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Advertising exposures for a seasonal good in a segmented market *

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Abstract. The optimal control problem of determining advertising efforts for a seasonal good in a heterogeneous market is considered. We characterize optimal advertising exposures under different conditions: the general situation in which several wide-spectrum media are available, under the assumption of additive advertising effects on goodwill evolution, the ideal situation in which the advertising process can reach selectively each segment and the more realistic one in which a single medium reaches several segments with different effectiveness.

Keywords: Marketing; Advertising; Optimal Control.

JEL Classification Numbers: M37, M31, C61.

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1 Introduction

In the paper we study a special marketing problem formulated like an optimization model [5]. In particular we consider the problem of determining advertising efforts for selling a seasonal good in a heterogeneous market. In order to address such a kind of market it is important to segment it. The decision on advertising in a segmented market has been studied by Viscolani [12] and Buratto et al. [1, 2]. Here we consider the advertising in a segmented market for a firm which sells a seasonal product. In particular the results presented in this paper complement the analysis done by Favaretto and Viscolani in [4], in which the effect of advertising on sales was assumed strictly concave: the linear model of the present study is a limit case with respect to that assumption. Moreover here we consider the possibility of using several media. Sorato and Viscolani [11] have studied the possibility of using several media but in a homogeneous market and in a static context.

The general problem is formulated as an optimal control problem. The optimal advertising exposures are characterized under different conditions: the general situation in which several wide-spectrum media are available, under the assumption of additive advertising effects on goodwill evolution, the ideal situation in which the advertising process can reach selectively each segment, and the more realistic one in which a single medium reaches several segments with different effectiveness.

The paper is organized as follows. In Section 2 we present the model of a segmented market and describe the motion equation for the goodwill evolution. In Section 3 we consider the situation in which the firm may use several advertising media to reach different segments with variable effectiveness and we assume an additive model to represent the total effective advertising intensity. We formulate the optimal control problem and we determine the special bang–bang structure of the optimal solution using the Pontryagin Maximum Principle. In Section 4 we assume that the company has a set of segment–specific media: one advertising medium for each market segment. We derive some general results and we propose a special case with two segments and a quadratic production cost function. In Section 5 we present the situation in which the company has to use a single advertising medium to reach several segments with variable effectiveness.

2 Market segmentation and goodwill evolution

We consider the optimal control problem of determining advertising efforts for selling a seasonal good in a heterogeneous market. For example fashion markets are characterized by heterogeneity: factors such as age, personal disposal income, lifestyle, and culture appear to influence a specific and increasingly fragmented market context [9]. In order to address such a kind of market it is important to segment it. As M. Easey says, "Market segmentation is where the larger market is heterogeneous and can be broken down into smaller units that are similar in character" [3, p. 253]. Typical finite segmentations are obtained using such attributes as *region*, *age*, *gender*, *occupation*, *generation*, *lifestyle*, *occasions*, ...[6, p. 288]. Bikini is a typical example of a seasonal product in a heterogeneous market.

Let the consumer population be partitioned into n groups (segments), each one specified by the value $i \in \{1, 2, \ldots n\}$ of a suitable parameter *(segmentation attribute)*. Let $G_i(t)$ represent the stock of goodwill of the product at time t, for the (consumers in the) i segment. We refer to the definition of *goodwill* given by Nerlove and Arrow [8] to describe the variable which summarizes the effects of present and past advertising on the demand; the goodwill needs an advertising effort to increase, while it is subject to a spontaneous decay. Here we assume that the goodwill evolution satisfies the set of independent ordinary differential equations

$$\dot{G}_i(t) \;=\; w_i(t) - \delta_i G_i(t) \,, \qquad i = 1, \ldots n \,, \tag{1}$$

where $\delta_i > 0$ represents the goodwill depreciation rate for the members of the consumer group i and $w_i(t)$ is the effective advertising intensity at time t directed to that same group. For each fixed value of the parameter $i \in \{1, \ldots n\}$, i.e. for each segment, the dynamics of the goodwill given by (1) is essentially the same as the one proposed in [8]. Here, consistent with the assumption of distinct goodwill variables for different market segments, we further assume that both the advertising intensity and the goodwill decay parameter may depend on the attribute value i.

