g \in \nu$.

Definition 4.2. A variety $\mathcal{V}$ of type ν is a *(semi-)Boolean-like variety* if every member of $\mathcal{V}$ is a (semi-)Boolean-like algebra with respect to the same term $q(x, y, z)$ and the same constants $0, 1$.

While Boolean algebras and Boolean rings are easily seen to be BlAs, we bet the reader will be curious to see examples of SBlAs which fail to be BlAs. Observe that, in general, orthomodular lattices, FL_{ew}-algebras, Heyting algebras or rings with unit are not even SBlAs; in fact, orthomodular lattices and FL_{ew}-algebras fail to satisfy Ax_1, rings with unit fail to satisfy Ax_2, and Heyting algebras fail to satisfy Ax_3 for $g(x,y) = x \to y$. The next two algebras, on the other hand, qualify as pertinent examples.

Example 4.3. Let $\mathbf{3} = (\{0,1,2\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(0,a,b) = q(2,a,b)$ for all $a,b \in \{0,1,2\}$. It can be checked that $\mathbf{3}$ is semi-Boolean-like. However, $c(2) = q(2,1,0) = 0 \neq 2$. Moreover, $\mathbf{3}$ is a nonsimple subdirectly irreducible algebra, with the middle congruence corresponding to the partition $\{\{1\},\{0,2\}\}$. Therefore $V(\mathbf{3})$ is not a discriminator variety.

Example 4.4. Let $\mathbf{3}' = (\{0,1,2\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(1,a,b) = q(2,a,b)$ for all $a,b \in \{0,1,2\}$. It can be checked analogously that $\mathbf{3}'$ is semi-Boolean-like and $V(\mathbf{3}')$ is not a discriminator variety.

It is interesting to observe that double-pointed discriminator varieties are always semi-Boolean-like varieties:

Proposition 4.5. *Any double-pointed discriminator variety $\mathcal{V}$ with switching term s is a semi-Boolean-like variety with respect to the term $q(e,x,y) = s(e,0,y,x)$.*

A few elementary properties of SBlAs follow.

Lemma 4.6. *Let* $\mathbf{A}$ *be a* SBlA*. Then for all $e,a,b \in A$:*

(1) $q(e,a,b) = q(c(e),a,b)$;
(2) $c(e) = c(c(e))$;
(3) $q(e',a,b) = q(e,b,a)$;
(4) $c(e)$ *and* e' *are central;*
(5) $a \vee (c(b))' = a \vee b'$.

Proof. (1) $q(c(e),a,b) = q(q(e,1,0),a,b) =_{\mathrm{Ax}_1} q(q(e,1,0), q(e,a,a), q(e,b,b))$ $=_{\mathrm{Ax}_3} q(e, q(1,a,b), q(0,a,b)) = q(e,a,b)$.
(2) From (1), for $a = 1, b = 0$.
(3) $q(e',a,b) = q(q(e,0,1),a,b) =_{\mathrm{Ax}_1} q(q(e,0,1), q(e,a,a), q(e,b,b))$ $=_{\mathrm{Ax}_3} q(e, q(0,a,b), q(1,a,b)) = q(e,b,a)$.
(4) Since $c(c(e)) = c(e)$ and $c(e') = q(e',1,0) = q(e,0,1) = e'$, we get our conclusion.
(5) From item (1) of the present Lemma it follows that $(c(b))' = b'$. □

4.1. Two characterisations. For the class of Boolean-like varieties the following result holds.

Proposition 4.7. *Let $\mathcal{V}$ be a double-pointed variety. Then the following conditions are equivalent:*

(1) *$\mathcal{V}$ is a Boolean-like variety;*
(2) *$\mathcal{V}$ is a discriminator variety such that $|A| = 2$ for every s.i. member $\mathbf{A}$ of $\mathcal{V}$;*
(3) *$\mathcal{V}$ is a* SBlA *variety for which $\wedge$ (resp. $\vee$) is commutative.*
(4) *$\mathcal{V}$ is a* SBlA *variety for which $\wedge$ and $\vee$ are both idempotent.*

Proof. (1) $\Rightarrow$ (2) $\mathcal{V}$ is a discriminator variety with switching term $s(x, y, z, u) \equiv ((x \oplus y) \wedge u) \vee ((x \oplus y)' \wedge z)$ (where $\oplus$ is symmetric difference). If $\mathbf{A} \in \mathcal{V}$ is s.i., then $|A| = 2$ because all its elements are central.
(2) $\Rightarrow$ (3) By Proposition 4.5 $\mathcal{V}$ is a semi-Boolean-like variety. Moreover, $\mathcal{V}$ is Boolean-like since every element in a s.i. member of $\mathcal{V}$ is either 0 or 1, and therefore central. The meet $\wedge$ (resp. join $\vee$) coincides with the commutative meet (resp. join) of the Boolean algebra $\mathrm{Ce}(\mathbf{A})$.
(3) $\Rightarrow$ (4) Since $c(a) = q(a, 1, 0) = q(1, a, 0) = a$, the conclusion follows.
(4) $\Rightarrow$ (1) $a = a \vee a = q(a, 1, a) = q(a, 1, a \wedge a) = q(a, 1, q(a, a, 0))) = q(a, 1, 0) = c(a)$. Then every element is central. ☐

What is the relationships between semi-Boolean-like varieties and Church varieties in which $c(x)$ is central for every x? The next proposition shows that the first concept is at least as strong, and the subsequent example demonstrates that it is actually stronger.

Proposition 4.8. *For a Church variety $\mathcal{V}$ (w.r.t. the term q), the following are equivalent:*

(1) *$\mathcal{V}$ is semi-Boolean-like;*
(2) *$\mathcal{V}$ satisfies the conditions:*
   (i) *for all $a, b, c \in \mathbf{A} \in \mathcal{V}$, $q(a, b, c) = q(c(a), b, c)$*
   (ii) *for all $a \in \mathbf{A} \in \mathcal{V}$, $c(a)$ is central.*
(3) *$\mathcal{V}$ satisfies the condition $2(i)$ and the following universal formula holds in every subdirectly irreducible member of $\mathcal{V}$:*

$$c(0) \approx 0 \;\bar{\wedge}\; c(1) \approx 1 \;\bar{\wedge}\; \forall x (c(x) \approx 0 \,\underline{\vee}\, c(x) \approx 1)$$

Proof. (1) implies (2) by Lemma 4.6. (2) clearly implies (3). For the remaining implication, in every s.i. $\mathbf{A} \in \mathcal{V}$, $c(a) \in \{0, 1\}$ is central for all a and since $q(a, b, c) = q(c(a), b, c)$ for all $b, c \in \mathbf{A}$, we conclude that a is semi-central. ☐

Example 4.9. *Let $\mathbf{A} = (\{0, 1, 2\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(2, 0, 0) = 1$ and $q(2, a, b) = 0$ if either $a \neq 0$ or $b \neq 0$. It can be seen that $c(a)$ is central for every $a \in A$, but $\mathbf{A}$ is not a* SBlA. *In fact, $q(c(2), 0, 0) = 0 \neq 1 = q(2, 0, 0)$.*

Corollary 4.10. *Let $\mathcal{V}$ be a Church variety (w.r.t. the term q) such that $c(a)$ is central for all $a \in \mathbf{A} \in \mathcal{V}$. Then $\mathcal{V}$ is a semi-Boolean-like variety w.r.t. the term $q_1(x, y, z) = q(c(x), y, z)$.*

4.2. A purely algebraic characterisation. One of the deepest results in the theory of discriminator varieties gives the following purely algebraic characterisation thereof: a variety is a discriminator variety iff it is congruence permutable, semisimple, and has equationally definable principal congruences (EDPC) [10]. One may wonder if a suitable analogue of this theorem holds in the case under investigation. Before answering the question in the affirmative, however, we need to enrich our toolbox with suitable adaptations of the preceding concepts. We start with a common generalisation of the notions of congruence permutability and τ-permutability [12], first introduced in [25].

