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Fixed-point-free 2-finite automorphism groups

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Abstract. A fixed-point-free group G of automorphisms of an abelian group is shown to be locally finite if any two elements of G generate a finite subgroup.

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A group G of automorphisms of a group $(A, +)$ is called *fixed-point-free*, if $g(a) \neq a$ for $1 \neq g \in G$ and $0 \neq a \in A$. One says that a group G is n-*finite*, if any n elements of G generate a finite subgroup; local finiteness means that this holds for every positive integer n. We prove the following results.

Theorem 1. *Let G be a fixed-point-free group of automorphisms of some abelian group. If G is 2-finite, then G is locally finite. Moreover, G is countable. In fact G/G' and G'/G'' have locally cyclic subgroups of index at most 2, and G'' is finite and isomorphic to $\mathrm{SL}(2, 5)$ or to the quaternion group Q_8 of order 8 or $|G''| \leq 2$.*

With the modified assumption that G is 1-finite, i.e., periodic (or even of finite exponent), G need not be locally finite, as the examples at the end of this paper show. For groups G as in Theorem 1, the subgroups generated by all elements of prime order are characterized in Sozutov [9, Theorem 1]. Our theorem does not readily follow from Theorem 3 in [9], since that Theorem 3 imposes a weak version of 2-finiteness on the whole Frobenius group, and not just on the Frobenius complement.

Corollary 2. *Let N be a nearfield such that the multiplicative group $N^\times$ of N is 2-finite. Then N is a locally finite nearfield.*

Corollary 3. *Let G be a sharply 2-transitive permutation group, and assume that G is 2-finite. Then G is locally finite, and if G is infinite, then*

$$G \leq \mathrm{A\Gamma L}(1, F) := \{x \mapsto a \cdot \alpha(x) + b \mid a, b \in F, a \neq 0, \alpha \in \mathrm{Aut}F\}$$

for some locally finite field F.

The locally finite nearfields are known in some detail (see [3] or [11, IV]). Corollaries 2 and 3 extend the main results of [10]. According to Zassenhaus [13, Satz 17], there are exactly seven finite sharply 2-transitive groups that are not subgroups of $\mathrm{A\Gamma L}(1, F)$ for a finite field F; see also [7, 20.3].

By a result of Jacobson, every skew field with periodic multiplicative group is locally finite (and commutative); see [1, Theorem 3.9.5]. Thus one might conjecture that the two corollaries hold with the weaker assumption of periodicity instead of 2-finiteness.

Proof of Theorem 1. The group G contains at most one involution g, since g fixes $a + g(a)$ for every element a of the abelian group acted on, hence $a + g(a) = 0$, and g is the inversion and belongs to the center of G. This implies that the finite 2-subgroups of G are cyclic or generalized quaternion groups; see [8, 5.3.6] or [12, 5.3.2]. For odd primes p, all finite p-subgroups of G are cyclic (see [8, 10.5.5] and [4, Lemma 2.6], or [14, Lemma 2]).

Every finite subgroup H of G has a normal subgroup Z such that all Sylow subgroups of Z are cyclic and

(a) H'' is trivial and $|H : Z| \leq 2$, or
(b) H'' has order 2 and $|Z| \equiv 2 \bmod 4$ and $H/Z \cong A_4$, or
(c) $H'' \cong Q_8$ and $|Z| \equiv 2 \bmod 4$ and $H/Z \cong S_4$, or
(d) $H'' \cong \mathrm{SL}(2, 5)$ and $|H : H''Z| \leq 2$;

see [12, 6.1.9, 6.1.11], [7, 18.2] for the solvable case, and [12, 6.3.1] or [7, 18.6] for the nonsolvable case (or [13, Sätze 6, 8, 16]).

All subgroups of order 3 are conjugate in G, by 2-finiteness and Sylow's theorem. According to Zhurtov [16, Lemma 8], G contains at most one subgroup isomorphic to $\mathrm{SL}(2, 5)$; this follows also from [15, Theorem 1] or [9, Theorem 1]. Hence the subgroup T generated by all copies of $\mathrm{SL}(2, 5)$ and all involutions in G is finite and normal in G. Below we show that the quotient $\Gamma := G/T$ is locally finite (which implies that G is locally finite).