In the following we will write the motion equations (1) in vector notation:

$$\dot{\mathbf{G}}(t) \;=\; -\mathrm{diag}(\delta)\,\mathbf{G}(t) + \mathbf{w}(t) \,, \tag{2}$$

where $\mathrm{diag}(\delta)$ is the diagonal matrix with diagonal entries $\delta_1, \ldots \delta_n$.

We consider a firm which produces (or purchases), advertises and sells a seasonal product. The feature of the product being seasonal amounts to assume that production and sales take place in two disjoint and consecutive time intervals: the production and the sales intervals. We consider the sales interval $[0,1]$ only and we want to determine the optimal advertising policy, in order to maximize the net profit. As the season is a short time horizon, we consider undiscounted costs and revenue.

The value of the goodwill vector at the initial time 0 is a known datum:

$$\mathbf{G}(0) \;=\; \mathbf{G}^0 \geq 0 \,. \tag{3}$$

The demand intensity in each segment i depends linearly on the goodwill function $G_i(t)$, so that the market sales until time t, $x(t)$, satisfies the differential equation

$$\dot{x}(t) \;=\; \langle \beta, \mathbf{G}(t) \rangle \,, \tag{4}$$

and the initial condition

$$x(0) \;=\; 0 \,. \tag{5}$$

The component $\beta_i > 0$ of the vector parameter β is the marginal demand of goodwill in segment i: its value depends on the dimension of the segment, i.e. number of potential consumers in it, and on the interest of those consumers to the product.

We observe that if the manufacturer chooses the simplest advertising policy, with a constant effort for all the time, $\mathbf{w}(t) = \bar{\mathbf{w}} \geq \mathbf{0}$, then $G_i(t) = \left(G_i^0 - \bar{w}_i/\delta_i\right) e^{-\delta_i t} + \bar{w}_i/\delta_i$, $i = 1, \ldots n$, and $x(1) = \bar{x}(\bar{\mathbf{w}})$, where

$$\bar{x}(\bar{\mathbf{w}}) \;=\; \sum_{i=1}^{n} \frac{\beta_i G_i^0}{\delta_i}\left(1 - e^{-\delta_i}\right) \;+\; \sum_{i=1}^{n} \frac{\beta_i \bar{w}_i}{\delta_i^2}\left(\delta_i + e^{-\delta_i} - 1\right) \,. \tag{6}$$

The first term in the representation (6) of $\bar{x}(\bar{\mathbf{w}})$ is the quantity of good $\bar{x}(\mathbf{0})$, which the manufacturer can sell without any advertising effort.

3 Several media with additive effect

We assume that the company may use several advertising media which reach different segments with variable effectiveness. Let $\mathbf{u}(t) \in R^m$, $m \geq 1$, be the advertising effort: its j-th component $u_j(t) \geq 0$ be the medium j advertising effort, $j \in \{1, \dots m\}$. We assume that the medium j effective advertising intensity at time $t \in [0,1]$ directed to segment i be

$$w_i^j(t) \;=\; \gamma_{ij} u_j(t)\,, \qquad i = 1, \dots n\,,\; j = 1, \dots m\,, \tag{7}$$

for some segment and medium specific parameters $\gamma_{ij} \geq 0$, such that $\sum_{i=1}^n \gamma_{ij} = 1$, for each $j \in \{1, \dots m\}$. We assume that the total effective advertising intensity at time t directed to segment i is $w_i(t) = \sum_{j=1}^m \gamma_{ij} u_j(t)\,,\; i = 1, \dots n$, so that, in vector notation, we have

$$\mathbf{w}(t) \;=\; \Gamma \mathbf{u}(t)\;, \tag{8}$$

where $\Gamma = (\gamma_{ij}) \in M_{n\times m}(R)$. We call medium j *segment-spectrum* the j-th column γ_j of matrix Γ. Equation (8) represents the *additive advertising effects assumption* (see [11]).