Definition 4.11. Let $\mathcal{V}$ be a variety of type ν, and let t, s be at most unary terms of the same type. $\mathcal{V}$ is (t, s)*-permutable* iff for every $\mathbf{A} \in \mathcal{V}$, every $\theta, \psi \in \mathrm{Con}(\mathbf{A})$ and every $a, b \in A$, $(t(a), s(b)) \in \theta \circ \psi$ iff $(t(a), s(b)) \in \psi \circ \theta$. In case $t = s$, we call $\mathcal{V}$ t*-permutable.*

If we let $t = s$ in the preceding definition be the identity, we get the standard notion of congruence permutability, while if we let a, b be the same element we recover Blok's and Raftery's concept of τ -permutability, at least for translations consisting of a single equation. A Maltsev-type characterisation of (t, s)-permutability is readily available [29]:

Theorem 4.12. *A variety $\mathcal{V}$ is (t, s)-permutable iff there exists a ternary term p such that*

$$\mathcal{V} \models p(x, s(y), y) \approx t(x) \text{ and } \mathcal{V} \models p(x, t(x), y) \approx s(y).$$

Definition 4.13. Let $\mathbf{A}$ be Church algebra. $\theta \in \mathrm{Con}(\mathbf{A})$ is called a *B-congruence* if $\mathbf{A}/\theta$ is a BlA.

We denote by $\theta_B(a, b)$ the smallest B-congruence collapsing a and b, and by $\mathrm{Con}_B(\mathbf{A})$ the complete lattice of B-congruences of $\mathbf{A}$. Moreover, we denote by $\Delta_B^{\mathbf{A}}$ the least B-congruence $\bigwedge\{\theta : \mathbf{A}/\theta \text{ is a BlA}\}$.

Lemma 4.14. *Let $\mathbf{A}$ be a Church algebra. Then we have:*

(1) $\Delta_B^{\mathbf{A}} \supseteq \ker(c^{\mathbf{A}})$*;*
(2) *The following lattices are isomorphic: (i)* $\mathrm{Con}_B(\mathbf{A})$*; (ii) The interval sublattice* $[\Delta_B^{\mathbf{A}}, A \times A]$ *of* $\mathrm{Con}(\mathbf{A})$*; (iii) The congruence lattice* $\mathrm{Con}(\mathbf{A}/\Delta_B^{\mathbf{A}})$ *of the BlA* $\mathbf{A}/\Delta_B^{\mathbf{A}}$.

Proof. (1) $\Delta_B^{\mathbf{A}} \supseteq \ker(c^{\mathbf{A}})$ since, if f is a homomorphism from $\mathbf{A}$ into a BlA $\mathbf{B}$, and $c(a) = c(b)$, then $f(a) = c(f(a)) = f(c(a)) = f(c(b)) = c(f(b)) = f(b)$. (2) follows because $\Delta_B^{\mathbf{A}}$ is the least B-congruence. □

We need another ingredient: a notion of a variety whose subdirectly irreducibles have no nontrivial B-congruence.

Definition 4.15. A Church variety $\mathcal{V}$ is *B-semisimple* iff in every s.i. member $\mathbf{A}$ of $\mathcal{V}$ the only B-congruences are $\Delta_B^{\mathbf{A}}$ and $\nabla^{\mathbf{A}}$.

Theorem 4.16. *Every semi-Boolean-like variety $\mathcal{V}$ is B-semisimple and c-permutable.*

Proof. Let $\mathbf{A} \in \mathcal{V}$. By Lemma 4.14 $\mathbf{A}/\Delta_B^{\mathbf{A}}$ is a Boolean-like algebra. Boolean-like varieties are congruence permutable, as witnessed by the term $(z \wedge (y' \vee x)) \vee (x \wedge (y' \vee z))$, because they are discriminator varieties. It is easy to show that the term

$$(c(z) \wedge (c(y)' \vee c(x))) \vee (c(x) \wedge (c(y)' \vee c(z)))$$

is a Maltsev term witnessing c-permutability for $\mathbf{A}$ according to Theorem 4.12.

We now show that $\mathcal{V}$ is B-semisimple. Let $\mathbf{A}$ be a s.i. (hence directly indecomposable) member of $\mathcal{V}$, and let a, b be distinct members of A. By Proposition 4.8 the elements 0 and 1 are the sole possible values for $c(a)$ and $c(b)$. Then, by Lemma 4.14(1) $\theta_B(a,b) = \Delta_B^{\mathbf{A}}$ if $c(a) = c(b)$, while $\theta_B(a,b) = \nabla^{\mathbf{A}}$ otherwise. □

To prove a converse to the preceding theorem, it is expedient to proceed as in [10] and define an analogue of the concept of quaternary deduction (QD) term, relativised to B-congruences.

Definition 4.17. Let $\mathcal{V}$ be a Church variety of type ν, and let t be a unary term of the same type. A quaternary term p is called a *t-quaternary deduction (t-QD) term* for $\mathbf{A} \in \mathcal{V}$ iff for all $a, b, d, f \in A$,

$$p(a,b,d,f) = \begin{cases} d \text{ if } t(a) = t(b) \\ f \text{ if } d \equiv_{\theta_B(a,b)} f \text{ and } t(a) \neq t(b) \\ \text{arbitrary otherwise} \end{cases}$$

p is called a t-QD term for $\mathcal{V}$ iff it is a t-QD term for any $\mathbf{A} \in \mathcal{V}$.

Lemma 4.18. *Let $\mathcal{V}$ be a Church variety. Then, for every $\mathbf{A} \in \mathcal{V}$, we have:*

(1) *The join semilattice $\mathrm{Cp}_B(\mathbf{A})$ of compact B-congruences is dually relatively pseudocomplemented;*
(2) *The join semilattice $\mathrm{Cp}_B(\mathbf{A})$ of compact B-congruences, with $\Delta^{\mathbf{A}}$ adjoined at the bottom, is dually relatively pseudocomplemented.*

Proof. (1) follows from Lemma 4.14.2 and the fact that the BlA $\mathbf{A}/\Delta_B^{\mathbf{A}}$ has EDPC. As regards (2), let us denote by $*$ the dual relative pseudocomplement operation in the join semilattice of compact B-congruences of $\mathbf{A}$. Let us introduce on $\mathrm{Con}_B(\mathbf{A}) \cup \{\Delta^{\mathbf{A}}\}$ a new binary operation $\widehat{*}$ by means of a case-splitting definition:

$$\theta \widehat{*} \varphi = \begin{cases} \theta * \varphi \text{ if } \varphi \nleq \theta \text{ and } \varphi, \theta \in \mathrm{Con}_B(\mathbf{A}); \\ \Delta \text{ if } \varphi \leq \theta; \\ \varphi \text{ if } \theta = \Delta. \end{cases}$$