Every finite subgroup $\widetilde{H}$ of Γ has a normal subgroup $\widetilde{Z}$ such that all Sylow subgroups of $\widetilde{Z}$ are cyclic and $\widetilde{H}/\widetilde{Z}$ is isomorphic to a quotient of A_4 or S_4, hence a $\{2, 3\}$-group. Repeatedly applying [8, 10.1.9] we obtain for $n \geq 5$ that the set $\widetilde{H}_n = \{h \in \widetilde{H} \mid$ no prime divisor of $|\langle h\rangle|$ is smaller than $n\} \subseteq \widetilde{Z}$ is a subgroup of $\widetilde{H}$. By 2-finiteness, this property of $\widetilde{H}$ carries over to Γ, and the set

$$\Gamma_n = \{g \in \Gamma \mid \text{no prime divisor of } |\langle g\rangle| \text{ is smaller than } n\}$$

is a (normal) subgroup of Γ for every integer $n \geq 5$.

We claim that $\mathrm{O}(\Gamma)$*, the largest normal subgroup of* Γ *consisting of elements of odd order, is locally finite.* For every prime p, all p-sections of $\mathrm{O}(\Gamma)$ are locally cyclic (by 2-finiteness). We have $\Gamma_5 \subseteq \mathrm{O}(\Gamma)$, and $\mathrm{O}(\Gamma)/\Gamma_5$ is a locally cyclic 3-group. For $n \geq 5$ the quotient Γ_n/Γ_{n+1} is a p-group (in fact, trivial unless $n = p$ is a prime) which is locally cyclic, as well. Since local finiteness is an extension property (see [8, 14.3.1]), the quotient $\mathrm{O}(\Gamma)/\Gamma_n$ is locally finite for $n \geq 5$. All Sylow subgroups of a finite subgroup S of $\mathrm{O}(\Gamma)/\Gamma_n$ are cyclic, hence $S'' = \{1\}$ by a result of Hölder, Burnside and Zassenhaus, see [8, 10.1.10] or [12, 5.4.1] or [13, Satz 5]. Since $\mathrm{O}(\Gamma)/\Gamma_n$ is locally finite, we infer that $\mathrm{O}(\Gamma)'' \subseteq \Gamma_n$ for $n \geq 5$, hence $\mathrm{O}(\Gamma)''$ is trivial. This implies that $\mathrm{O}(\Gamma)$ is locally finite (because periodic abelian groups are locally finite and local finiteness is an extension property).

Now we claim that the $\{2,3\}$*-group* $\bar{\Gamma} := \Gamma/\mathrm{O}(\Gamma)$ *is locally finite.* Suppose $x \in \bar{\Gamma}$ has order $3k$ with $k \in \{2,3\}$. By definition of $\mathrm{O}(\Gamma)$, the subgroup $\langle x^k \rangle$ of order 3 is not normal in $\bar{\Gamma}$, hence $\langle x^k \rangle \neq \langle y^k \rangle$ for some conjugate y of x. The finite group $\bar{H} = \langle x, y \rangle$ has type (b) or (c), since subgroups of $\bar{\Gamma}$ of type (a) have a normal Sylow 3-subgroup by [8, 10.1.9] and therefore only one subgroup of order 3. The corresponding subgroup $\bar{Z}$ has odd order, hence $\bar{Z} = \mathrm{O}(\bar{H})$ is a 3-group. For $k = 3$ we obtain $\langle x^3 \rangle = \bar{Z} = \langle y^3 \rangle$, which is a contradiction. For $k = 2$ we have $x^3 \notin \bar{Z}$, hence $x^2 \in \bar{Z}$, and analogously $y^2 \in \bar{Z}$; hence $\bar{H}/\bar{Z} \cong A_4, S_4$ is generated by two involutions, which is absurd. Thus $\bar{\Gamma}$ has no element of order 6 or 9, and $\bar{Z} = \mathrm{O}(\bar{H})$ is trivial for finite subgroups $\bar{H}$ of $\bar{\Gamma}$ of type (b), (c).