The goodwill evolution is driven by the advertising efforts (the control functions) $u_i(t) \geq 0$ with constant marginal costs $\kappa_i > 0$, so that the total advertising cost rate at time t is $\langle \kappa, \mathbf{u}(t) \rangle$.

Let $s : R_+ \to R$ be a strictly increasing and concave function, where $s(x)$ represents the company profit, gross of advertising costs, from selling the quantity x of good. In particular we assume that $s(\cdot)$ is twice continuously differentiable, with $s'(\cdot) > 0$ and $s''(\cdot) < 0$. The *advertising problem* of maximizing the overall profit

$$J(\mathbf{u}) \;=\; s\left(x(1)\right) - \int_0^1 \langle \kappa, \mathbf{u}(t) \rangle \, dt \tag{9}$$

subject to the conditions (2–5), (8), and the control constraints

$$u_j(t) \;\in\; [0, \bar{u}_j]\,, \qquad j = 1, \dots m\,, \tag{10}$$

for some $\bar{u}_j > 0$, is an optimal control problem. The associated Hamiltonian is

$$H(\mathbf{G}, x, \mathbf{u}, \lambda, \mu) \;=\; -\langle \kappa, \mathbf{u} \rangle + \langle \lambda, \Gamma\mathbf{u} - \mathrm{diag}(\delta)\,\mathbf{G} \rangle + \mu \langle \beta, \mathbf{G} \rangle\;, \tag{11}$$

which is a linear function of $(\mathbf{G}, x, \mathbf{u})$. Using the Pontryagin Maximum Principle (see e.g. [10, p. 182]) we obtain that the optimal advertising effort with medium j must satisfy

$$u_j(t) \;=\; \begin{cases} \bar{u}_j, & \text{if } \langle \lambda(t), \gamma_j \rangle > \kappa_j\,, \\ 0, & \text{if } \langle \lambda(t), \gamma_j \rangle < \kappa_j\,; \end{cases} \tag{12}$$

the adjoint function $\mu(t)$ is constant and has the value

$$\mu(t) \;=\; \mu \;=\; s'(x(1)) \;>\; 0\,; \tag{13}$$

the adjoint function $\lambda(t)$ satisfies the linear differential equation

$$\dot{\lambda}(t) \;=\; \mathrm{diag}(\delta)\,\lambda(t) - \mu\beta\,, \tag{14}$$

and the transversality condition

$$\lambda(1) \;=\; \mathbf{0}\,, \tag{15}$$

so that $\lambda_i(t)$ is the positive and monotonically decreasing function

$$\lambda_i(t) \;=\; \frac{\mu\beta_i}{\delta_i}\left(1 - e^{\delta_i(t-1)}\right). \tag{16}$$

Therefore, for all j,

- either $-\kappa_j + \langle\lambda(0), \gamma_j\rangle > 0$, and there exists a unique $\tau_j \in (0,1)$ such that

$$\langle\lambda(\tau_j), \gamma_j\rangle \;=\; \kappa_j\,, \tag{17}$$

- or $-\kappa_j + \langle\lambda(0), \gamma_j\rangle \le 0$, and we define $\tau_j = 0$.

Now, in view of (12), the j-th component of the control is

$$u_j^*(t) \;=\; \begin{cases} \bar{u}_j, & t \in [0, \tau_j]\,, \\ 0, & t \in (\tau_j, 1]\,. \end{cases} \tag{18}$$

We remark that $\tau_j < 1$, for all j, because of (16) and the reasoning which has led us to define τ_j: it is optimal not to advertise just before the end of the sale interval, no matter which medium is considered.