It can be checked that this operation is well-defined and that for any $\theta, \varphi, \psi \in \mathrm{Con}_B(\mathbf{A}) \cup \{\Delta^{\mathbf{A}}\}$, $\varphi \leq \theta \vee \psi$ iff $\theta \widehat{*} \varphi \leq \psi$. □

Theorem 4.19. *If a Church variety $\mathcal{V}$ is c-permutable, then $\mathcal{V}$ has a c-QD term p.*

Proof. Let $\mathbf{F}$ be the 4-generated $\mathcal{V}$-free algebra over free generators x, y, z, w. By Lemma 4.18, $\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)$ and $(\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)^2) = (\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)) \mathbin{\widehat{*}} \theta_B(z,w)$ both exist in the join semilattice of compact B-congruences of $\mathbf{F}$ with $\Delta^{\mathbf{A}}$ adjoined at the bottom and by the theory of dually relatively pseudocomplemented semilattices $(z,w) \in (\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)) \vee (\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)^2)$. Since $\mathbf{F}$ is c-permutable and the preceding congruence is a B-congruence, we have that $(z,w) \in (\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)) \circ (\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)^2)$. Therefore there is a quaternary term p such that

$$w(\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w))p(x,y,z,w)(\theta_B(x,y) \mathbin{\widehat{*}} \theta_B(z,w)^2)z.$$

Now, evaluate all this over an arbitrary algebra $\mathbf{A} \in \mathcal{V}$. If $c(a) = c(b)$, then $(\theta_B(a,b) \mathbin{\widehat{*}} \theta_B(d,f)^2) = (\Delta_B^{\mathbf{A}} \mathbin{\widehat{*}} \theta_B(d,f)^2) = \theta_B(d,f) \mathbin{\widehat{*}} \theta_B(d,f) = \Delta^{\mathbf{A}}$, whence $\mathbf{A}$ satisfies $p(a,b,d,f) = d$. If $(d,f) \in \theta_B(a,b)$, then $\theta_B(a,b) \mathbin{\widehat{*}} \theta_B(d,f) = \Delta^{\mathbf{A}}$ and so $p(a,b,d,f) = f$. □

Theorem 4.20. *For $\mathcal{V}$ a Church variety the following are equivalent:*

(1) *$\mathcal{V}$ is a semi-Boolean-like variety;*
(2) *$\mathcal{V}$ is c-permutable and is B-semisimple.*

Proof. (1) implies (2) because of Theorem 4.16. For the converse direction, let $\mathcal{V}$ have the indicated properties. By Theorem 4.19 $\mathcal{V}$ has a c-QD term p. Now, let $\mathbf{A}$ be a s.i. member of $\mathcal{V}$. Then, if $c(a) = c(b)$, $p(a,b,d,f) = d$. If $c(a) \neq c(b)$, then $\theta_B(a,b) = \nabla^{\mathbf{A}}$ by Boolean semisimplicity and thus $d \equiv_{\theta_B(a,b)} f$, whence $p(a,b,d,f) = f$.

Let $q_1(x,y,z) = p(x,0,z,y)$ and $c_1(x) = q_1(x,1,0)$. From Proposition 4.8 it follows that $\mathcal{V}$ is a semi-Boolean-like variety w.r.t the term q_1, if we observe the following two facts:

(1) $\mathcal{V} \models q_1(x,y,z) \approx q_1(c_1(x),y,z)$ as this identity holds in all s.i. members of $\mathcal{V}$.

(2) For all $a \in A$, where $\mathbf{A}$ is s.i., $c_1(a) = p(a,0,0,1)$ is equal either to 0 or to 1. □

4.3. The pure variety $\mathcal{SB}l\mathcal{A}_0$. We now turn our attention to the variety $\mathcal{SB}l\mathcal{A}_0$, consisting of all the pure term reducts of semi-Boolean-like algebras. The variety $\mathcal{SB}l\mathcal{A}_0$ is axiomatised by the two identities defining Church algebras and by Ax_1-Ax_3 above. As a consequence of Proposition 4.7, the subvariety $\mathcal{B}l\mathcal{A}_0$ of $\mathcal{SB}l\mathcal{A}_0$ consisting of pure Boolean-like algebras is term equivalent to the variety $\mathcal{BA}$ of Boolean algebras.

If $\mathbf{A}$ is a member of $\mathcal{SB}l\mathcal{A}_0$, we denote by $c[\mathbf{A}]$ the partial subalgebra of $\mathbf{A}$ with universe $\{c(a) : a \in A\}$.

Proposition 4.21. *Let $\mathbf{A}$ be a member of $\mathcal{SB}l\mathcal{A}_0$. Then*

(1) *$c^{\mathbf{A}}$ is an idempotent endomorphism of $\mathbf{A}$;*

(2) $c[\mathbf{A}]$ *is a Boolean-like algebra.*

Proof. By Proposition 4.8 $c(a)$ is central for every $a \in A$, so that $c(c(a)) = c(a)$. Then $c[\mathbf{A}]$ is a Boolean-like algebra if we show that $c^{\mathbf{A}} : A \to A$ is an endomorphism:

$$c(q(a,b,d)) = q(q(a,b,d),1,0) =_{Ax3} q(a,c(b),c(d)) =_{\mathrm{L.4.6.1}} q(c(a),c(b),c(d)).$$

□

The algebras **3** and **3′** of Examples 4.3 and 4.4 are not only semi-Boolean-like algebras which fail to be Boolean-like, but they also jointly generate the pure variety $\mathcal{SB}l\mathcal{A}_0$.

Theorem 4.22. $V(\{\mathbf{3},\mathbf{3}'\}) = \mathcal{SB}l\mathcal{A}_0$.

Proof. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathcal{SB}l\mathcal{A}_0$. First, observe that, in the light of Proposition 4.21.2, the kernel $\ker(c)$ of the term operation $c^{\mathbf{A}}$ determines a retract $\mathbf{A}/\ker(c)$ which is isomorphic to the 2-element Boolean-like algebra by Proposition 4.8.3. Let now $\mathbf{A}$ be s.i. but not simple; then there is an element $a \notin \{0,1\}$ such that $c(a) \in \{0,1\}$. We suppose ex absurdo that there are two distinct such elements a, b, and go through a case-splitting argument. If $c(a) = c(b) = 1$, consider the equivalence relations θ_1 and θ_2, which coincide with the diagonal except that $1\theta_1 a$ and $1\theta_2 b$, respectively. Taking into account that, $q(a,y,z) =_{\mathrm{L.4.6.1}} q(c(a),y,z) = q(1,y,z) = y$ (for all $y, z \in A$) and similarly for b, an elementary check will ensure that both θ_1 and θ_2 are congruences on $\mathbf{A}$, such that $\theta_1 \wedge \theta_2 = \Delta$, against the hypothesis. In the other three possible cases we argue analogously, replacing 1 by 0 in the definition of congruences when necessary. Thus, $a = b$, that is $\mathbf{A}$ is either the algebra **3**, or the algebra **3′**. □

From the point of view of its congruence properties, $\mathcal{SB}l\mathcal{A}_0$ is anything but well-behaved. In fact:

Theorem 4.23. $\mathcal{SB}l\mathcal{A}_0$ *has no congruence identities.*

Proof. Consider the class $\mathcal{K}$ of all finite algebras in $\mathcal{SB}l\mathcal{A}_0$ which satisfy the condition $x \neq 1 \Rightarrow q(x,y,z) \approx z$. This class is nonempty. More than that, for every positive integer n there is a member of $\mathcal{K}$ with n elements: given an arbitrary n-element set A, simply construct the table for q according to the condition above and check that this does not conflict with the axioms of $\mathcal{SB}l\mathcal{A}_0$. Now let $\mathbf{A} \in \mathcal{K}$, and let θ be the equivalence on A corresponding to the partition $\{\{1\}, A\backslash\{1\}\}$. It can be checked that every subpartition of θ corresponds to a congruence, which means that the lattice of congruences of $\mathbf{A}$ coincides with the full lattice of partitions of $A\backslash\{1\}$. This is enough to yield the desired result. □

$\mathcal{SB}l\mathcal{A}_0$, having no congruence identities in virtue of Theorem 4.23, fails in particular to be congruence distributive or even congruence modular; it also lacks properties such as congruence permutability (or even congruence n-permutability for any n) which are not expressible in the language of lattices but are known to imply the existence of congruence identities of some sort or another. On the other hand, 0-permutability or 1-permutability — which respectively coincide, as we have seen, with 0-subtractivity and 1-subtractivity — are not out of the question in principle. The next example, however, will mercilessly dash our hopes.

Example 4.24. Let $\mathbf{A} = (\{0,1,2,3\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(1,a,b) = q(2,a,b) = q(3,a,b)$ for all $a, b \in \{0,1,2,3\}$. It can be checked that $\mathbf{A}$ is a SBlA. However, the congruences $\theta = \{\{0\},\{1,2\},\{3\}\}$ and $\psi = \{\{0\},\{2,3\},\{1\}\}$ fail to permute at 1, for $(2,1) \in \theta$ and $(2,3) \in \psi$, yet $1/\psi \cap 3/\theta = \emptyset$. Therefore, $\mathbf{A}$ fails to be 1-subtractive.

On the other hand, $\mathcal{SB}l\mathcal{A}_0$ — and, more generally, every semi-Boolean-like variety — is *quasi-subtractive* in the sense of Definition 2.6.

Lemma 4.25. *Every semi-Boolean-like variety* $\mathcal{V}$ *is* 1*-quasi-subtractive with witness terms* $x \to y = y \vee x'$ *and* $\Box x = c(x)$.

Proof. We have to make sure that the equations Q1-Q4 in Definition 2.6 hold true. We check them one by one. So, for the remainder of this proof let $\mathbf{A}$ be a generic SBlA and let $a, b \in A$.

Q1. $\Box a \to a = a \vee (c(a))' =_{\text{L.4.6.5}} a \vee a' = q(a,1,q(a,0,1)) =_{\text{Ax}_2} q(a,1,1) =_{\text{Ax}_1} 1$.

Q2. $1 \to a = a \vee 1' = q(a,1,q(1,0,1)) = q(a,1,0) = c(a) = \Box a$.

Q3. $\Box(a \to b) = c(q(b,1,a')) =_{\text{P.4.21(i)}} q(c(b),c(1),c(a')) =_{\text{L.4.6}} q(b,1,a') = a \to b$.

Q4. $\Box(a \to b) \to (\Box a \to \Box b) =_{\text{Q3}} \Box\Box(a \to b) \to (\Box a \to \Box b) =_{\text{P.4.21(i)}} \Box(\Box a \to \Box b) \to (\Box a \to \Box b) =_{\text{Q1}} 1$.

□

What about regularity properties for $\mathcal{SB}l\mathcal{A}_0$? Point regularity implies congruence modularity [21] and can therefore be ruled out. Clearly, the stronger property of τ-regularity, for $\tau = \{c(x) \approx 1\}$, fails to hold as well. Nonetheless, a weaker but still somewhat pleasing result turns out to be true:

Proposition 4.26. $\mathcal{SB}l\mathcal{A}_0$ *is weakly* $\{c(x) \approx 1\}$*-regular.*

Proof. It suffices to prove that the $\{c(x) \approx 1\}$-assertional logic of $\mathcal{SB}l\mathcal{A}_0$ coincides with the 1-assertional logic of the pure Boolean-like variety $\mathcal{B}l\mathcal{A}_0$, which is, by Proposition 4.7, the 1-assertional logic of a 1-regular variety and thus is strongly and finitely algebraisable by results in [17]. Therefore, we want to show that, given a set of terms Γ and a term t,

$$\{c(s) \approx 1 : s \in \Gamma\} \models_{\mathcal{SB}l\mathcal{A}_0} c(t) \approx 1 \text{ iff } \{s \approx 1 : s \in \Gamma\} \models_{\mathcal{B}l\mathcal{A}_0} t \approx 1.$$

Left to right. Let $\mathbf{A} \in \mathcal{B}l\mathcal{A}_0$, $\overline{a} \in A$ and $s^{\mathbf{A}}(\overline{a}) = 1$ for all $s \in \Gamma$. Then $c^{\mathbf{A}}\left(s^{\mathbf{A}}(\overline{a})\right) = 1$ for all $s \in \Gamma$, and since Boolean-like algebras are in particular semi-Boolean-like, $c^{\mathbf{A}}\left(t^{\mathbf{A}}(\overline{a})\right) = 1$. However, as $\mathbf{A} \in \mathcal{B}l\mathcal{A}_0$, this implies $t^{\mathbf{A}}(\overline{a}) = 1$.

Right to left. Let $\mathbf{A} \in \mathcal{SB}l\mathcal{A}_0$, $\overline{a} \in A$ and $c^{\mathbf{A}}\left(s^{\mathbf{A}}(\overline{a})\right) = 1$ for all $s \in \Gamma$. Now, by Proposition 4.21, $s^{\mathbf{A}}\left(c^{\mathbf{A}}(\overline{a})\right) = c^{\mathbf{A}}\left(s^{\mathbf{A}}(\overline{a})\right) \in c[\mathbf{A}] \in \mathcal{B}l\mathcal{A}_0$, whence $t^{\mathbf{A}}\left(c^{\mathbf{A}}(\overline{a})\right) = 1$ and, going backwards, $c^{\mathbf{A}}\left(t^{\mathbf{A}}(\overline{a})\right) = 1$. □

Let $\mathbf{A}$ be a member of $\mathcal{SB}l\mathcal{A}_0$. As a consequence of Proposition 4.21(i), $\ker(c^{\mathbf{A}})$ is the least B-congruence of $\mathbf{A}$ (see Definition 4.13).

It follows therefore from Theorem 2.8, and from Lemma 4.14, Lemma 4.25 and Proposition 4.26 that:

Corollary 4.27. *In every member $\mathbf{A}$ of $\mathcal{SB}l\mathcal{A}_0$, the following lattices are isomorphic:*

(1) *The lattice of $\mathcal{SB}l\mathcal{A}_0$-open filters of $\mathbf{A}$;*
(2) *The lattice of B-congruences of $\mathbf{A}$;*
(3) *The congruence lattice of the $\mathcal{B}l\mathcal{A}_0$ $c[\mathbf{A}]$.*

This result is of limited usefulness unless we are in a position to characterise $\mathcal{SB}l\mathcal{A}_0$-open filters in an effective way. The next Proposition does the job. First, observe that the variety of pure Boolean-like algebras is term equivalent to the variety $\mathcal{BA}$ of Boolean algebras (cp. [18]).