By Zhurtov [16, Lemma 8], the elements of order 3 in $\bar{\Gamma}$ generate a locally finite (normal) subgroup; since the quotient is a 2-group which is locally finite (any two squares commute), this implies that $\bar{\Gamma}$ is locally finite. We offer the following alternative argument. If $x, y \in \bar{\Gamma}$ are 2-elements and $\langle x, y \rangle$ is of type (a), then x^2 and y^2 centralize the 3-group $\mathrm{O}(\langle x, y \rangle)$ and belong to a cyclic group of squares of 2-elements. Thus the set $\Delta := \{x^2 \mid x \in \bar{\Gamma} \text{ is a 2-element}\}$ is an abelian (normal, locally finite) subgroup of $\bar{\Gamma}$. Every element of $\bar{\Gamma}/\Delta$ has order 1, 2 or 3, hence $\bar{\Gamma}/\Delta$ is locally finite by a theorem of B. H. Neumann [5] (in fact, $\bar{\Gamma}/\Delta$ is finite by [4, Lemma 2.4], since its 2-subgroups have order at most 4). Thus $\bar{\Gamma}$ is locally finite.

We conclude that $\Gamma = G/T$ is locally finite, and so is G, as T is finite. Every finite subgroup H of the locally finite group G satisfies $|H''| \leq 120$, hence G'' is finite and coincides with one of the groups H'' listed above.

The finite subgroups of the abelian groups G/G' and G'/G'' are cyclic or have a cyclic subgroup of index 2. Hence G/G' and G'/G'' have locally cyclic subgroups of index at most 2. This implies that G/G' and G'/G'' are countable, and so is G, as G'' is finite. □

Proof of Corollary 2. The multiplicative group $N^\times$ acts faithfully on the additive group $(N, +)$ as a fixed-point-free automorphism group. By Theorem 1, $N^\times$ is locally finite. This implies that N is a locally finite nearfield; see Wähling [10, Satz 2]. □

Proof of Corollary 3. According to the theorem in Collins [2], G contains a sharply transitive abelian normal subgroup $(N, +)$. Thus G is a semidirect product NG_0 where $G_0 \leq \mathrm{Aut}(N, +)$ is fixed-point-free (and transitive on the set $N\backslash\{0\}$). Therefore $G = \{x \mapsto a \circ x + b \mid a, b \in N, a \neq 0\}$ for a suitable nearfield-multiplication $\circ$ on $(N, +)$. Since $G_0 \cong (N\backslash\{0\}, \circ)$ is 2-finite, Corollary 2 implies that the nearfield $(N, +, \circ)$ is locally finite.

Every infinite, locally finite nearfield is a 'regular' nearfield constructed from a locally finite field; see [3, Theorem 2.2] and its proof, or [11, IV, 9.5a]. This means that there exists a field multiplication $\cdot$ such that $F = (N, +, \cdot)$ is a locally finite field and for $a, x \in N$ one has $a \circ x = a \cdot \alpha(x)$, where $\alpha \in \mathrm{Aut}F$ depends on a only. $\square$

Monstrous examples. Let $p > 2$ be a prime number. For every integer t such that p^t is sufficiently large, there exists a finitely generated infinite group G of exponent p^t such that the center C of G has order p and contains all elements of order p in G; see [6, Theorems 31.2, 31.3, 31.5 and (the proof of) 31.7]. If $t = 2$ and if the prime p is sufficiently large, the quotient G/C is a so-called Tarski monster: it is infinite, simple and all its proper nontrivial subgroups are cyclic of order p (see [6, Section 28]).

Let $C = \langle c \rangle$ and let I be the ideal generated by the central element $1 + c + \cdots + c^{p-1}$ in the rational group ring $\mathbb{Q}G$. The natural action of G on $\mathbb{Q}G$ yields a faithful action of G on $\mathbb{Q}G/I$ which is fixed-point-free, because every element $a + I \in \mathbb{Q}G/I$ fixed by a nontrivial element of G is fixed also by c, hence $pa \in a + ca + \cdots + c^{p-1}a + I \subseteq Ia + I = I$ and $a \in I$.

This action of G is not transitive on the non-zero elements of $\mathbb{Q}G/I$; indeed, a non-zero element cannot be mapped to its negative (as G contains no involution), and $1 + I$ cannot be mapped to $1 - c + I$.

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