We notice that if μ increases, then by (16) $\lambda_i(t)$ increases for all $t < 1$ and all i; hence by (17) τ_j increases for all j; and eventually $x(1)$ increases. It follows that, as μ grows from 0 to $+\infty$, $s'(x(1))$ moves decreasing from $s'(\bar{x}(\mathbf{0})) > 0$ to $s'(\bar{x}(\Gamma\bar{\mathbf{u}}))$, and there exists a unique solution μ^* to the transversality condition (13). Such value μ^* determines a unique solution to the necessary Pontryagin conditions.

In view of the fact that the Hamiltonian (11) is concave in $(\mathbf{G}, x, \mathbf{u})$, and $s(x)$ is concave in x, we know that the solution found is in fact optimal (see e.g. [10, p. 182]). We conclude that the advertising problem has the unique optimal control $\mathbf{u}^*(t)$ given by (18).

We have obtained that the total effective advertising intensity directed to any market segment is a monotonically decreasing, stepwise function, with m positive values at most, and the value 0 in a neighborhood of the final time 1. This qualitative result is in agreement with those obtained in [4].

3.1 Optimal solution

In view of equation (18), the unique optimal control has a special bang-bang structure, characterized by the m switch times τ_j, $j = 1, \ldots m$; these are advertising stopping times which we call *advertising exposures*.

Let us denote by x^* the sold product quantity $x(1)$ associated with the optimal solution. Then, in view of the equations (13) and (16), the characterization of the optimal advertising exposures reads as follows:
for all $j = 1, \ldots m$,

- either $\tau_j^* \in (0,1)$ and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i}\gamma_{ij}\left(1-e^{-\delta_i(1-\tau_j^*)}\right) \;=\; \frac{\kappa_j}{s'(x^*)}\,, \tag{19}$$

- or $\tau_j^* = 0$ and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i}\gamma_{ij}\left(1-e^{-\delta_i}\right) \;\le\; \frac{\kappa_j}{s'(x^*)}\,. \tag{20}$$

After recalling the definition (6) of $\bar{x}(\mathbf{w})$, we observe that $s'(x^*) \le s'(\bar{x}(\mathbf{0}))$, where $\bar{x}(\mathbf{0})$ is the minimum quantity of good the manufacturer can sell if he does not advertise towards any segment, and therefore there exists an upper bound $\bar{t}_j$ to the optimal exposure τ_j, i.e. $\tau_j \le \bar{t}_j$, which is characterized as follows:

- either $\bar{t}_j \in (0,1)$ and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i}\gamma_{ij}\left(1-e^{-\delta_i(1-\bar{t}_j)}\right) \;=\; \frac{\kappa_j}{s'(\bar{x}(\mathbf{0}))}\,, \tag{21}$$

- or $\bar{t}_j = 0$ and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i}\gamma_{ij}\left(1-e^{-\delta_i}\right) \;\le\; \frac{\kappa_j}{s'(\bar{x}(\mathbf{0}))}\,. \tag{22}$$

We have, in particular, that

$$\tau = \mathbf{0} \quad \Leftrightarrow \quad \bar{\mathbf{t}} = \mathbf{0}\,, \tag{23}$$

where we have denoted by $\bar{\mathbf{t}}$ the vector of upper bounds $\bar{t}_j$. Proposition (23) is obvious for the part $\Leftarrow$; as for the implication $\Rightarrow$, we observe that $\tau = \mathbf{0}$ requires that equation (22) holds for all j.

We notice that the upper bound $\bar{t}_j$, for any given j, can be determined independently of the other components of $\bar{\mathbf{t}}$, as the conditions (21)–(22) depend only on the medium j parameters. Moreover, no integration of the motion equations (1) and (2) is needed to determine $\bar{\mathbf{t}}$, as the elementary datum $\bar{x}(\mathbf{0})$ is sufficient information on the sold product.

Both observations are false for determining the optimal advertising exposure τ_j, because the conditions (19)–(20) involve the knowledge of all components of τ through the optimal sales value x^*, and we must integrate the equations (1) and (2) to know x^*.