Proposition 4.28. *Let $\mathbf{A} \in \mathcal{SB}l\mathcal{A}_0$, and let $F \subseteq A$. Then t.f.a.e.:*

(1) *$F = (c^{\mathbf{A}})^{-1}(H)$ for some Boolean filter H of the $\mathcal{B}l\mathcal{A}_0$ $c[\mathbf{A}]$;*
(2) *F is a $\mathcal{SB}l\mathcal{A}_0$-open filter of $\mathbf{A}$;*
(3) *F satisfies the conditions (F1) $1 \in F$; (F2) $a, b \in F \Rightarrow a \wedge b \in F$; (F3) $a \in F, b \in A \Rightarrow a \vee b, b \vee a \in F$; (F4) $c(a) \in F \Rightarrow a \in F$;*
(4) *F satisfies F1, F4, and (G1)*

$$a, b \in F, d \in A \Rightarrow q(a, b, d) \in F.$$

Proof. (1) ⇒ (2) We must now show that: i) F is closed w.r.t. all open filter terms; ii) F is close d w.r.t. the two-way necessitation rule. As regards ii), $c(a) \in F$ iff $c(c(a)) = c(a) \in H$ iff $a \in F$. As regards i), let $p(x_1, \ldots, x_n, \overline{y})$ be an open filter term in the variables $x_1, \ldots, x_n$, and let $a_1, \ldots, a_n \in F$. The Boolean filter H determines a congruence θ_H on $c[\mathbf{A}]$. Since $c(a_i) \in H$ for every i, $c(a_i)/\theta_H = 1/\theta_H$, and then, as p is an open filter term, $c(p(\overline{a}, \overline{b}))/\theta_H = 1/\theta_H$, whence $c(p(\overline{a}, \overline{b}))/\theta_H \in H$ and so $p(\overline{a}, \overline{b}) \in F$.

(2) ⇒ (3) It suffices to check that (i) $x \wedge y$ is a $\mathcal{SB}l\mathcal{A}_0$-open filter term in x, y; (ii) $x \vee y$ is a $\mathcal{SB}l\mathcal{A}_0$-open filter term in x; (iii) $x \vee y$ is a $\mathcal{SB}l\mathcal{A}_0$-open filter term in y. These conditions are readily seen to hold by Proposition 4.21(i) and by $\mathbf{A} \in \mathcal{SB}l\mathcal{A}_0$. The remaining conditions follow from the definition of open filter.

(3) ⇒ (4) For a start, we claim that $a, b \in F, d \in A \Rightarrow q(a, b, c(d)) \in F$. In fact, let $a, b \in F$. Using F2 and F3, we get that

$$(a \vee d) \wedge (a' \vee b) = q(q(a, 1, d), q(a', 1, b), 0) \in F.$$

However, by Lemma 4.6.3, $q(a', 1, b) = q(a, b, 1)$. So

$$q(q(a, 1, d), q(a, b, 1), 0) = q(q(a, 1, d), q(a, b, 1), q(a, 0, 0)) \in F.$$

It follows that $q(a, q(1, b, 0), q(d, 1, 0)) \in F$, whence $q(a, b, c(d)) \in F$. Having established our claim, we proceed to prove our main conclusion. If $a, b \in F$, by F1 and F2 we have $c(a), c(b) \in F$. Our claim then implies that $q(c(a), c(b), c(d)) = c(q(a, b, d)) \in F$, and we have our conclusion by F4.

(4 ⇒ 1) By F1, F4 and G1 (for $b = 1, d = 0$) the set F is closed w.r.t. the two-way necessitation rule. Let $H = \{c(a) : a \in F\}$. Then $F \supseteq H$ and $F = (c^{\mathbf{A}})^{-1}(H)$. To show that H is a Boolean filter, we use two times G1 for $d = 0$ and for $b = 1$. □

5. Double-pointed discriminator varieties

One of the most interesting applications of the concepts defined hereto arises when studying discriminator varieties in the double-pointed case. These notions appear from the very beginning as intimately related, and the aim of this section is making the nature of this relationship as clear as possible.

5.1. Idempotent semi-Boolean-like algebras. Whereas idempotency of both join and meet is enough to enforce a Boolean-like behaviour ina SBlA, idempotency of join alone (or meet alone) is not: the algebras **3** and **3′** of Examples 4.3 and 4.4 are respective counterexamples. Therefore, we may look for some middle ground between these concepts by adding either one of the idempotency identities.

Definition 5.1. A SBlA is *meet-idempotent* if it satisfies the following identity:

(Ax_5) $x \wedge x \approx x$.

Henceforth, we will use the abbreviation *idempotent* in place of the more cumbersome *meet-idempotent*. The next theorem characterises idempotent semi-Boolean-like varieties in the context of semi-Boolean-like varieties.

Theorem 5.2. *Let $\mathcal{V}$ be a semi-Boolean-like variety. Then the following conditions are equivalent:*

(1) *$\mathcal{V}$ is idempotent;*
(2) *$\mathcal{V}$ is a unary discriminator variety*[3] *w.r.t. c;*

[3] Recall that the unary discriminator on a double-pointed set A (with constants $0, 1$) is a unary function u on A such that $u(0) = 0$ and $u(a) = 1$ for $a \neq 0$. A variety $\mathcal{V}$ of type ν is a unary discriminator variety iff there is a unary term of type ν realising the unary discriminator in all s.i. members of $\mathcal{V}$.

(3) *The identity* $x \vee x \approx c(x)$ *holds in* $\mathcal{V}$*;*
(4) $\mathcal{V}$ *is* 0*-subtractive with witness term* $x - y$*.*

Proof. (1) $\Rightarrow$ (2) Let $\mathbf{A} \in \mathcal{V}$ be subdirectly irreducible and let $a \neq 0 \in A$. Assume, by contraposition, that $c(a) = 0$. Then we have: $a = a \wedge a = q(a, a, 0) =_{L.4.6.1} q(c(a), a, 0) = q(0, a, 0) = 0$. This contradicts the assumption $a \neq 0$.

(2) $\Rightarrow$ (3) Let $\mathbf{A} \in \mathcal{V}$ be s.i. and $a \in A$. If $a \neq 0$ then $a \vee a = q(a, 1, a) =_{L.4.6.1} q(c(a), 1, a) = q(1, 1, a) = 1 = c(a)$. If $a = 0$ then $0 \vee 0 = q(0, 1, 0) = 0 = c(0)$.

(3) $\Rightarrow$ (4) Let $\mathbf{A} \in \mathcal{V}$ be s.i. and $a \in A$. We distinguish two cases. If $c(a) = 1$ then $a - a = q(a', a, 0) =_{L.4.6.3} q(a, 0, a) =_{L.4.6.1} q(c(a), 0, a) = q(1, 0, a) = 0$. If $c(a) = 0$ the conclusion easily follows if we show $a = 0$. In fact, $0 = c(a) = a \vee a = q(a, 1, a) =_{L.4.6.1} q(c(a), 1, a) = q(0, 1, a) = a$.