Let $\mathbf{G}(t;\tau) = \mathbf{G}(t;\tau_1,...,\tau_m)$ be the unique solution to the motion equation (2) with

$$\mathbf{w}(t) \;=\; \sum_{j=1}^{m} \gamma_j u_j^*(t)\,; \tag{24}$$

we have that

$$\mathbf{G}(t;\tau) = e^{-\mathrm{diag}(\delta)t}\mathbf{G}^{\mathbf{0}} + \sum_{j=1}^{m} \bar{u}_j e^{-\mathrm{diag}(\delta)t}\int_0^{\min\{\tau_j,t\}} e^{\mathrm{diag}(\delta)(s)}\gamma_j ds\,. \tag{25}$$

After integrating the motion equation (4), we obtain that the quantity of good sold in the interval $[0,1]$ is

$$x(1;\tau) \;=\; \bar{x}(\mathbf{0}) + \sum_{i=1}^{n} \frac{\beta_i}{\delta_i^2} \sum_{j=1}^{m} \bar{u}_j \gamma_{ij} \left(\delta_i \tau_j + e^{-\delta_i} - e^{\delta_i(\tau_j - 1)} \right) . \tag{26}$$

4 Advertising timing with total segment–resolution

Here we consider the advertising timing problem with one advertising medium for each market segment, and so designed as to reach that segment only. In other terms the manufacturer has a set of segment-specific media: this situation is expressed by Γ being the nth order identity matrix, $\Gamma = I$, i.e.

$$\gamma_{ii} = 1 \,, \quad \gamma_{ij} = 0 \,, \; i \neq j \,. \tag{27}$$

Assuming that $\Gamma = I$ amounts to assuming that the firm may control an advertising process, with such a high segment-resolution, as to be able to reach each segment with the desired intensity. This is the *total segment-resolution* assumption, which, in the extreme, is characteristic of *micromarketing* [7, p.380].

From the results of Section 3.1 and, in particular, formulas (19–20) for τ, and (21–22) for the upper bound vector $\bar{\mathbf{t}}$, we obtain the following statement.
Let us define, preliminarily, the functions

$$T_j(y) \;=\; 1 + \frac{1}{\delta_j} \ln \left(1 - \frac{\delta_j \kappa_j}{\beta_j y} \right) , \qquad y \;>\; \frac{\delta_j \kappa_j}{\beta_j} , \quad j = 1, \ldots n \,. \tag{28}$$

If τ is an optimal solution, with associated sold product quantity $x^* \;=\; x(1,\tau)$, then, for all $j = 1, \ldots n$,

$$\tau_j \;\leq\; \bar{t}_j \;=\; \begin{cases} T_j\left(s'\left(\bar{x}(\mathbf{0})\right)\right) , & s'\left(\bar{x}(\mathbf{0})\right) \;>\; \frac{\delta_j \kappa_j}{\beta_j \left(1 - e^{-\delta_j}\right)} , \\ 0 , & s'\left(\bar{x}(\mathbf{0})\right) \;\leq\; \frac{\delta_j \kappa_j}{\beta_j \left(1 - e^{-\delta_j}\right)} ; \end{cases} \tag{29}$$

moreover

- either $\tau_j \in (0,1)$ and
$$\tau_j \;=\; T_j\left(s'\left(x^*\right)\right) , \tag{30}$$
- or $\tau_j = 0$ and
$$s'\left(x^*\right) \;\leq\; \frac{\delta_j \kappa_j}{\beta_j \left(1 - e^{-\delta_j}\right)} . \tag{31}$$

We notice, in particular, that

$$\tau = \mathbf{0} \quad \Leftrightarrow \quad \bar{\mathbf{t}} = \mathbf{0} \quad \Leftrightarrow \quad s'\left(\bar{x}(\mathbf{0})\right) \;\leq\; \frac{\delta_j \kappa_j}{\beta_j \left(1 - e^{-\delta_j}\right)} , \quad \text{for all } j \,. \tag{32}$$

EXAMPLE Let us consider a two–segment market and the profit $s(x) = x - c(x)$, with quadratic production cost, i.e.

$$c(x) = \frac{1}{2}\omega x^2, \tag{33}$$

where $\omega > 0$. Let $\omega = 0.01$ and

$$G_1^0 = G_2^0 = 2, \quad \delta_1 = 0.001, \quad \delta_2 = 0.002, \quad \beta_1 = 2, \quad \beta_2 = 3, \quad \bar{u}_1 = \bar{u}_2 = 2$$

be fixed, whereas we consider different values for the advertising cost parameters κ_1, κ_2. We obtain that the minimum and maximum observable demands for the seasonal good are

$$\bar{x}(\mathbf{0}) \simeq 9.992, \qquad \bar{x}(\bar{\mathbf{u}}) \simeq 14.989,$$

and observe that

$$c'(\bar{x}(\mathbf{0})) = \omega\bar{x}(\mathbf{0}) \simeq 0.1 < 1,$$

so that it may be profitable to produce at a higher level than $\bar{x}(\mathbf{0})$ and advertise accordingly. Table 1 provides the essential resuls for a set of choices of the advertising cost parameters κ_1, κ_2. We observe that in all cases the times $\bar{t}_1, \bar{t}_2$ are rather good excess approximations

Table 1: Optimal advertising exposures

κ_1	κ_2	$\bar{t}_1$	$\bar{t}_2$	τ_1	τ_2	x^*	profit
2.0	3.0	0	0	0	0	9.992	9.473
2.0	2.0	0	0.259	0	0.248	11.294	9.665
1.5	2.0	0.166	0.259	0.149	0.243	11.825	9.705
1.0	2.0	0.444	0.259	0.428	0.237	12.588	9.993
1.0	1.5	0.444	0.444	0.423	0.423	13.324	10.322
1.0	1.0	0.444	0.630	0.419	0.613	13.866	10.840
0.5	0.5	0.722	0.815	0.707	0.805	14.703	12.111
0.3	0.3	0.833	0.889	0.824	0.882	14.886	12.754
0.1	0.1	0.944	0.963	0.941	0.961	14.978	13.476

of the optimal advertising exposures τ_1, τ_2.

5 Using a single medium

Let us consider the situation in which the company has to use a single advertising medium which reaches several segments with variable effectiveness.

The matrix Γ is $n \times 1$: it has a column only. For simplicity we may drop the advertising medium index $j = 1$.

If τ is an optimal solution, with associated sold product quantity $x^* = x(1, \tau)$, then,

- either $\tau \in (0,1)$ and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i} \gamma_i \left(1 - e^{-\delta_i (1-\tau)}\right) = \frac{\kappa}{s'(x^*)}, \tag{34}$$

and the upper bound $\bar{t}$ is characterized by

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i} \gamma_i \left(1 - e^{-\delta_i (1-\bar{t})}\right) = \frac{\kappa}{s'(\bar{x}(\mathbf{0}))}, \tag{35}$$

- or $\tau = 0$ (so that $x^* = \bar{x}(\mathbf{0})$) and

$$\sum_{i=1}^{n} \frac{\beta_i}{\delta_i} \gamma_i \left(1 - e^{-\delta_i}\right) \leq \frac{\kappa}{s'(\bar{x}(\mathbf{0}))}. \tag{36}$$

6 Conclusion

We have brought some market segmentation concepts into the statement of an advertising problem for a seasonal product.

We have considered three kinds of situations: several wide-spectrum media are available and they have an additive effects on goodwill evolution; the advertising process can reach selectively each segment; a single medium reaches several segments with different effectiveness. In these cases we have formulated the general optimal control problem and have obtained the unique optimal solution.

For the sake of simplicity we have assumed that the objective is the profit in one season, without any consideration of any further firm activity. The absence of any constraint on the state at terminal time is justified by such restrictive assumption. A natural step in this direction should include some constraints on the goodwill at the final time. The associated transversality conditions would change the qualitative features of optimal solutions.

Further directions of analysis, with practical implications, are to consider sets of media with special features in order to represent real life situations and using numerical simulation to analyze the solution sensitivity to characteristic parameters of the model in special instances.

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