(4) $\Rightarrow$ (1) $a \wedge a = q(a, a, 0) = q(a, a, a - a) = q(a, a, a' \wedge a) = q(a, a, q(a, 0, a)) = q(a, a, a) = a$. □

The pure idempotent semi-Boolean-like variety $\mathcal{ISB}l\mathcal{A}_0$, consisting of all the pure term reducts of idempotent SBlAs, is of course axiomatised by the two identities characterising Church algebras and by Ax_1-Ax_3 plus Ax_5. The proof of Theorem 4.22 immediately implies that:

Theorem 5.3. $V(\mathbf{3}') = \mathcal{ISB}l\mathcal{A}_0$.

5.2. Some characterisations of double-pointed discriminator varieties. If a double-pointed variety $\mathcal{V}$ is a discriminator variety with switching term s, we already know from Proposition 4.5 that $\mathcal{V}$ is a semi-Boolean-like variety with respect to the term $q(e, x, y) = s(e, 0, y, x)$. It is moreover immediate to check that:

Proposition 5.4. *If* $\mathcal{V}$ *is a discriminator variety with switching term* s*, then*

- $\mathcal{V}$ *is a variety of idempotent* SBlA*s.*
- $\mathbf{A} \in \mathcal{V}$ *is simple iff* $\mathbf{A}$ *satisfies* $\forall x(c(x) \approx 0 \veebar c(x) \approx 1)$*, where* $c(x) = s(x, 0, 0, 1)$*.*

A series of different characterisations of double-pointed discriminator varieties follows next.

Lemma 5.5. *Let* $\mathcal{V}$ *be a double-pointed variety. Then,* $\mathcal{V}$ *is discriminator if, and only if, the following conditions hold:*

(1) $\mathcal{V}$ *is* 0*-regular with a witness binary term* $d(x, y)$*;*
(2) $\mathcal{V}$ *has a unary discriminator* $u(x)$*;*
(3) *There is a binary term* $x + y$ *such that* $0 + y \approx y + 0 \approx y$ *holds in* $\mathcal{V}$*;*
(4) *There is a binary term* $x \cdot y$ *such that* $0 \cdot y \approx 0$ *and* $1 \cdot y \approx y$ *hold in* $\mathcal{V}$*;*
(5) *There is a unary term* x' *such that* $0' \approx 1$ *and* $1' \approx 0$ *hold in* $\mathcal{V}$*.*

Proof. ($\Leftarrow$) The term $t(x,y,z) = (u(d(x,y)) \cdot x) + ((u(d(x,y)))' \cdot z)$ is a ternary discriminator term.

($\Rightarrow$) if $\mathcal{V}$ is discriminator with switching term s, define

(1) $d(x,y) = s(x,y,0,1)$
(2) $u(x) = s(x,0,0,1)$
(3) $x + y = s(x,0,y,x)$
(4) $x \cdot y = s(x,0,0,y)$
(5) $x' = s(x,0,1,0)$.

□

Theorem 5.6. *Let $\mathcal{V}$ be a double-pointed variety. Then $\mathcal{V}$ is discriminator if, and only if, $\mathcal{V}$ is* 0*-regular and idempotent semi-Boolean-like.*

Proof. ($\Rightarrow$) By Lemma 5.5 and Proposition 5.4.

($\Leftarrow$) Let $q(x,y,z)$ be the Church term for the variety $\mathcal{V}$. Since $\mathcal{V}$ is 0-regular there exist binary terms $d_1(x,y), \dots, d_n(x,y)$ such that $\mathcal{V}$ satisfies $d_i(x,x) \approx 0$ $(i = 1, \dots, n)$ and the following implication:

$$d_1(x,y) \approx 0, \dots, d_n(x,y) \approx 0 \;\Rightarrow\; x \approx y.$$

Since $\mathcal{V}$ is semi-Boolean-like the term operation $x \vee y = q(x,1,y)$ is associative. We define:

(1) $d(x,y) = d_1(x,y) \vee d_2(x,y) \vee \dots \vee d_n(x,y)$
(2) $u(x) = q(x,1,0)$
(3) $x + y = q(x,x,y)$
(4) $x \cdot y = q(x,y,0)$
(5) $x' = q(x,0,1)$.

We now show that the above term operations satisfy items (1)–(5) of Lemma 5.5. We confine ourselves to the nontrivial items. First, $x + 0 = q(x,x,0) = x \wedge x = x$. Moreover, by Theorem 5.2 $\mathcal{V}$ is a unary discriminator variety w.r.t. $u(x)$. We now show that $\mathcal{V}$ is 0-regular with witness term $d(x,y)$. Let $\mathbf{A} \in \mathcal{V}$, $a \neq b \in A$ and i be the least index such that $d_i(a,b) \neq 0$. We distinguish two cases.

(i) $\mathbf{A}$ is subdirectly irreducible. Any element $x \vee y$ is different from 0 whenever $x \neq 0$: $q(x,1,y) = q(c(x),1,y) = q(1,1,y) = 1$. Then from $d_i(a,b) \neq 0$ it follows that $d(a,b) \neq 0$.

(ii) $\mathbf{A}$ is not subdirectly irreducible. Then $\mathbf{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras $\mathbf{A}_i \in \mathcal{V}$ $(i \in I)$. Then $d(a,b) = 0$ in $\mathbf{A}$ iff $d(a_i,b_i) = 0$ in each member $\mathbf{A}_i$ of the subdirect product iff $a_i = b_i$, for all $i \in I$, iff $a = b$ in $\mathbf{A}$. □

Corollary 5.7. *A double-pointed variety of type ν is a discriminator variety if, and only if, for suitable terms $q(x,y,z)$, $w(x,y,z)$ and $d(x,y)$, it satisfies the following identities:*

- $x \approx q(1,x,y) \approx q(0,y,x) \approx q(y,x,x) \approx q(x,x,0)$;
- $q(x,q(x,y_1,y_2),z) \approx q(x,y_1,z) \approx q(x,y_1,q(x,y_2,z))$;

- $q(x, g(\overline{y}), g(\overline{z})) \approx g(q(x, y_1, z_1), \ldots, q(x, y_n, z_n))$, *for every* $g \in \nu$*;*
- $d(x, x) \approx 0$*;*
- $x \approx w(x, y, 0) \approx w(y, x, d(y, x))$*.*

Proof. Immediate from the preceding theorem, taking into account Fichtner's Maltsev-type characterisation of point regular varieties, see [4]. □

Theorem 5.8. *Let $\mathcal{V}$ be a double-pointed variety. Then $\mathcal{V}$ is a discriminator variety if, and only if, $\mathcal{V}$ is an idempotent semi-Boolean-like variety and there exists a binary term $u(x, y)$ such that the identity $u(x, x) \approx 0$ holds in $\mathcal{V}$ and the implication $x \neq y \Rightarrow u(x, y) \approx x$ holds in every subdirectly irreducible member of $\mathcal{V}$.*

Proof. ($\Rightarrow$) Define $q(x, y, z) = s(x, 0, z, y)$ and $u(x, y) = t(x, y, 0)$, where s and t are respectively the switching term and the ternary discriminator term for $\mathcal{V}$.

($\Leftarrow$) By Theorem 5.6 it is sufficient to define $d(x, y)$ as follows

$$d(x, y) = u(x, y) \vee u(y, x).$$

Assume that $\mathbf{A} \in \mathcal{V}$ is subdirectly irreducible and $a \neq b \in A$. Then $d(a, b) = u(a, b) \vee u(b, a) = a \vee b$. Since either a or b is different from 0, then we have that $d(a, b) = a \vee b \neq 0$. If $\mathbf{A}$ is not subdirectly irreducible, then we argue as in the proof of Theorem 5.6. □

Acknowledgment. We acknowledge hereby the contribution by Matthew Spinks, whose suggestions helped us to correct numerous inaccuracies in a first draft of this paper. We also thank Peter Jipsen, Anatolij Dvurečenskij, and two anonymous referees for their comments.

References

[1] Aglianò P., Ursini A.: On subtractive varieties II: General properties. Algebra Universalis **36**, 222–259 (1996)
[2] Aglianò P., Ursini A.: On subtractive varieties III: From ideals to congruences. Algebra Universalis **37**, 296–333 (1997)
[3] Aglianò P., Ursini A.: On subtractive varieties IV: Definability of principal ideals. Algebra Universalis **38**, 355–389 (1997)
[4] Barbour G.D., Raftery J.G.: On the degrees of permutability of subregular varieties. Czechoslovak Mathematical Journal **47**, 317–325 (1997).
[5] Barendregt H. P.: The Lambda Calculus: Its Syntax and Semantics. North Holland, Amsterdam (1984)
[6] Bergman G.M.: Actions of Boolean rings on sets. Algebra Universalis **28**, 153–187 (1991)
[7] Bignall R.J., Leech J.: Skew Boolean algebras and discriminator varieties. Algebra Universalis **33**, 387–398 (1995)
[8] Bigelow D., Burris S.: Boolean algebras of factor congruences. Acta Scientiarum Mathematicarum **54**, 11–20 (1990)
[9] Blok W.J., Pigozzi D.: Algebraizable Logics. Memoirs of the AMS, no. 396, American Mathematical Society, Providence, RI (1989)
[10] Blok W.J., Pigozzi D., On the structure of varieties with equationally definable principal congruences III. Algebra Universalis **32**, 545–608 (1994)

[11] Blok W.J., Pigozzi D.: On the structure of varieties with equationally definable principal congruences IV. Algebra Universalis **31**, 1–35 (1994)
[12] Blok W.J., Raftery J.G.: Assertionally equivalent quasivarieties. International Journal of Algebra and Computation **18**, 589–681 (2008)
[13] Burris S.N., Sankappanavar H.P.: A Course in Universal Algebra. Springer, Berlin (1981)
[14] Burris S.N., Werner H.: Sheaf constructions and their elementary properties. Transactions of the American Mathematical Society **248**, 269–309 (1979)
[15] Burris S.N., Werner H.: Remarks on Boolean products. Algebra Universalis **10**, 333–344 (1980)
[16] Comer S.: Representations by algebras of sections over Boolean spaces. Pacific Journal of Mathematics **38**, 29–38 (1971)
[17] Czelakowski, J.: Equivalential logics I. Studia Logica **45**, 227–236 (1981)
[18] Dicker R.M.: A set of independent axioms for Boolean algebra. Proceedings of the London Mathematical Society, **3**, 20–30 (1963)
[19] Galatos N., Jipsen P., Kowalski T., Ono H.: Residuated Lattices: An Algebraic Glimpse on Substructural Logics. Elsevier, Amsterdam (2007)
[20] Givant S., Halmos P.: Introduction to Boolean algebras. Springer, Berlin (2009)
[21] Hagermann J., Hermann C.: A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity. Archive for Mathematics **32**, 234–245 (1979)
[22] Jackson M., Stokes T.: Semigroups with if-then-else and halting programs. International Journal of Algebra and Computation **19**, 937–961 (2009)
[23] Jipsen P.: Generalizations of Boolean products for lattice-ordered algebras. Annals of Pure and Applied Logic **161**, 228–234 (2009)
[24] Jónsson B.: Congruence distributive varieties. Mathematica Japonica **42**, 353–402 (1995)
[25] Kowalski T., Paoli F.: Joins and subdirect products of varieties. Algebra Universalis **65**, 371–391 (2011)
[26] Kowalski T., Paoli F., Spinks M.: Quasi-subtractive varieties. Journal of Symbolic Logic **76**, 1261–1286 (2011)
[27] Kozen D.C.: On Hoare Logic and Kleene algebra with Tests. ACM Transactions on Computational Logic **1**, 60–76 (2000)
[28] Ledda A., Konig M., Paoli F., Giuntini R.: MV algebras and quantum computation. Studia Logica **82**, 245–270 (2006)
[29] Ledda A., Paoli F., Spinks M., Kowalski T.: Quasi-subtractive varieties 2 (in preparation)
[30] Manzonetto G., Salibra A.: Boolean algebras for lambda calculus. In: Proceedings of the 21th IEEE Symposium on Logic in Computer Science (LICS 2006), IEEE Computer Society, pp. 317–326 (2006)
[31] Manzonetto G., Salibra A.: From λ-calculus to universal algebra and back. In: Proceedings of MFCS'08 Symposium, Lecture Notes in Computer Science, vol. 5162, pp. 479–490 (2008)
[32] Manzonetto G., Salibra A.: Applying universal algebra to lambda calculus. Journal of Logic and Computation **20**, 877–915 (2010)
[33] Manzonetto G., Salibra A.: Lattices of equational theories as Church algebras. In: Drossos C. et al. (eds.) Proceedings of the Seventh Panhellenic Logic Symposium, Patras University Press, pp. 117–121 (2009)
[34] McCarthy J.: A basis for a mathematical theory of computation. In: Braffort P. and Hirschberg D. (eds.) Computer Programming and Formal Systems, North-Holland, Amsterdam, pp. 33–70 (1963)
[35] McKenzie R.N., McNulty G.F., Taylor W.F.: Algebras, Lattices, Varieties, vol. I. Wadsworth Brooks, Monterey, California (1987)
[36] Peirce R.S.: Modules over Commutative Regular Rings. Memoirs of the American Mathematical Society (1967)
[37] Pratt V.R.: Dynamic Algebras: Examples, Constructions, Applications. Technical Report MIT/LCS/TM-138, M.I.T. Laboratory for Computer Science, July 1979.

[38] Ursini A.: On subtractive varieties I. Algebra Universalis **31**, 204–222 (1994)
[39] Vaggione D.: Varieties in which the Pierce stalks are directly indecomposable. Journal of Algebra **184**, 424–434 (1996)
[40] Vaggione D.: $\mathcal{V}$ with factorable congruences and $\mathcal{V} = I\Gamma^a(\mathcal{V}_{DI})$ imply $\mathcal{V}$ is a discriminator variety. Acta Scientiarum Mathematicarum **62**, 359–368 (1996)
[41] Werner H.: Discriminator Algebras. Studien zur Algebra und ihre Anwendungen, Band 6, Akademie-Verlag, Berlin (1978)

Appendix

In this appendix we present a proof of Theorem 3.9. First, since Church algebras are Pierce algebras, from [39, Theorem 5(e)] we obtain

Lemma 5.9. *Let* $\mathbf{A}$ *and* $\mathbf{B}_i$ $(i \in I)$ *be Church algebras. If* $\mathbf{A} \leq \Pi_{i\in I}\mathbf{B}_i$ *is a subdirect product and* $e = (e_i : i \in I) \in A$*, then* e *is central in* $\mathbf{A}$ *iff* e *is central in* $\Pi_{i\in I}\mathbf{B}_i$ *iff* e_i *is central in each* $\mathbf{B}_i$*.*

Proof of Theorem 3.9. (1) $\Rightarrow$ (2) follows from [8, Proposition 3.4].

(2) $\Rightarrow$ (3) Let $e \in \mathbf{A} \in \mathcal{V}$, and $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by e, where $B = \{t^{\mathbf{A}}(e) : t \in T_\nu(x)\}$. We show that e is central in $\mathbf{A}$ iff it is central in $\mathbf{B}$. Suppose, by way of contradiction, that e is central in $\mathbf{B}$ but not central in $\mathbf{A}$. By Birkhoff's theorem $\mathbf{A}$ is a subdirect product $\Pi_{i\in J}\mathbf{A}/\theta_i$ of s.i. algebras $\mathbf{A}/\theta_i$. Since e is not central in $\mathbf{A}$, then by Lemma 5.9 there exists $j \in J$ such that e/θ_j is not central in the s.i. algebra $\mathbf{A}/\theta_j$. Since $\mathbf{B}/\theta_j$ is a subalgebra of the s.i. $\mathbf{A}/\theta_j$, then by hypothesis $\mathbf{B}/\theta_j$ is directly indecomposable. As e is central in $\mathbf{B}$, then e/θ_j is central in $\mathbf{B}/\theta_j$, and either $e \equiv_{\theta_j} 0$ or $e \equiv_{\theta_j} 1/\theta_j$. This contradicts the fact that e/θ_j is not central in $\mathbf{A}/\theta_j$. Then e is central in $\mathbf{A}$ and to check this it is sufficient to check that e is central in $B = \{t^{\mathbf{A}}(e) : t \in T_\nu(x)\}$. It follows that e is central in $\mathbf{A}$ if, and only if, the identities of Theorem 3.9(iii) are satisfied with $t, t_1, t_2, \overline{u}, \overline{v}$ ranging over the full set $T_\nu(x)$ and with the function symbol f ranging over the full signature ν. We denote by $\Pi(x) = \{q(x, t(x), t(x)) = t(x), \dots\}$ the set of these identities.

We now prove that central elements can be defined by a finite subset of $\Pi(x)$. Let Δ be the first-order formulas axiomatizing $\mathcal{V}_{DI}$. We consider a new similarity type $\nu' = \nu \cup \{m\}$ by adding to ν a new constant m. Let

$$\mathcal{K} = \{\mathbf{C} : \mathbf{C} = (\mathbf{A}, m^{\mathbf{C}}) \text{ with } \mathbf{A} \in \mathcal{V} \text{ and } m^{\mathbf{C}} \in A \text{ is a central element of } \mathbf{A}\}.$$

$\mathcal{K}$ is a variety of type ν' because it is axiomatized by the identities $Eq(\mathcal{V})$ axiomatizing $\mathcal{V}$ plus the identities $\Pi(m)$ axiomatizing that m is central. Moreover, $\mathbf{C} = (\mathbf{A}, m^{\mathbf{C}}) \in \mathcal{K}$ is directly indecomposable iff $\mathbf{A}$ is directly indecomposable in $\mathcal{V}$. It follows that $\mathcal{K}_{DI}$ is axiomatized, relative to $\mathcal{K}$, by Δ. We have

$$\Delta \cup \Pi(m) \models m \approx 0 \veebar m \approx 1.$$

By compactness there exists a finite subset $\Pi_0(m)$ of $\Pi(m)$ such that

$$\Delta \cup \Pi_0(m) \models m \approx 0 \veebar m \approx 1.$$

Assume now that an algebra $\mathbf{C} = (\mathbf{A}, m) \models \Pi_0(m)$ with $\mathbf{A} \in \mathcal{V}$, but m is not central in $\mathbf{A}$. Since $\mathbf{A}$ can be represented as subdirect product of s.i. algebras, then by Lemma 5.9 there exists a s.i. algebra $\mathbf{A}/\theta$ such that m/θ is not central in $\mathbf{A}/\theta$. From $(\mathbf{A}, m) \models \Pi_0(m)$ it follows that $(\mathbf{A}/\theta, m/\theta) \models \Pi_0(m)$. Since $\mathbf{A}/\theta$ is also directly indecomposable, we have that $(\mathbf{A}/\theta, m/\theta) \models \Delta \cup \Pi_0(m)$ that implies $(\mathbf{A}/\theta, m/\theta) \models m/\theta \approx 0/\theta \veebar m/\theta \approx 1/\theta$. Contradiction. It follows that, if $(\mathbf{A}, m) \models \Pi_0(m)$ with $\mathbf{A} \in \mathcal{V}$, then m is central in $\mathbf{A}$. In other words, for every algebra $\mathbf{A} \in \mathcal{V}$, m is central in $\mathbf{A}$ iff $(\mathbf{A}, m) \models \Pi_0(m)$. Since $\Pi_0(m) \subseteq \Pi(m)$ is finite, then we get (3).

(3) $\Rightarrow$ (1) Assume, by way of contradiction, that $\mathbf{A}/\theta_I$ is not directly indecomposable. Then there exists a nontrivial central element $a/\theta_I \in \mathbf{A}/\theta_I$. Consider the finite set Π of identities $t(x) \approx u(x)$ defining centrality in the variety $\mathcal{V}$. As $\mathbf{A}/\theta_I \models t(a/\theta_I) = u(a/\theta_I)$ for every $t \approx u \in \Pi$ and $\theta_I = \bigcup_{e \in I} \theta(e, 0)$, then there exists a central element $e \in I$ such that $t(a)\ \theta(e, 0)\ u(a)$ for every $t \approx u \in \Pi$. Define $\phi = \theta(e, 0)$, $\overline{\phi} = \theta(e, 1)$, $\psi = \theta(a/\phi, 0/\phi)$ and $\overline{\psi} = \theta(a/\phi, 1/\phi)$. Then a/ϕ is central in $\mathbf{A}/\phi$. Since $\mathbf{A} = \mathbf{A}/\phi \times \mathbf{A}/\overline{\phi}$ and $\mathbf{A}/\phi = (\mathbf{A}/\phi)/\psi \times (\mathbf{A}/\phi)/\overline{\psi}$, then we get the following decomposition of $\mathbf{A}$:

$$\mathbf{A} = (\mathbf{A}/\phi)/\psi \times [(\mathbf{A}/\phi)/\overline{\psi} \times \mathbf{A}/\overline{\phi}].$$

Therefore, there exists a central element $d \in A$ such that $(\mathbf{A}/\phi)/\psi = \mathbf{A}/\theta(d, 0)$. From $\phi = \theta(e, 0) \subseteq \theta(d, 0)$ it follows that $e \leq d$, so that $\theta(d, 0) \in I$. Moreover, by the definition of $\psi = \theta(a/\phi, 0/\phi)$ we obtain that $a\ \theta(d, 0)\ 0$ and then $a/\theta_I = 0/\theta_I$. A contradiction.

Antonino Salibra
Università Ca'Foscari Venezia
e-mail: salibra@si.unive.it

Antonio Ledda
Università di Cagliari
e-mail: ledda@unica.it

Francesco Paoli
Università di Cagliari
e-mail: paoli@unica.it

Tomasz Kowalski
Department of Mathematics and Statistics, La Trobe University
e-mail: t.kowalski@latrobe.edu.au